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ZERO-DIVISOR GRAPHS OF MODULES VIA MODULE HOMOMORPHISMS

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ABSTRACT. In this paper, using module endomorphisms, we extend the concept of the zero-divisor graph of a ring to a module over an arbitrary commutative ring. The main aim of this article is studying the interplay of module-theoretic properties of a module with graph properties of its zero-divisor graph.

1. Introduction. Throughout this paper, let R be a commutative ring with non-zero identity and $Z(R)$ the set of its zero-divisors. Also we set $Z^*(R) := Z(R)\backslash\{0\}$. The concept of a zero-divisor graph of a commutative ring was introduced and studied by Beck in [**6**]. He let all elements of the ring be vertices of the graph and was interested mainly in colorings. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices in the set of non-zero zero-divisors of R and, for two distinct elements x and y in $Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Moreover, a non-empty graph $\Gamma(R)$ is finite if and only if R is finite. (See [**5,** Theorem 2.2].) The above definition of $\Gamma(R)$ and the emphasis on studying the interplay between graph-theoretic properties of $\Gamma(R)$ and ring-theoretic properties of R are from [**5**].

For example, in [**5,** Theorem 2.3], it was proved that $\Gamma(R)$ is connected with $\operatorname{diam}(\Gamma(R)) \leqslant 3$. There are several papers devoted to studying the properties of zero-divisor graphs. (See [**2, 3, 4, 9, 10, 14**].) For an R-module M, consider the zero-divisor graph of $R(+)M$, where $R(+)M$ is the idealization of M. Redmond, in [**15**], defined the zero-divisor graph of M as the subgraph of $\Gamma(R(+)M)$ with vertices

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in the set $0(+)M := \{(0, m) \mid m \in M\}$. So, for all $m, m' \in M$, the vertices $(0, m)$ and $(0, m')$ are adjacent. Recently, Behboodi, in **[7]**, introduced the sets of weak zero-divisors, zero-divisors and strong zero-divisors of M, denoted by $Z_*(M)$, $Z(M)$ and $Z^*(M)$, respectively. Also, he associated three (simple) graphs $\Gamma_*(M)$, $\Gamma(M)$ and $\Gamma^*(M)$ to M with vertices in $Z_*(M)$, $Z(M)$ and $Z^*(M)$, respectively, and the vertices x and y are adjacent if and only if $I_xI_yM = 0$, where, for an element z in M, I_z is the ideal $\operatorname{Ann} R(M/Rz) := \{r \in R \mid rM \subseteq Rz\}$.

In this paper, for an R-module M, using endomorphisms on M, we assign a zero-divisor graph $\mathcal{H}_R(\Gamma(M))$ to M. We show that the graph $\mathcal{H}_R(\Gamma(M))$ coincides with the zero-divisor graph of R when $M = R$ and $\Gamma(R)$ is not a singleton. In Section 2, we study some basic properties of the zero-divisor graph $\mathcal{H}_R(\Gamma(M))$. For instance, we show that if $\psi : R \to S$ is a ring epimorphism, then the graphs $\mathcal{H}_R(\Gamma(M))$ and $\mathcal{H}_S(\Gamma(M))$ coincide. In Section 3, we study the zero-divisor graphs of decomposable modules and the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, for a prime number p. Also, we use the concept of the tensor product of graphs for studying the graph $\mathcal{H}_{R^k}(\Gamma(M))$, whenever M has a decomposition $M = M_1 \oplus \cdots \oplus M_k$ for some submodules $M_1, \ldots, M_k$ of M. Moreover, we study the planarity of the graph $\mathcal{H}_R(\Gamma(M))$ in several cases.

We recall that, for a graph G, the set of vertices is denoted by $V(G)$. Moreover, if $P = x_0 - \cdots - x_k$ is a path and $k \geqslant 2$, then the graph $C := P + x_k - x_0$ is called a *cycle*. The above cycle C may be written as $x_0 - \cdots - x_k - x_0$. The length of a cycle is its number of edges (or vertices). The minimum length of a cycle (contained) in a graph G is the girth $\mathrm{g}\,(G)$ of G. The distance $\mathrm{d}\,(x, y)$ in G of two distinct vertices x and y is the length of a shortest path from x to y in G. If no such path exists, we set $d(x, y) := \infty$. The greatest distance between any two vertices in G is the diameter of G, which is denoted by $\mathrm{diam}\,(G)$. If all the vertices of G are pairwise adjacent, then G is complete. A complete graph on n vertices is denoted by K_n. The greatest integer r such that $K_r \subseteq G$ is the clique number $\omega(G)$ of G. Let $r \geqslant 2$ be an integer. A graph $G = (V, E)$ is called *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes and the vertices in the same partition class must not be adjacent. Instead of 2-partite, one usually says bipartite. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete*. Note that a graph is bipartite if and only if it contains no odd cycle. The complete r-partite graph is denoted by $K_{n_1,\ldots,n_r}$, where n_i is the cardinality of the ith partition of V. Graphs of the form $K_{1,n}$ are called *stars*; the vertex in a singleton partition class of this $K_{1,n}$ is the star's center. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends.

For a general reference on ring theory we use [**13**], and for a general reference on graph theory we use [**11**].

2. Definition and basic properties. In this section, by using the concept of endomorphisms of an R-module M, we define a zero-divisor graph of M, denoted by $\mathcal{H}_R(\Gamma(M))$, which is a generalization of the zero-divisor graph of a commutative ring. To do this, we first establish our notation. For any $f, g \in \mathrm{End}_R(M)$, $\mathrm{IK}\,(f,g)$ is the Cartesian product of $\mathrm{Im}\,(f) \cap \mathrm{Ker}\,(g)$ and $\mathrm{Ker}\,(f) \cap \mathrm{Im}\,(g)$, and we put

$$\mathrm{IK}(M) := \bigcup_{f,g\in \mathrm{End}_R(M)} \mathrm{IK}(f,g).$$

Now, we describe the zero-divisor graph $\mathcal{H}_R(\Gamma(M))$. For any two non-zero distinct elements $m, m' \in M$, we say that, in the graph $\mathcal{H}_R(\Gamma(M))$, m and m' are adjacent if and only if $(m, m') \in \mathrm{IK}\,(M)$, and moreover, $m \in M$ is a vertex in $\mathcal{H}_R(\Gamma(M))$ if there exists $m' \in M$ such that m and m' are adjacent. If we omit the word "distinct" in the definition of $\mathcal{H}(\Gamma(M))$, we obtain the graph $H(\Gamma(M))$; this graph may have loops.

The following theorem shows that $\mathcal{H}_R(\Gamma(M))$ is a generalization of the concept of the zero-divisor graph of R.

Theorem 2.1. *For any commutative ring R, if $\Gamma(R)$ is not a singleton, then we have $\mathcal{H}_R(\Gamma(R)) = \Gamma(R)$.*

Proof. Let r and r' be adjacent vertices in $\mathcal{H}_R(\Gamma(R))$. So there are $f, g \in \mathrm{End}_R(R)$ such that $(r, r') \in \mathrm{IK}\,(f, g)$. Put $c := f(1)$ and $c' := g(1)$. Then $f(s) = cs$ and $g(s) = c's$ for all $s \in R$. Thus, $c'r = 0 = cr'$, $ct = r$ and $c't' = r'$ for some $t, t' \in R$. Hence, $rr' = 0$. This means that r and r' are adjacent in $\Gamma(R)$.

Now suppose that r and r' are adjacent vertices in $\Gamma(R)$. Hence, $rr' = 0$. Consider the endomorphisms $\varphi, \psi : R \to R$ induced by multiplication by r and r', respectively. So $(r, r') \in \mathrm{IK}\,(\varphi, \psi)$, and hence r and r' are adjacent in $\mathcal{H}_R(\Gamma(R))$. Also note that $\Gamma(R)$ and $\mathcal{H}_R(\Gamma(R))$ have the same vertices. □

Lemma 2.2. *Let r_1 and r_2 be two elements of R and $m \in M$ such that $r_1 r_2 m = 0$. Then $(r_1 m, r_2 m) \in \mathrm{IK}\,(M)$.*

Proof. Suppose that $f, g : M \to M$ are given by multiplication by r_1 and r_2, respectively. Now it is easy to see that $(r_1 m, r_2 m) \in \mathrm{IK}\,(f, g)$. □

Note that, in Lemma 2.2, if $r_1 m$ and $r_2 m$ are non-zero distinct elements of M, then $r_1 m$ and $r_2 m$ are adjacent in $\mathcal{H}_R(\Gamma(M))$.

Remark 2.3. Suppose that $Z(R)$ is an ideal of R and there exists an element $m \in M$ such that $Z(R) \cap \mathrm{Ann}_R(m) = \{0\}$. Then, in view of Lemma 2.2,

$$\mathrm{g}\,(\mathcal{H}_R(\Gamma(M))) \leqslant \mathrm{g}\,(\Gamma(R)).$$

In the rest of this section, we study properties of the zero-divisor graph of modules through change of rings. Recall that, for an S-module M and a ring homomorphism $\psi : R \to S$, one can construct an R-module structure on M by the multiplication $rm := \psi(r)m$ for all $r \in R$ and $m \in M$. In the following proposition, we compare the graphs $\mathcal{H}_R(\Gamma(M))$ and $\mathcal{H}_S(\Gamma(M))$.

Proposition 2.4. *Suppose that $\psi : R \to S$ is a ring homomorphism and M is an S-module. Then $\mathcal{H}_S(\Gamma(M))$ is an induced subgraph of $\mathcal{H}_R(\Gamma(M))$, where M has the R-module structure induced by ψ.*

Proof. By using the structure of M as an R-module, it is easy to check that every S-endomorphism on M is an R-endomorphism. This implies that the adjacency is preserved from the S-module M to the R-module structure of M. □

Theorem 2.5. *Assume that $\psi : R \to S$ is a ring epimorphism. Then*

$$\mathcal{H}_R(\Gamma(M)) = \mathcal{H}_S(\Gamma(M)).$$

Proof. It is routine to check that $\mathrm{End}_R(M) = \mathrm{End}_S(M)$. Hence, the graphs $\mathcal{H}_R(\Gamma(M))$ and $\mathcal{H}_S(\Gamma(M))$ coincide. □

Corollary 2.6. *Let M be an R-module and I an ideal of R with $I \subseteq \mathrm{Ann}_R(M)$. Then $\mathcal{H}_R(\Gamma(M)) = \mathcal{H}_{R/I}(\Gamma(M))$.*

Proof. Consider the natural ring epimorphism $R \to R/I$. The result now follows from Theorem 2.5. □

3. Zero-divisor graph of certain modules. In this section, for a $\mathbb{Z}$-module M, we study the zero-divisor graph $\mathcal{H}_{\mathbb{Z}}(\Gamma(M))$. We begin with the following remark, which is an immediate consequence of Corollary 2.6 and Theorem 2.1.

Remark 3.1. For every positive integer n, if $\Gamma(\mathbb{Z}_n)$ is not a singleton, then $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_n)) = \Gamma(\mathbb{Z}_n)$.

Now, we recall the definition of a refinement of a simple graph.

Definition 3.2. A simple graph G is called a refinement of a simple graph H if $V(G) = V(H)$ and $E(H) \subseteq E(G)$.

Theorem 3.3. *Let p be a prime number and n be a positive integer greater than 1 such that $p^n \neq 4$. Then:*

(i) *the zero-divisor graph $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p^n}))$ is a refinement of a star graph with center p^{n-1};*
(ii) *the graph $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p^n}))$ is connected with diameter at most 2; and,*
(iii) *if $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p^n}))$ has a cycle, then $\mathrm{g}\,(\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p^n})) = 3$.*

Proof. Part (i) follows from the Corollary 2.6 and the fact that the set of zero-divisors of $\mathbb{Z}_{p^n}$ is an ideal generated by p. The claims in (ii) and (iii) immediately follow from (i). □

Recall that an R-module M is decomposable if $M \cong M_1 \oplus \cdots \oplus M_k$, for some non-zero submodules $M_1, \ldots, M_k$ of M with $k > 1$. Suppose that M is both an Artinian and a Noetherian R-module and f is an endomorphism of M. Put $f^{\infty}(M) := \bigcap_{n=1}^{\infty} f^n(M)$ and $f^{-\infty}(0) := \bigcup_{n=1}^{\infty} \mathrm{Ker}\,(f^n)$, where f^n is the composition of n-times of f. Then, by Fitting's lemma, $M = f^{-\infty}(0) \oplus f^{\infty}(M)$. (See [**12,** page 113].) Such a decomposition is called *Fitting's decomposition.* Hence if M is finite, then M has a Fitting decomposition. This allows us to study the decomposable modules.

Suppose that M decomposes as $M_1 \oplus \cdots \oplus M_k$. Then any element $m \in M$ can be represented uniquely by $(m_1, \ldots, m_k)$, where $m_i \in M_i$ for each $i = 1, \ldots, k$. We define the support of m as follows:

$$\mathrm{Supp}\,(m) := \{i \mid m_i \neq 0\}.$$

Also, note that, in this situation, M has an R^k-module structure by the multiplication

$$(r_1, \ldots, r_k)(m_1, \ldots, m_k) = (r_1 m_1, \ldots, r_k m_k),$$

for all $(r_1, \ldots, r_k) \in R^k$ and $(m_1, \ldots, m_k) \in M$.

The following theorem shows that there exists a strong connection between the graphs $H_R(\Gamma(M_1)), \ldots, H_R(\Gamma(M_k))$ and the graph $H_{R^k}(\Gamma(M))$.

Theorem 3.4. *Suppose that an R-module M has a decomposition $M = M_1 \oplus \ldots \oplus M_k$ and that $m = (m_1, \ldots, m_k)$ and $n = (n_1, \ldots, n_k)$ are non-zero elements of M. Then m and n are adjacent in $H_{R^k}(\Gamma(M))$ if and only if, for each $i \in \mathrm{Supp}\,(m) \cap \mathrm{Supp}\,(n)$, n_i and m_i are adjacent in $H_R(\Gamma(M_i))$.*

Proof. Suppose that, for $i \in \mathrm{Supp}\,(m) \cap \mathrm{Supp}\,(n)$, n_i and m_i are adjacent in $H_R(\Gamma(M_i))$. Hence, there are homomorphisms $f_i, g_i \in \mathrm{End}_R(M_i)$ such that $(m_i, n_i) \in \mathrm{IK}\,(g_i, f_i)$. Also, for $1 \leqslant i \leqslant k$ with $i \notin \mathrm{Supp}\,(m) \cap \mathrm{Supp}\,(n)$, we have that either m_i or n_i is zero and consequently $(m_i, n_i) \in \mathrm{IK}(0, id)$ or $(m_i, n_i) \in \mathrm{IK}\,(id, 0)$, where id is the identity endomorphism on M. Thus, for each $i = 1, \ldots, k$, there are homomorphisms $f_i, g_i \in \mathrm{End}_R(M_i)$ such that $(m_i, n_i) \in \mathrm{IK}(f_i, g_i)$. Put $f := (f_1, \ldots, f_k)$ and $g := (g_1, \ldots, g_k)$. Clearly, these homomorphisms satisfy the adjacency conditions for m and n in the graph $H_{R^k}(\Gamma(M))$.

Conversely, assume that m and n are adjacent vertices in $H_{R^k}(\Gamma(M))$. Since by [**8,** Theorem 2.6.8 (iii)],

$$\mathrm{End}_{R^k}(M) \cong \mathrm{End}_R(M_1) \oplus \cdots \oplus \mathrm{End}_R(M_k),$$

there are $f, g \in \mathrm{End}_{R^k}(M)$ with $(m, n) \in \mathrm{IK}\,(f, g)$ of the form $f = (f_1, \ldots, f_k)$, $g = (g_1, \ldots, g_k)$ where $f_i, g_i \in \mathrm{End}_R(M_i)$, for $i = 1, \ldots, k$. Now, for $i \in \mathrm{Supp}\,(m) \cap \mathrm{Supp}\,(n)$, $m_i \neq 0 \neq n_i$. It is routine to check that $(m_i, n_i) \in \mathrm{IK}\,(f_i, g_i)$, and so m_i and n_i are adjacent vertices in $H_R(\Gamma(M_i))$. □

Remark 3.5. Let M be an R-module such that $M = M_1 \oplus \cdots \oplus M_k$ for some submodules $M_1, \ldots, M_k$. Then, for a positive integer i with $1 \leqslant i \leqslant k$, by using the following multiplication, M has an R^i-module structure

$$(r_1, \ldots, r_i)(m_1, \ldots, m_k) := (r_1, \ldots, r_i, 0, \ldots, 0)(m_1, \ldots, m_k),$$

for all $(m_1, \ldots, m_k) \in M$ and $(r_1, \ldots, r_i) \in R^i$. So $\mathrm{End}_{R^{i+1}}(M) \subseteq \mathrm{End}_{R^i}(M)$. This implies that $\mathcal{H}_{R^{i+1}}(\Gamma(M))$ can be considered as an induced subgraph of $\mathcal{H}_{R^i}(\Gamma(M))$, and so it is easy to verify that we have the following chain of subgraphs of $\mathcal{H}_R(\Gamma(M))$

$$\mathcal{H}_{R^k}(\Gamma(M)) \subseteq \cdots \subseteq \mathcal{H}_R(\Gamma(M)).$$

Corollary 3.6. *Suppose that an R-module M has a decomposition $M_1 \oplus \cdots \oplus M_k$ such that $M_i \neq 0$ for all i with $1 \leq i \leq k$. Then every two distinct non-zero elements $m = (m_1, \ldots, m_k)$ and $n = (n_1, \ldots, n_k)$ with* $\mathrm{Supp}\,(m) \cap \mathrm{Supp}\,(n) = \emptyset$ *are adjacent in $\mathcal{H}_R(\Gamma(M))$.*

In view of Corollary 3.6, the maximum number of summands in the decomposition of M is a lower bound for the clique number of the graph $\mathcal{H}_R(\Gamma(M))$. Now, we show that the tensor product of graphs is a powerful tool for studying the zero-divisor graph of module. To this end, we first recall the definition of the tensor product of two graphs.

Definition 3.7. The tensor product $G \otimes H$ of graphs G and H is a graph such that

- the vertex set of $G \otimes H$ is the Cartesian product $V(G) \times V(H)$; and,

- any two vertices (u, u') and (v, v') are adjacent in $G \otimes H$ if and only if u is adjacent to v and u' is adjacent to v'.

Notation 3.8. For an R-module M, if we add vertex 0 to vertex set of $H_R(\Gamma(M))$, then we obtain the graph $H^*_R(\Gamma(M))$. In this graph 0 is adjacent to all vertices.

Remark 3.9.

(1) Let M have a decomposition $M_1 \oplus \cdots \oplus M_k$. Hence, by Theorem 3.4, it is easy to see that

$$H^*_{R^k}(\Gamma(M)) = H^*_R(\Gamma(M_1)) \otimes \cdots \otimes H^*_R(\Gamma(M_k)).$$

(2) Let M be a simple R-module. Then $H_R(\Gamma(M))$ is the empty graph.

(3) Let $M \cong \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{\alpha_k}}$ be a finite $\mathbb{Z}$-module such that $p_1, \ldots, p_k$ are prime numbers for all $i = 1, \ldots, k$. Then by (1),

$$H^*_{\mathbb{Z}^k}(\Gamma(M)) = H^*_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p_1^{\alpha_1}})) \otimes \cdots \otimes H^*_{\mathbb{Z}}(\Gamma(\mathbb{Z}_{p_k^{\alpha_k}})).$$

Proposition 3.10. *Let M be an R-module, and let x and y be adjacent vertices in $\mathcal{H}_R(\Gamma(M))$. Then, for each $r, s \in R$, $(sx, ry) \in \mathrm{IK}\,(M)$.*

Proof. Suppose that x and y are adjacent vertices in $\mathcal{H}_R(\Gamma(M))$. We need only show that $(x, ry) \in \mathrm{IK}\,(M)$, for all $r \in R$. To this end, suppose that f and g are endomorphisms on M such that $(x, y) \in \mathrm{IK}\,(f, g)$. Since $\mathrm{Ker}\,(g) \cap \mathrm{Im}\,(f)$ is a submodule of M, for any element $r \in R$, $ry \in \mathrm{Ker}\,(f) \cap \mathrm{Im}\,(g)$. Hence, f and g satisfy the required conditions for adjacency of x and ry. So $(x, ry) \in \mathrm{IK}\,(M)$. □

In the rest of the paper, we study the planarity of the zero-divisor graph of M. We begin with the following examples.

Example 3.11.

(a) We show that the zero-divisor graph of $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3))$ contains $K_{3,3}$ as a subgraph, and so it is not planar. Set

$$V_1 := \{\alpha_1 = (1, 0, 0), \alpha_2 = (1, 1, 0), \alpha_3 = (0, 1, 0)\}$$

and

$$V_2 := \{\beta_1 = (1,1,1), \beta_2 = (0,0,1), \beta_3 = (0,0,2)\}.$$

Now, in view of Corollary 3.6, we need only show that $\{\alpha_1, \beta_1\}$, $\{\alpha_2, \beta_1\}, \{\alpha_3, \beta_1\}$ are edges in $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3))$. Consider the elements $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ in $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then, for adjacency of α_1 and β_1, we define $f_1(e_1) = f_1(e_2) = e_1$, $f_1(e_3) = g_1(e_1) = g_1(e_3) = 0$ and $g_1(e_2) = e_1 + e_2 + e_3$. Hence, $(\alpha_1, \beta_1) \in \mathrm{IK}\,(f_1, g_1)$, and so $\{\alpha_1, \beta_1\} \in E(\mathcal{H}_{\mathbb{Z}}(\Gamma(M)))$. For $\{\alpha_2, \beta_1\}$, consider the endomorphisms f_2 and g_2 given by $f_2(e_1) = f_2(e_2) = e_1 + e_2$, $g_2(e_1) = e_1 + e_2 + e_3$, $g_2(e_2) = e_1 + e_2 + 2e_3$ and $f_2(e_3) = g_2(e_3) = 0$. Then $(\alpha_2, \beta_1) \in \mathrm{IK}\,(f_2, g_2)$. Finally, set $f_3(e_1) = f_3(e_2) = e_2$, $g_3(e_1) = e_1 + e_2 + e_3$ and $f_3(e_3) = g_3(e_2) = g_3(e_3) = 0$. Thus, $(\alpha_3, \beta_1) \in IK(f_3, g_3)$ as required.

(b) It is routine to check that the zero-divisor graph $\mathcal{H}_{\mathbb{Z}}(\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2))$ is isomorphic to the complete graph K_7, and so it is not planar.

Theorem 3.12. *Suppose that an R-module M has the decomposition $M = M_1 \oplus \cdots \oplus M_k$, for some non-zero R-module M_i with $1 \leqslant i \leqslant k$. Then*:

(a) *if $k \geqslant 4$, then $\mathcal{H}_R(\Gamma(M))$ is not planar*;
(b) *if $k = 3$, $M \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $M \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, as $\mathbb{Z}$-modules, then $\mathcal{H}_R(\Gamma(M))$ is not planar*; *and,*
(c) *if $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$, then $\mathcal{H}_{\mathbb{Z}}(\Gamma(M))$ is not planar, but when we consider M as a ring, the zero-divisor graph $\Gamma(M)$ is planar.*

Proof.

(a) Consider the subsets $V_1 := M_1 \oplus M_2 \oplus \{0\} \oplus \cdots \oplus \{0\}$ and $V_2 := \{0\} \oplus \cdots \oplus \{0\} \oplus M_{k-1} \oplus M_k$ of M. By Corollary 3.6, all non-zero elements of V_1 are adjacent to all non-zero elements of V_2. Since $|V_i \backslash \{0\}| \geqslant 3$, for $i = 1, 2$, $\mathcal{H}_R(\Gamma(M))$ contains $K_{3,3}$ as a subgraph, and so it is not planar.
(b) Suppose that $|M_i| > 3$ for some i with $1 \leqslant i \leqslant 3$. Without loss of generality, one can assume that $i = 1$. Set $V_1 := M_1 \oplus \{0\} \oplus \{0\}$ and $V_2 := \{0\} \oplus M_2 \oplus M_3$. Again, by Corollary 3.6, all non-zero

elements of V_1 are adjacent to all non-zero elements of V_2. Since $|V_i\backslash\{0\}| \geqslant 3$, for $i = 1, 2$, $\mathcal{H}_R(\Gamma(M))$ contains $K_{3,3}$ as a subgraph. Hence, we may assume that, for each i with $1 \leqslant i \leqslant 3$, $|M_i| \leqslant 3$ and that $|M_1| \leqslant |M_2| \leqslant |M_3|$. Now, if $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ or $M \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, as $\mathbb{Z}$-modules, then it is not hard to see that $\mathcal{H}_{\mathbb{Z}}(\Gamma(M))$ has $\Gamma(M)$ as a subgraph, and so, by [**1,** Case 2, page 171], is not planar.

(c) It follows from Examples 3.12 and [**3,** Theorem 5.1(b)]. □

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