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A CLASS OF GROUPS DETERMINED BY THEIR 3-S-RINGS

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ABSTRACT. We study the class of groups G satisfying the condition that, for every ordered pair $x, y \in G$, one of the following is true: (1) $xy = yx$; (2) x and y are conjugate; (3) $x^y = x^{-1}$; (4) $y^x = y^{-1}$. We describe all such groups completely and give a further condition that characterizes these groups in terms of their 3-S-rings.

1. Introduction. When Frobenius developed the concept of a character table for non-abelian groups, he also studied [**4**] the k-characters of a group, these being functions defined on the k-classes for $k = 2, 3, \ldots$, where 1-classes are conjugacy classes and 1-characters are the usual group characters.

The center of the group ring of a finite group G, $Z(\mathbb{C}G)$, is also the S-ring over G determined by the 1-classes. For $k > 1$, generalized centralizer rings, called k-S-rings, are defined (see Section 2 for details) by taking the S-ring over G^k determined by the k-classes of G. It is a standard result to show that the 1-characters, i.e., the character table, can be determined from the centralizer ring $Z(\mathbb{C}G)$. However, for $k = 2$, it is possible to find groups with the same 2-S-rings, but different 2-characters, and to find groups with the same 2-characters, but different 2-S-rings [**5**]. We note that the second author has recently shown that the 3-S-ring of a finite group G determines G [**9**].

The fact that $Z(\mathbb{C}G)$ is a commutative ring plays a key role in calculating the character table from $Z(\mathbb{C}G)$. Further, commutative S-rings have been studied in [**11**]. In this paper we characterize groups which have a commutative 3-S-ring, which we will call $\mathfrak{S}^{(3)}$-com groups. We prove the following results:

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Theorem 1.1. *If a finite group G is $\mathfrak{S}^{(3)}$-com, then for all ordered pairs $x, y \in G$ we have at least one of:*

(1) $xy = yx$;
(2) *x and y are conjugate*;
(3) $x^y = x^{-1}$;
(4) $y^x = y^{-1}$.

A finite group G will be called a pre-$\mathfrak{S}^{(3)}$-com group if it satisfies the conclusion of Theorem 1.1. We determine all pre-$\mathfrak{S}^{(3)}$-com groups:

Theorem 1.2. *A finite group G is a pre-$\mathfrak{S}^{(3)}$-com group if and only if G satisfies one of the following conditions:*

(1) *G is abelian*;
(2) *G is the generalized dihedral group of an abelian group N of odd order, i.e., $G = N \rtimes_\phi C_2$ where C_2 is the cyclic group of order 2 and its generator conjugates elements of N to their inverses*;
(3) *$G \cong Q_8 \times C_2^r$, where Q_8 is the quaternion group of order 8 and $r \geq 0$.*

We note that each of the above groups have irreducible characters of degree at most 2, such groups having been characterized by Amitsur [**1**]. We refer to the above three types as *abelian*, *dihedral* and 2-*group* types.

Clearly the abelian groups are $\mathfrak{S}^{(3)}$-com. We prove the following results:

Theorem 1.3. *No non-abelian finite 2-group has a commutative 3-S-ring.*

Theorem 1.4. *If G is a generalized dihedral of order $2n$ with n odd, then G is $\mathfrak{S}^{(3)}$-com.*

From these, our main result follows immediately:

Theorem 1.5. *A finite non-abelian group G is $\mathfrak{S}^{(3)}$-com exactly when G is generalized dihedral of order $2n$, n odd.*

As a corollary of our main result, we are able to answer the corresponding question regarding commutative 4-S-rings.

Theorem 1.6. *A finite group G has a commutative 4-S-ring if and only if G is abelian.*

2. S-rings and k-S-rings. For $X = \{x_1, \dots, x_k\} \subseteq G$, $|X| = k$, we let $\overline{X} = x_1 + \cdots + x_k \in \mathbb{C}G$. We now define the concept of a k-S-ring (from [**5**]), first recalling the definition of an S-ring [**3, 10, 13**]:

An S-*ring* over a group G is a sub-ring of $\mathbb{C}G$ which is constructed from a partition $\mathbb{S} = \{\Gamma_1, \Gamma_2, \dots, \Gamma_m\}$ of the elements of G: $G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, with $\Gamma_1 = \{e\}$, satisfying:

(1) If $i \geq 1$ and $\Gamma_i = \{g_1, \dots, g_s\}$, then there is some $j \geq 1$ such that $\Gamma_i^{-1} := \{g_1^{-1}, \dots, g_s^{-1}\}$ is equal to Γ_j;
(2) If $i, j \geq 1$, then $\overline{\Gamma}_i \overline{\Gamma}_j = \sum_k \lambda_{ijk} \overline{\Gamma}_k$ where λ_{ijk} is a non-negative integer for all i, j, k.

The Γ_i are called the *principal sets* of the S-ring. The $\overline{\Gamma_i}$ are called the *principal elements*. The S-ring is thus $\langle \overline{\Gamma_1}, \dots, \overline{\Gamma_m} \rangle$.

Let G be a finite group. Fix $k \geq 1$, and let the symmetric group Σ_k act on G^k by permuting entries:

$$(g_1, g_2, \dots, g_k)\sigma = (g_{(1)\sigma}, g_{(2)\sigma}, \dots, g_{(k)\sigma})$$

Let G act on G^k by diagonal conjugation:

$$(g_1, g_2, \dots, g_k)^g = (g_1^g, g_2^g, \dots, g_k^g).$$

Let $\widetilde{G}_k$ denote the permutation group generated by these actions of Σ_k and G on G^k. Then the $\widetilde{G}_k$-orbits of the action are called k-classes, with the $\widetilde{G}_k$-orbit of $(g_1, g_2, \dots, g_k) \in G^k$ being denoted $C_G^{(k)}(g_1, g_2, \dots, g_k)$. The k-classes determine an S-ring over G^k which we call the *k-S-ring of G* and which we denote by $\mathfrak{S}_G^{(k)}$.

In the case $k = 1$, we see that $\mathfrak{S}_G^{(1)}$ is just the centralizer ring $Z(\mathbb{C}\mathbb{G})$. Thus, we think of the k-S-rings of G as generalized centralizer rings.

The *k-classes* as defined here are the same k-classes used by Frobenius in his study of k-characters. In the definition of the 2-character table of a group the notion of the 2-S-ring is used [**8**]. See also [**12**] for a discussion of k-S-rings and k-characters.

In [**5**], the authors have shown that there are non-isomorphic groups with the same 2-S-ring.

3. Proof of Theorem 1.2. We have the following conventions. All groups will be assumed finite. We will be constantly referring to conditions (1)–(4) of Theorem 1.1 for an ordered pair of elements $x, y \in G$. We let $\sim_H$ denote conjugacy in the subgroup H of G. We let $|x|$ denote the order of $x \in G$. As usual, we have: $x^y = y^{-1}xy$, $(x, y) = x^{-1}y^{-1}xy$. We let y^G denote the conjugacy class of $y \in G$. We let 1 denote the identity element of the group G.

Suppose that G is a group satisfying (1), (2) or (3) of Theorem 1.2. We show that G is pre-$\mathfrak{S}^{(3)}$-com i.e., that any pair $x, y \in G$ satisfies one of (1), (2), (3) or (4). This is clear if G is abelian. For a generalized dihedral dihedral group $G = N \rtimes C_2$, with the order of N odd, $G \setminus N$ consists entirely of involutions which conjugate elements of N to their inverses [**7,** page 57]. This means that, for any involution $t \in G$, $n^t = n^{-1}$ so that $t^n = n^{-2}t$. Since $|N|$ is odd and N is abelian, the map $n \mapsto n^2$ gives a surjection of N, so $t^G = G \setminus N$, i.e., any two involutions must be conjugate. Finally, suppose $G \cong Q_8 \times C_2^r$. Let $x, y \in Q_8 \leq G$ be generators for the subgroup Q_8, so that $x^y = x^{-1}, y^x = y^{-1}, x^4 = y^4 = 1$. If $u, v \in C_2^r$, then $(xu)^{yv} = x^y u = x^{-1}u = (xu)^{-1}$ and $(xyu)^{yv} = (xy)^y u = (xyu)^{-1}$, and all other cases follow similarly. Thus, any $g, h \in G$ either commute or satisfy $g^h = g^{-1}$.

So all of the groups of types (1), (2) and (3) of Theorem 1.2 are pre-$\mathfrak{S}^{(3)}$-com groups, and it remains to show the converse.

Throughout the remainder of this section we will let G be a pre-$\mathfrak{S}^{(3)}$-com group. Let N be the (possibly trivial) set of elements of G which have odd order. Notice that, if $y \in N$ and $x \in G$ satisfy $x^y = x^{-1}$, then $x = x^{y^{|y|}} = x^{-1}$. The following two results follow immediately:

Lemma 3.1. *If $x, y \in N$ then $xy = yx$ or $x \sim y$.*

Proof. Since $x \neq 1$ has odd order, we know $x \neq x^{-1}$, and the same is true for y, so it follows from the remark above that the pair x, y cannot satisfy (3) or (4). ☐

Lemma 3.2. *If $y \in N$ and $x \in G \setminus N$, then $xy = yx$ or $y^x = y^{-1}$.*

Proof. Cases (2) and (3) cannot be true if $y \neq 1$, so only Cases (1) and (4) remain. ☐

We also have the following:

Lemma 3.3. *If, for some $y \in N$, we have $xy = yx$ for all $x \in G \setminus N$, then y is central.*

Proof. In this situation, Lemma 3.1 shows that y commutes with every element not in its conjugacy class y^G, i.e., $G \setminus y^G \subseteq C_G(y)$. But, since $|C_G(y)| = |G|/|y^G|$ this gives $|G| - |y^G| \leq |G|/|y^G|$, or equivalently that $|y^G| \leq |G|/(|G| - |y^G|) \leq 2$, and if we have equality, then $|C_G(y)| = 2$, giving a contradiction, since y has odd order. ☐

Corollary 3.4. *If G is a pre-$\mathfrak{S}^{(3)}$-com group of odd order, then G is abelian.*

Proof. The requirements of Lemma 3.3 are vacuously satisfied, so every element of G is central. ☐

Let $|G| = 2^k m$ where m is odd. We handle the cases where $m \neq 1$, and the 2-group case separately.

Lemma 3.5. *If, for some $y \in N$, we have $y^x = y^{-1}$ for some $x \in G \backslash N$, then $y^G = \{y, y^{-1}\}$.*

Proof. From Lemma 3.1, we know that for any $g \in N$, either $yg = gy$ or $y \sim g$. From Lemma 3.2, we know that for any $h \in G \backslash N$, either h and y commute or $y^h = y^{-1}$. So all elements of G not in y^G either commute with y or conjugate y to its inverse. So we have $G \backslash y^G \subset C_G(y) \cup C_G(y)x$ or, more precisely, $G \setminus y^G \cup \{y, y^{-1}\} \subset C_G(y) \cup C_G(y)x$. This means that $|G| - |y^G| + 2 \leq 2|C_G(y)| = 2|G|/|y^G|$. We can simplify this inequality to get $|G|(|y^G| - 2) \leq |y^G|^2 - 2|y^G|$. If we assume $|y^G| > 2$, then dividing by $|y^G| - 2$ gives $|G| \leq |y^G|$, which is a contradiction. Thus, $|y^G| \leq 2$, and since $y^{-1} \sim y$ we have $y^G = \{y, y^{-1}\}$. ☐

Theorem 3.6. *Let G be a pre-$\mathfrak{S}^{(3)}$-com group with $|G| = 2^k m$ where $k \geq 1$ and $m \neq 1$ is odd. Suppose that, for some $y \in N$, $t \in G \setminus N$ we have $y^t = y^{-1}$. Then $G = N \rtimes C_2$ is generalized dihedral.*

Proof. We know from Lemma 3.5 that $|G : C_G(y)| = 2$, so $C_G(y)$ is normal in G. Assume that, for some $x \in C_G(y)$, we have $x^t = x$. Then $t^{xy} = t^y$ which cannot be t or t^{-1}, since y has odd order. So $(xy)^t = (xy)^{-1}$, but $(xy)^t = xy^{-1}$, so in fact we must have $x = x^{-1}$. This means we have $h^t = h^{-1}$ for any $h \in C_G(y)$. Also, if $h, g \in C_G(y)$, then

$$h^{-1}g^{-1} = (gh)^{-1} = (gh)^x = g^x h^x = g^{-1}h^{-1},$$

so $C_G(y)$ is abelian.

Next we show that $C_G(y) = N$. If $C_G(y)$ has even order, then $G \setminus C_G(y)$ is not a single class of involutions, because the map $g \to g^2$ is not a surjection of $C_G(y)$. Choose $gt \in G \setminus C_G(y)$ which is not conjugate to yt. For involutions, (3) and (4) reduce to (1), and so gt and yt must commute; therefore, their product $gtyt = gy^{-1}$ must be an involution. But this is impossible, since $C_G(y)$ is abelian and y has odd order. So $C_G(y) = N$, and this depends only on the fact that t inverts y. We showed that t inverts every element of $C_G(y)$, so for any $h \in C_G(y) = N$ we have $C_G(h) = N$; we thus see that in fact $C_G(N) = N$.

Now there is an involution s in $G \setminus N$, and $N = C_G(N)$, so from Lemma 3.5 we know $h^s = h^{-1}$ for any $h \in N$. So $G = \langle N, s\rangle$ and is generalized dihedral. □

Theorem 3.7. *If G is a pre-$\mathfrak{S}^{(3)}$-com group with $|G| = 2^k m$ where $k \geq 1$ and $m \neq 1$ is odd, and for any $y \in N$ we have $xy = yx$ for all $x \in G \setminus N$, then G is abelian.*

Proof. From Lemma 3.3, it follows that all elements of odd order are central, and N is in fact a subgroup of the center of G.

We will also use the fact that if $x, y \in G$ have order a power of two, then $xm \sim yn$ for $n, m \in N \setminus \{1\}$ only if $n = m$. This is true because $z^{-1}xmz = yn$ implies $z^{-1}xz = ynm^{-1}$, where the term on the left has order a power of two, and the term on the right must also, but this only occurs if $nm^{-1} = 1$.

Suppose, by way of contradiction, that there are $x, y \in G \setminus N$ which do not commute. Then their 2-parts [**6,** page 134] also cannot commute. Let s be the 2-part of x, and t is the 2-part of y.

Choose $n, m \in N \setminus \{1\}$ with $n \neq m$. Then $ns \not\sim mt$, and they cannot commute, so we must have either $(ns)^{mt} = (ns)^{-1}$ or $(mt)^{ns} = (mt)^{-1}$. In the first case we get

$$s^{-1}n^{-1} = (ns)^{mt} = n^{mt}s^{mt} = ns^t,$$

so that $s^t = s^{-1}(n^2)^{-1}$. But the left hand side has order a power of 2 and the right hand side does not, since s and n commute, giving a contradiction. And similarly the second case is also impossible.

So G must be abelian. $\square$

The rest of the proof of Theorem 1.2, the 2-group case, is by induction on $|G|$. As shown earlier, Q_8 is pre-$\mathfrak{S}^{(3)}$-com. If the dihedral group of order 8 is presented as $D_4 = \langle r, s | r^4 = s^2 = 1, r^s = r^3 \rangle$, then $s \not\sim sr$ are both of order 2 and $s^{sr} = sr^2$, $(sr)^s = sr^3$, so D_4 is not a pre-$\mathfrak{S}^{(3)}$-com group. Assume we have a non-abelian 2-group G which is pre-$\mathfrak{S}^{(3)}$-com with $|G| > 8$ and that all 2-groups of smaller order which are also pre-$\mathfrak{S}^{(3)}$-com are abelian or of type (3).

If $z \in Z(G)$, $|z| = 2$, then $G/\langle z \rangle$ is also a pre-$\mathfrak{S}^{(3)}$-com group with smaller order. Thus, the induction shows that either

(A) $G/\langle z \rangle$ is abelian; or

(B) $G/\langle z \rangle \cong Q_8 \times C_2^r, r \geq 0$.

Suppose we have (A). Then the classes of G are either central elements or cosets of the central subgroup $\langle z \rangle$. In particular, elements of the same class commute. Thus we can write G as a disjoint union of classes:

$$G = Z(G) \cup g_1\langle z \rangle \cup g_2\langle z \rangle \cup \cdots \cup g_s\langle z \rangle.$$

Lemma 3.8.

(1) *If* $|g_i| = |g_j| = 2$, $i \neq j$, *then* $(g_i, g_j) = 1$.
(2) *If* $|g_i| = 2, |g_j| > 2$, *then* $(g_i, g_j) = 1$.

In particular, all involutions are central in G.

Proof.

(1) If, for the pair of involutions g_i, g_j, we have (1), (3) or (4), then $(g_i, g_j) = 1$. However, we cannot have (2), since $i \neq j$. This gives (1).
(2) Without loss of generality, let $i = 1$, $j = 2$. For the pair g_1, g_2 we cannot have (2), and (3) implies that $(g_1, g_2) = 1$.

So suppose that we have (4): $g_2^{g_1} = g_2^{-1}$. We note that $g_1 g_2\langle z\rangle$ is either a central set or is a conjugacy class. If $g_1 g_2$ is central, then $(g_1, g_1 g_2) = 1$, showing that $(g_1, g_2) = 1$. So now, suppose that $g_1 g_2\langle z\rangle$ is a class. Note that, in fact, $g_1 g_2\langle z\rangle \neq g_1\langle z\rangle$. Thus, the pair $g_1, g_1 g_2$ does not satisfy (2) (in G). If g_1 and g_2 satisfy (3), i.e., $g_1^{g_1 g_2} = g_1^{-1} = g_1$, then we have $(g_1, g_2) = 1$, as required. Lastly, if $g_1, g_1 g_2$ satisfies (4), then

$$(g_1 g_2)^{g_1} = g_2^{-1} g_1^{-1} = g_2^{-1} g_1.$$

Then, using $g_2^{g_1} = g_2^{-1}$, we see that this gives $g_1 g_2^{-1} = g_1 g_2^{g_1} = (g_1 g_2)^{g_1} = g_2^{-1} g_1$, showing that $(g_1, g_2) = 1$. $\square$

Lemma 3.9. *Let $x, y \in G$, where $(x, y) \neq 1$. Then we have*

$$x^4 = y^4 = 1, \quad x^2 = y^2 = z, \tag{3.1}$$
$$(x, z) = 1, \quad (y, z) = 1, \quad x^y = x^{-1}, \quad y^x = y^{-1},$$

and $\langle x, y\rangle \cong Q_8$.

Proof. If we can show these relations, then certainly $\langle x, y\rangle \cong Q_8$. Since $z \in Z(G)$, we have $(x, z) = (y, z) = 1$.

For the pair x, y, we do not have (1). If we have (2), $x \sim y$, then $G/\langle z\rangle$ abelian means that either $x = y$ or $x = yz$, and in either case we have $(x, y) = 1$, a contradiction. Thus, we must have $x^y = x^{-1}$ or $y^x = y^{-1}$. By symmetry, there is no loss in assuming $x^y = x^{-1}$. But $x^y \neq x$ implies that $x^y = xz$. Then we have $xz = x^y = x^{-1}$, giving $x^2 = z$ and $x^4 = 1$.

Now consider the pair yx, y^x. They cannot satisfy (1) because $(yx, y^x) = 1$ implies $(x, y) = 1$. If we have (2), $yx \sim y^x$; then we have $yx = y^x z$ and so $(yx, y^x) = 1$ again. If we have (3) for this pair, then

$$x^{-1}y^{-1}x \cdot yx \cdot x^{-1}yx = x^{-1}y^{-1},$$

and so $z = x^2 = y^{-2}$ and we are done, since $y^x = y^{-1}$ follows.

If we have (4), then

$$x^{-1}y^{-1} \cdot x^{-1}yx \cdot yx = x^{-1}y^{-1}x,$$

which gives $yz = y^x = y^{-1}$, which in turn gives $y^2 = z$ and $y^4 = 1$. $\square$

Lemma 3.10. *Let $x, y \in G$ where $(x, y) \neq 1$. Let $u \in C_G(\langle x, y\rangle)$. Then $u^2 = 1$. In particular, $Z(G) = C_G(\langle x, y\rangle)$ is an elementary 2-group.*

Proof. By Lemma 3.9, we see that x, y satisfy (3.1). Consider the pair xu, yu. This pair cannot satisfy (1) or (2). If we have (3), then $(xu)^{yu} = u^{-1}x^{-1}$ gives $x^y u = x^{-1}u^{-1}$. But, from the above, we have $x^y = x^{-1}$, and so $u^2 = 1$. We similarly obtain $u^2 = 1$ if we have (4) for this pair. This shows that $C_G(\langle x, y\rangle)$ has exponent 2 and so is an elementary 2-group.

We clearly have $Z(G) \subseteq C_G(\langle x, y\rangle)$, and if $u \in C_G(\langle x, y\rangle)$, then $u^2 = 1$ and so Lemma 3.8 shows that $u \in Z(G)$. $\square$

Proposition 3.11. *Let $x, y \in G$ where $[x, y] \neq 1$. Then $G = \langle x, y, C_G(x, y)\rangle \cong Q_8 \times C_2^r$.*

Proof. Lemma 3.9 shows that x and y satisfy (3.1). Let $w \in G \setminus \langle x, y, C_G(x, y)\rangle$; then one of $(x, w), (y, w)$ is non-trivial. Assume, without loss, that $(w, x) \neq 1$. By Lemma 3.9, we have the relations

$$w^4 = 1, \quad w^2 = z, \quad (w, z) = 1, \quad x^w = x^{-1}, \quad w^x = w^{-1}. \tag{3.2}$$

If we also have $(y, w) \neq 1$, then by Lemma 3.9, we have the relations (3.1) and

$$y^w = x^{-1}, \quad w^y = w^{-1}. \tag{3.3}$$

The group satisfying (3.1)–(3.3) has the property that $xyz \in C_G(x, y)$ and so $w \in \langle x, y, C_G(x, y)\rangle$, a contradiction.

If $(w, y) = 1$, then the group satisfying (3.1), (3.2) and $(y, w) = 1$ has $yw \in C_G(x, y)$, and so $w \in \langle x, y, C_G(x, y)\rangle$, a contradiction. This shows that $G = \langle x, y, C_G(x, y)\rangle$.

From Lemma 3.10, we have $Z(G) = C_G(\langle x, y\rangle) = C_2^{r+1}$, $r \geq 0$. Now $Z(\langle x, y\rangle) = \langle z\rangle \subset Z(G)$, and so we may write $Z(G) = \langle z\rangle \times C_2^r$; it follows that $G = \langle x, y, C_G(x, y)\rangle = \langle x, y\rangle \times C_2^r$, as required. □

This concludes consideration of (A).

Now suppose that we have (B): Since $G/\langle z\rangle = Q_8 \times C_2^r$, there are $x, y \in G$ such that $\pi(\langle x, y\rangle) = Q_8$. Here $\pi : G \to G/\langle z\rangle$ is the projection. Thus, $H = \langle x, y, z\rangle$ is a normal subgroup of G of order 16, where $H/\langle z\rangle \cong Q_8$. One can check that the only possibilities for H are: (I) $G = Q_8 \times C_2$; and (II) the group

$$J = \langle x, y, u, v | x^2 = v, y^2 = u, u^2, v^2, y^x = yu, (u, x), (u, y), (v, x), (v, y)\rangle.$$

We now look at each case separately; we dismiss case (II) first as it is easier.

(II) $H = J$. Here one can check that if the pair x, xy satisfies any of (1), (3), (4), then $|H| = 8$, a contradiction. However, $x \sim_H xy$ implies $x \sim_{Q_8} xy$, a contradiction. Thus the case $H = J$ does not happen.

(I) $H = Q_8 \times C_2 = \langle x, y\rangle \times \langle u\rangle$. Then the only possibilities for z are (a) $z = u$ or (b) $z = ux^2$, since these are the only elements of H of order 2 with $H/\langle z\rangle$ non-abelian. In both cases, we see that x, y satisfy the relations $x^4 = y^4 = 1, x^y = x^{-1}, y^x = y^{-1}$.

Lemma 3.12. *If $u \in C_G(\langle x, y\rangle)$, then $u^2 = 1$. In particular, $C_G(\langle x, y\rangle) \cong C_2^s$.*

Proof. If $u \in H$, then $u \in Z(H) = \langle x^2, z\rangle$, and we certainly have $u^2 = 1$.

If $u \notin H$, then we consider the pair ux, uy. Now $ux \not\sim uy$, as one can see by considering the quotient $G \to Q_8 \times C_2^r \to Q_8$. Clearly, we have $(ux, uy) \neq 1$. If we have (3), then $(ux)^{uy} = u^{-1}x^{-1}$ gives $ux^{-1} = ux^y = (ux)^{uy} = u^{-1}x^{-1}$, giving $u^2 = 1$. Condition (4) similarly gives $u^2 = 1$.

The last statement follows from the fact that $C_G(\langle x, y\rangle)$ has exponent 2. □

Lemma 3.13. $G = \langle x, y, C_G(\langle x, y\rangle)\rangle$.

Proof. Let $u \in G \setminus \langle x, y, C_G(\langle x, y\rangle)\rangle$. Then either $(x, u) \neq 1$ or $(y, u) \neq 1$. Assume, without loss, that $(x, u) \neq 1$. We also have $u \notin H$, so that $u \not\sim w$ for all $w \in H \lhd G$. Thus, the pair x, u satisfies (3) or (4). Further, the pair y, u satisfies one of (1), (3), (4). We consider the six cases so determined.

(3), (1): $x^u = x^{-1}$, $y^u = y$. Here we have $u^x = x^2 u = y^2 u$, so that $(yu)^x = y^{-1} y^2 u = yu$. Thus, $yu \in C_G(\langle x, y, \rangle)$ and so $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$, a contradiction

(3), (3): $x^u = x^{-1}$, $y^u = y^{-1}$. Here we have $(xyu)^x = xy^{-1}(y^2 u) = xyu$ and $(xyu)^y = x^{-1} y x^2 u = xyu$, giving $xyu \in C_G(\langle x, y\rangle)$, and so $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(3), (4): $x^u = x^{-1}$, $u^y = u^{-1}$. Here we consider the pair xy, u. If $(xy, u) = 1$, then $xyu \in C_G(\langle x, y\rangle)$; if $(xy)^u = (xy)^{-1}$, then $yu \in C_G(\langle x, y\rangle)$. Thus, in each case we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$. We are left with the case $u^{xy} = u^{-1}$. Here, we consider the pair u, yu. If we impose any of the relations (1), (3), (4) on u, yu, then we get $|\langle x, y\rangle| = 4$, a contradiction.

(4), (1): $u^x = u^{-1}$, $y^u = y$. Here we consider the pair x, xu. If $(x, xu) = 1$, then $u \in C_G(\langle x, y\rangle)$. If $x^{xu} = x^{-1}$ or $(xu)^x = (xu)^{-1}$, then $yu \in C_G(\langle x, y\rangle)$. Thus, in each case we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(4), (3): $u^x = u^{-1}$, $y^u = y^{-1}$. Here we consider the pair xy^{-1}, u. If $(xy^{-1}, u) = 1$, then $xyu \in C_G(\langle x, y\rangle)$. If $(xy^{-1})^u = (xy^{-1})^{-1}$, then $xu \in C_G(\langle x, y\rangle)$. If $u^{xy^{-1}} = u^{-1}$, then $|\langle x, y\rangle| = 4$, a contradiction. Thus, again, we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(4), (4): $u^x = u^{-1}$, $u^y = u^{-1}$. Here we consider the pair x, xu. If $(x, xu) = 1$, then $u \in C_G(\langle x, y\rangle)$. If $x^{xu} = x^{-1}$ or $(xu)^x = (xu)^{-1}$, then $xyu \in C_G(\langle x, y\rangle)$.

This concludes consideration of all cases. □

It follows easily from the fact that $H = Q_8 \times C_2 = \langle x, y\rangle \times \langle u\rangle \lhd G$, together with Lemmas 3.12 and 3.13 that $G = Q_8 \times C_2^r$.

This concludes the proof of (B) and so of Theorem 1.2. □

4. Commutative 3-S-rings.

Theorem 4.1. *Let G be a group with commutative* 3-S-*ring. Then G is pre-$\mathfrak{S}^{(3)}$*-com.

Proof. Let $g, h \in G$. We wish to show that the pair x, y satisfies one of $(1), \ldots, (4)$. We consider the elements $x = (g, 1, g), y = (h, h, 1) \in G^3$ and let $C^{(3)}(x), C^{(3)}(y) \in \mathfrak{S}_G^{(3)}$ denote their 3-classes. We have $xy = (gh, h, g)$ as a term of $C(x) \cdot C(y)$ and so (gh, h, g) is also a term of $C(y) \cdot C(x)$.

Now the elements of $C^{(3)}(x)$ have the form

$$\text{(i) } (g^a, 1, g^a), \quad \text{(ii) } (g^a, g^a, 1), \quad (3)\ (1, g^a, g^a),$$

for some $a \in G$, and the elements of $C^{(3)}(y)$ have the form

$$\text{(i}'\text{) } (h^b, 1, h^b), \quad \text{(ii}'\text{) } (h^b, h^b, 1), \quad \text{(iii}'\text{) } (1, h^b, h^b),$$

for some $b \in G$. It follows that $xy = (gh, g, h)$ is realized in $C(y) \cdot C(x)$ as one of the following possibilities:

Case (i), (i'): Here

$$(h^b, 1, h^b)(g^a, 1, g^a) = (gh, h, g),$$

and so $h = 1$, giving $(g, h) = 1$.

Case (i), (ii'): Here

$$(h^b, h^b, 1)(g^a, 1, g^a) = (gh, h, g),$$

and so $h^b g^a = gh$, $h^b = h$, $g^a = g$, giving $(g, h) = 1$.

Case (i), (iii'): Here

$$(1, h^b, h^b)(g^a, 1, g^a) = (gh, h, g),$$

and so $g^a = gh$, $h^b = h$, $h^b g^a = g$ giving $g^a = h^{-1} g = gh$, so that $h^g = h^{-1}$.

Case (ii), (i'): Here

$$(h^b, 1, h^b)(g^a, g^a, 1) = (gh, h, g),$$

and so $g^a = h$, giving $g \sim h$.

Case (ii), (ii'): Here

$$(h^b, h^b, 1)(g^a, g^a, 1) = (gh, h, g),$$

and so $g = 1$, giving $(g, h) = 1$.

Case (ii), (iii'): Here

$$(1, h^b, h^b)(g^a, g^a, 1) = (gh, h, g),$$

and so $h^b = g$, giving $g \sim h$.

Case (iii), (i'): Here

$$(h^b, 1, h^b)(1, g^a, g^a) = (gh, h, g),$$

and so $g^a = h$, giving $g \sim h$.

Case (iii), (ii'): Here

$$(h^b, h^b, 1)(1, g^a, g^a) = (gh, h, g),$$

and so $h^b = gh$, $h^b g^a = h$, $g^a = g$, giving $gh = h^b = hg^{-1}$. We thus have $g^h = g^{-1}$.

Case (iii), (iii'): Here

$$(1, h^b, h^b)(1, g^a, g^a) = (gh, h, g),$$

and so $gh = 1$. We thus have $(g, h) = 1$.

This concludes consideration of all cases. □

Lemma 4.2. *A group of the form $G = Q_8 \times C_2^r$ does not have a commutative 3-S-ring.*

Proof. Let $\pi : G = Q_8 \times C_2^r \to Q_8$ be the projection. Then π induces a homomorphism of 3-S-rings, $\pi : \mathfrak{S}^{(3)}(G) \to \mathfrak{S}^{(3)}(Q_8)$. Since $\pi(\mathfrak{S}^{(3)}(G)) = \mathfrak{S}^{(3)}(Q_8)$, we need only show that $\mathfrak{S}^{(3)}(Q_8)$ is not commutative. Suppose that $Q_8 = \langle x, y \rangle$, where x, y satisfy the relations (3.1). Let α be the 3-class of $(x, x, 1)$ and β the 3-class of $(xy, xy, 1)$. Then one can check that $\alpha\beta \neq \beta\alpha$. □

Lemma 4.3. *A generalized dihedral group $G = N \rtimes C_2$ with order of N odd has a commutative 3-S-ring.*

Proof. Let $C_2 = \langle x \rangle$. Elements of G will be written nx^ε, $n \in N$, $\varepsilon = 0, 1$. If $\alpha \in G^3$, then $C^{(3)}(\alpha)$ contains an element of one of the following four types:

(A): (n_1, n_2, n_3), (B): $(n_1 x, n_2, n_3)$,
(C): $(n_1 x, n_2 x, n_3)$, (D): $(n_1 x, n_2 x, n_3 x)$.

Here $n_i \in N$, $i = 1, 2, 3$.

Let $\alpha, \beta \in G^3$. We thus have some cases to consider to show that $C^{(3)}(\alpha)C^{(3)}(\beta) = C^{(3)}(\beta)C^{(3)}(\alpha)$:

Case (A) $\times$ (A). Here $\alpha = (n_1, n_2, n_3)$, $\beta = (n_1', n_2', n_3')$ and in this case we have $\alpha\beta = \beta\alpha$, so we certainly have $C^{(3)}(\alpha)C^{(3)}(\beta) = C^{(3)}(\beta)C^{(3)}(\alpha)$.

Case (A) $\times$ (B). Here $\alpha = (n_1, n_2, n_3)$, $\beta = (n_1' x, n_2', n_3')$. To prove this case we just need to show that $\alpha\beta \in C^{(3)}(\beta) \cdot C^{(3)}(\alpha)$. Now, for $n \in N$, we have $x^n = n^{-2}x$ and so

$$(n_1' x, n_2', n_3')^n = (n_1' n^{-2} x, n_2', n_3'),$$

Since $|N|$ is odd and N is abelian, the map $n \mapsto n^2$ gives a surjection of N, and so the element $n \in N$ can be chosen so that

$$\beta^n \alpha = (n_1' n^{-2} x, n_2', n_3')(n_1, n_2, n_3)$$

is equal to

$$(n_1 n_1' x, n_2 n_2', n_3 n_3') = \alpha\beta.$$

Case (A) $\times$ (C). Here $\alpha = (n_1, n_2, n_3)$, $\beta = (n_1' x, n_2' x, n_3')$. Let $\beta' = \beta x$, so that β' has type (B). Then, from the above case $(A) \times (B)$, we have $\alpha\beta' = \beta'\alpha$. Thus, we have

$$\alpha\beta = \alpha\beta' x = \beta'\alpha x = \beta' x \cdot x\alpha x = \beta\alpha^x \in C^{(3)}(\beta) \cdot C^{(3)}(\alpha), \tag{4.1}$$

as required.

Case (A) $\times$ (D). Here $\alpha = (n_1, n_2, n_3)$, $\beta = (n_1' x, n_2' x, n_3' x)$. Let $\beta' = \beta x$, so that β' has type (A). Then, from the case $(A) \times (A)$, we have $\alpha\beta' = \beta'\alpha$. Thus, (4.1) again gives this case.

Case (B) $\times$ (B). Here we need to consider subcases:

(i) $\alpha = (n_1x, n_2, n_3)$, $\beta = (n_1'x, n_2', n_3')$. Then $\alpha\beta = (n_1(n_1')^{-1}, n_2n_2', n_3n_3')$. But

$$\beta^n\alpha = (n_1'n^{-2}x, n_2', n_3')(n_1x, n_2, n_3) = (n_1'n^{-2}n_1^{-1}, n_2n_2', n_3n_3'),$$

and we can find $n \in N$ such that this is equal to $\alpha\beta$, as required.

(ii) $\alpha = (n_1x, n_2, n_3)$, $\beta = (n_1', n_2'x, n_3')$. For $n, m \in N$, we have

$$\begin{aligned}\beta^m\alpha^n &= (n_1', n_2'm^{-2}x, n_3')(n_1n^{-2}x, n_2, n_3)\\ &= (n_1'n_1n^{-2}x, n_2'm^{-2}n_2^{-1}x, n_3n_3'),\end{aligned}$$

and we can choose $n, m \in N$ such that this is equal to $\alpha\beta = (n_1(n_1')^{-1}x, n_2n_2'x, n_3n_3')$.

Case (B) $\times$ (C). Here $\alpha = (n_1x, n_2, n_3)$, $\beta = (n_1'x, n_2'x, n_3')$. Let $\beta' = \beta x$. Then β' has type (B) and so we have $\alpha\beta' = \beta'\alpha$. The result now follows from (4.1).

The remainder of the cases can be proved by reducing to cases that we have already considered, and then using (4.1). □

Theorems 1.3, 1.4 and 1.5 follow from Theorem 1.1 and Lemmas 4.2 and 4.3. □

5. Commutative 4-S-rings.

Here we prove Theorem 1.6.

For any finite group G, the 4-S-ring contains a sub-ring isomorphic to the 3-S-ring [**5,** Theorem 1.1]. So a group G can have a commutative 4-S-ring only if G is $\mathfrak{S}^{(3)}$-com. So to show that only abelian groups have commutative 4-S-rings, it suffices to show that the generalized dihedral groups $N \rtimes C_2$ with N having odd order do not have commutative 4-S-rings.

Let $G = N \rtimes C_2$ with $C_2 = \langle t\rangle$, where N has odd order. Let $y \in N \setminus \{1\}$. Let K_1 be the 4-class of (y, y, y^{-1}, y^{-1}) in G^4. Since $y^G = \{y, y^{-1}\}$, this class contains only the 6 elements of G^4 obtained by permuting the entries. Let K_2 be the 4-class of $(1, t, t, t)$. For each $x \in N$, there are 4 elements of K_2 corresponding to the 4 possible permutations of $(1, xt, xt, xt)$.

The element $(y, yt, y^{-1}t, y^{-1}t) = (y, y, y^{-1}, y^{-1})(1, t, t, t)$ is a term in the product $\overline{K}_1\overline{K}_2$. Suppose $(y, yt, y^{-1}t, y^{-1}t) = kl$ where $k \in K_2$,

$l \in K_1$. Because $(xt)y = xy^{-1}t$ and $(xt)y^{-1} = xyt$, the first entry of k must be 1 (it cannot have a term with t), and hence $k = (1, xt, xt, xt)$ for some $x \in N$. It follows that l has first entry y and so l is one of (y, y, y^{-1}, y^{-1}), (y, y^{-1}, y, y^{-1}), and (y, y^{-1}, y^{-1}, y). So we need to determine whether there is an $x \in N$ for which one of the following can occur:

Case 1: $(y, yt, y^{-1}t, y^{-1}t) = (1, xt, xt, xt)(y, y, y^{-1}, y^{-1}) = (y, xy^{-1}t, xyt, xyt)$. In this case, we get $yt = xy^{-1}t$ and $y^{-1}t = xyt$ so that $yt = x(xyt) = x^2yt$ so $x^2 = 1$, giving $x = 1$. But then we have $yt = xy^{-1}t = y^{-1}t$ and so $y^2 = 1$, a contradiction.

Case 2: $(y, yt, y^{-1}t, y^{-1}t) = (1, xt, xt, xt)(y, y^{-1}, y, y^{-1}) = (y, xyt, xy^{-1}t, xyt)$. In this case, we get $yt = xyt = y^{-1}t$, a contradiction since $y \in N$.

Case 3: $(y, yt, y^{-1}t, y^{-1}t) = (1, xt, xt, xt)(y, y^{-1}, y^{-1}, y) = (y, xyt, xyt, xy^{-1}t)$. In this case, we get $yt = xyt = y^{-1}t$, again contradicting that $y \in N$.

So $(y, yt, y^{-1}t, y^{-1}t)$ cannot be a term in the product $\overline{K}_2\overline{K}_1$.

So the product $\overline{K}_1\overline{K}_2$ contains a term which is not a term of $\overline{K}_2\overline{K}_1$. Thus, these products are not equal, and so the 4-S-ring of G cannot be commutative. $\square$

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