ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 2, 2014

THE K-THEORY OF REAL GRAPH C*-ALGEBRAS

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ABSTRACT. In this paper, we will introduce real graph algebras and develop the theory to the point of being able to calculate the K-theory of such algebras. The K-theory situation is significantly more complicated than in the case for complex graph algebras. To develop the long exact sequence to compute the K-theory of a real graph algebra, we need to develop a generalized theory of crossed products for real C*-algebras for groups with involution. We also need to deal with the additional algebraic intricacies related to the period-8 real K-theory using united K-theory. Ultimately, we prove that the K-theory of a real graph algebra is recoverable from the K-theory of the corresponding complex graph algebra.

1. Introduction. In this paper, we will introduce real graph algebras and develop the theory to the point of being able to calculate the K-theory of such algebras. As in the complex case, there is a long exact sequence which has the K-theory of an AF-algebra as two out of three terms and the K-theory of the graph algebra as the third. Unlike in the complex case, this long exact sequence does not necessarily collapse into a four-term exact sequence, since K_1 and K_2 do not necessarily vanish for a real AF-algebra. However, the exact sequence will in all cases be enough to determine the K-theory of the graph algebra. Indeed, we prove that two graphs have the same real K-theory if and only if they have the same complex K-theory. These calculations are done in the context of united K-theory, the invariant consisting of real, complex and self-conjugate K-theory, introduced in [**1**]. A succinct overview of united K-theory can be found in [**4**, Section 2].

Our main result indicates that although the functor factors through the category of real C*-algebras from graphs to complex C*-algebras, there is no additional information about the graph detected by the K-theory of the real graph algebra. In the purely infinite simple case, applying the classification theorems of [**4, 8**], we can go further to

2010 AMS *Mathematics subject classification.* Primary 46L80.
Keywords and phrases. Graph Algebras, K-theory, real C*-algebras.
Received by the editors on September 20, 2011, and in revised form on October 10, 2011.

DOI:10.1216/RMJ-2014-44-2-397

conclude that there is no additional information detected by the real C*-algebra itself.

A *real C*-algebra* is a Banach *-algebra A over the field $\mathbf{R}$ of real numbers satisfying the C*-condition $\|a^*a\| = \|a\|^2$ and the condition that $1 + a^*a$ is invertible for all $a \in A$. For each real C*-algebra A, the complexification $A_{\mathbf{C}} = A + iA$ has a unique norm making it a complex C*-algebra. Furthermore, there is a conjugate-linear involution on $A_{\mathbf{C}}$ given by $a + ib \mapsto a - ib$; and A is recovered as the fixed point set in $A_{\mathbf{C}}$ under the involution. In fact, the category of real C*-algebras is equivalent to the category of pairs $(A, \overline{\cdot})$ where A is a complex C*-algebra with a conjugate-linear involution $a \mapsto \overline{a}$. We will refer to such a pair as a *complex C*-algebra with real structure*. We will frequently move back and forth between these categories.

One of the motivating problems in the field is to determine up to isomorphism all of the real forms of a given complex C*-algebra. The number of such structures can be 0 (for a complex C*-algebra not isomorphic to its own opposite as in [**9**]), it can be 1 (for example, $\mathcal{O}_n$ where n is odd as shown in [**4**]), it can be 2 (for example, $\mathcal{O}_n$ where n is even), and it can be more than 2 (for example, $\mathcal{O}_n \otimes \mathcal{K}$ for odd). The best general result we have so far along these lines is that every purely infinite, simple, separable, nuclear complex C*-algebra satisfying the universal coefficient theorem has at least one real structure [**2**] and that such real structures are classified by united K-theory [**4**].

Though any arbitrary graph algebra may yet have several real structures, the results in this paper imply that, in the purely infinite simple case, there is only one real structure in which the associated real C*-algebra is itself a graph algebra over $\mathbf{R}$.

The other important development of this paper is a generalized notion of a crossed product for real C*-algebras. Crossed products for real C*-algebras are introduced in [**13**, subsection 1.3] in the case where a locally compact group G acts directly on a real C*-algebra. The author of that book remarks that, in that context, the action of the dual group does not restrict to the real crossed product, so Takai's duality theorem does not hold for real C*-algebras. Here we present a more general notion of a crossed product for real C*-algebras for the situation when a group has an involution and the group acts on the complexification of the real C*-algebra in a way that intertwines the involutions. Within this context, we prove a duality theorem for real C*-algebras generalizing Takai duality. We use this result to obtain a dual Pimsner-Voiculescu long exact sequence which leads to the long exact sequence for the K-theory of a real graph algebra.

The development of crossed products will take place in Sections 2 and 3 while Section 4 contains the material on graph algebras.

2. Crossed products. A *real C*-dynamical system* with a real structure is a quintuple $(A, \overline{\cdot}, G, \overline{\cdot}, \alpha)$ where $(A, \overline{\cdot})$ is a complex C*-algebra with a conjugate-linear involution; $(G, \overline{\cdot})$ is a locally compact group with involutive endomorphism; and α is a continuous group action of G on A which intertwines the involutions. The continuity requirement is that the function $G \to \operatorname{Aut} A$ is continuous where $\operatorname{Aut} A$ is given the point norm topology. The intertwining requirement is described as follows. The involution on A induces an involution on $\operatorname{Aut} A$ by $\overline{\alpha}(a) = \overline{\alpha(\overline{a})}$. Then we require that the map $G \to \operatorname{Aut} A$ commute with the involutions; that is, $\overline{\alpha_s} = \alpha_{\overline{s}}$ for all $s \in G$. This amounts to requiring that $\overline{\alpha_s(a)} = \alpha_{\overline{s}}(\overline{a})$ for all $a \in A$ and $s \in G$.

Let $C_c(G, A)$ denote the set of continuous functions with compact support from G to A. This set is dense in the standard crossed product $A \rtimes_\alpha G$ with the C*-norm.

Lemma 1. *Let $(G, \overline{\cdot})$ be a locally compact group with involution. Then $\mu(E) = \mu(\overline{E})$ where μ is Haar measure and E is any measurable set.*

Proof. We note that the measure ν on G defined by $\nu(E) = \mu(\overline{E})$ is left translation invariant since

$$\nu(gE) = \mu(\overline{gE}) = \mu(\overline{g}\overline{E}) = \mu(\overline{E}) = \nu(E).$$

By the uniqueness of the Haar measure, then, ν is equal to μ, up to a positive constant. Find a compact set $K \subset G$ with non-zero measure. Then $K' = K \cup \overline{K}$ satisfies

$$\nu(K') = \mu(\overline{K'}) = \mu(K'),$$

proving that $\mu = \nu$ in general. $\square$

Theorem 2. *Let $(A, \overline{\cdot}, G, \overline{\cdot}, \alpha)$ be a real C*-dynamical system. Then there is a conjugate-linear involution on $A \rtimes_\alpha G$ such that $\overline{f}(s) = \overline{f(\overline{s})}$ for all $f \in C_c(G, A)$.*

Proof. It is routine to check that the involution on $C_c(G, A)$ defined by $\overline{f}(s) = \overline{f(\overline{s})}$ is a $*$-homomorphism (with respect to the convolution product and $*$-structure on $C_c(G, A)$). Furthermore, using Lemma 1, we have that $\|\overline{f}\|_1 = \|f\|_1$ for all $f \in C_c(G, A)$.

Choose a fixed conjugate-linear involution (which is necessarily an isometry) on $B(\mathcal{H})$, denoted by $a \mapsto \overline{a}$ for any $a \in B(\mathcal{H})$. Then, for any representation $\pi: C_c(G, A) \to B(\mathcal{H})$ we define a representation $\overline{\pi}: C_c(G, A) \to B(\mathcal{H})$ by $\overline{\pi}(f) = \overline{\pi(\overline{f})}$. Since $\pi \mapsto \overline{\pi}$ is a bijection on the set of irreducible representations of $C_c(G, A)$, we have

$$\|f\| = \sup_\pi \|\pi(f)\| = \sup_\pi \|\overline{\pi}(f)\| = \sup_\pi \|\overline{\pi(\overline{f})}\| = \sup_\pi \|\pi(\overline{f})\| = \|\overline{f}\|.$$

Therefore, $f \mapsto \overline{f}$ is an isometry of $C_c(G, A)$ and extends by continuity to a conjugate-linear involution on $A \rtimes_\alpha G$. $\square$

Let $(A \rtimes_\alpha G)^\#$ be the real C*-algebra associated with the conjugate-linear involution on $A \rtimes_\alpha G$ of Theorem 2. That is, $(A \rtimes_\alpha G)^\#$ is the set of elements in $A \rtimes_\alpha G$ fixed by the involution. Then $(A \rtimes_\alpha G)^\#$ has a dense subalgebra consisting of all functions $f \in C_c(G, A)$ such that $f(\overline{s}) = \overline{f(s)}$.

In the special case that the involution on G is trivial, then the action of G restricts to the fixed point algebra of A under $\overline{\cdot}$. That is, G acts on the underlying real C*-algebra in A, and the corresponding real C*-algebra $(A \rtimes_\alpha G)^\#$ is isomorphic to the crossed product construction for real C*-algebras described in [**13**].

We recall that, if A is a complex C*-algebra, then any conjugate-linear involution on A extends to one on the multiplier algebra $M(A)$. (This is equivalent to the comments in [**3**, Section 3] that, if A is a real C*-algebra, then A has a real multiplier algebra $M(A)$ such that $M(A_{\mathbf{C}}) = M(A)_{\mathbf{C}}$.) Recall from [**12**, Corollary 2.51] that if $\phi: A \to B$ is a surjective homomorphism of complex C*-algebras, then there exists a unique extension $\widetilde{\phi}: M(A) \to M(B)$; and we note furthermore that if ϕ respects given real structures on A and B, then $\widetilde{\phi}$ respects the induced real structures on $M(A)$ and $M(B)$.

If (A, G, α) is a complex C*-dynamical system, a covariant homomorphism i from (A, G, α) to $M(B)$ is a pair consisting of a C*-algebra homomorphism $i_A\colon A \to M(B)$ and a strictly continuous group homomorphism $i_G\colon G \to UM(B)$ such that $i(\alpha_s(a)) = i(s)i(a)i(s)^*$ for $a \in A$ and $s \in G$ (disregarding the subscripts from our notation when convenient and unambiguous). We say i is nondegenerate if $i(A)B$ is dense in B.

If $(A, \overline{\cdot}, G, \overline{\cdot}, \alpha)$ is a real C*-dynamical system, we say that a *real covariant homomorphism* is a covariant homomorphism $i = (i_A, i_G)$ from (A, G, α) to $M(B)$ that respects the real structures in the sense that both i_A and i_G commute with the appropriate involutions. That is, we require $i_A(\overline{a}) = \overline{i_A(a)}$ and $i_G(\overline{s}) = \overline{i_G(s)}$ for all $a \in A$ and $s \in G$. We will prove that $A \rtimes_\alpha G$ with its real structure is the universal object with respect to covariant homomorphisms respecting the real structure.

Before we discuss the universal properties of C*-dynamical systems with real structures, we review the universal property in the complex case. The following is a slight reworking of [**14**, Theorem 2.61].

Theorem 3. *Let (A, G, α) be a C*-dynamical system.*

(1) *There is a nondegenerate covariant homomorphism i from (A, G, α) to $M(A \rtimes_\alpha G)$ such that $i(A)i(C^*(G))$ is a dense subset of $A \rtimes_\alpha G$.*

(2) *For every nondegenerate covariant homomorphism j from (A, G, α) to $M(B)$ such that $j(A)j(C^*(G)) \subseteq B$, there exists a unique nondegenerate homomorphism $\phi\colon A \rtimes_\alpha G \to B$ such that $\widetilde{\phi} \circ i = j$.*

(3) *$A \rtimes_\alpha G$ is the unique C*-algebra up to isomorphism satisfying* (1) *and* (2).

Proof. Statement (1) is [**14**, Remark 2.62]. Statement (2) follows from [**14**, Remark 2.62] using a faithful nondegenerate representation of B. Finally, statement (3) can be proven by a standard uniqueness argument for universal objects. ◻

We now present the version of the same theorem for C*-dynamical systems with real structure.

Theorem 4. *Let $(A, \overline{\,\cdot\,}, G, \overline{\,\cdot\,}, \alpha)$ be a real C^*-dynamical system.*

(1) *There is a nondegenerate covariant homomorphism i from (A, G, α) to $M(A \rtimes_\alpha G)$ respecting the real structures such that $i(A)i(C^*(G))$ is a dense subset of $A \rtimes_\alpha G$.*

(2) *Let B be a complex C^*-algebra with real structure. For every nondegenerate covariant homomorphism j from (A, G, α) to $M(B)$ respecting the real structures such that $j(A)j(C^*(G)) \subseteq B$, there exists a unique nondegenerate homomorphism $\phi\colon A \rtimes_\alpha G \to B$ respecting the real structures such that $\widetilde{\phi} \circ i = j$.*

(3) *$A \rtimes_\alpha G$ is the unique C^*-algebra with real structure up to isomorphism satisfying* (1) *and* (2).

Proof. For (1) we need only show that the covariant homomorphism i from Theorem 3 respects the real structures. The covariant homomorphism i from (A, G, α) to $M(A \rtimes_\alpha G)$ is given by the formulas

$$\begin{aligned} &i_A\colon A \to M(A \rtimes_\alpha G) \quad \text{given by} \quad i_A(a)(f)(s) = af(s) \\ &i_G\colon G \to M(A \rtimes_\alpha G) \quad \text{given by} \quad i_G(t)(f)(s) = \alpha_t(f(t^{-1}s)) \end{aligned}$$

for $a \in A$, $t, s \in G$, and $f \in C_c(G, A)$ (see [**14**, Proposition 2.34]). We show that i_A and i_G respect the respective real structures of A, G, and $A \rtimes_\alpha G$. That is, $i_A(\overline{a}) = \overline{(i_A(a))}$ and $i_G(\overline{t}) = \overline{(i_G(t))}$. Indeed for a, t, s and f, as above,

$$i(\overline{a})(f)(s) = \overline{a}f(s) = \overline{a\overline{f(s)}} = \overline{a\overline{f}(\overline{s})} = \overline{i(a)(\overline{f})(\overline{s})} = \overline{i(a)}(f)(s)$$

and

$$\begin{aligned} i(\overline{t})(f)(s) &= \alpha_{\overline{t}}(f(\overline{t}^{-1}s)) = \overline{\alpha_t(\overline{f}(t^{-1}\overline{s}))} = \overline{i(t)(\overline{f})(\overline{s})} \\ &\overline{i(t)(\overline{f})}(s) = \overline{i(t)}(f)(s). \end{aligned}$$

Thus, $i(\overline{a}) = \overline{i(a)}$ and $i(\overline{t}) = \overline{i(t)}$ proving (1).

For (2) we need to show that the homomorphism ϕ from Theorem 3 respects the real structures. By [**14**, Proposition 2.39], we know that ϕ is given by the formula

$$\phi(f) = \int_G j_A(f(s))j_G(s)\,d\mu(s)$$

for $f \in C_c(G, \mathbf{C})$. Then we have

$$\begin{aligned}\phi(\overline{f}) &= \int_G j(\overline{f}(s))j(s)\,d\mu(s) = \int_G j(\overline{f(\overline{s})})j(s)\,d\mu(s)\\ &= \int_G j(\overline{f(s)})j(\overline{s})\,d\mu(s) \quad \text{(by Lemma 1)}\\ &= \overline{\int_G j(f(s))j(s)\,d\mu(s)} = \overline{\phi(f)},\end{aligned}$$

showing that $\phi(\overline{f}) = \overline{\phi(f)}$ and proving (2).

As above, (3) is a standard uniqueness argument. □

By taking the involution on G to be the identity, the previous theorem reduces to the following result phrased in terms of real C*-algebras.

Theorem 5. *Let (A, G, α) be a C*-dynamical system, where A is a real C*-algebra.*

(1) *There is a nondegenerate covariant homomorphism i from (A, G, α) to $M(A \rtimes_\alpha G)$ such that $i(A)i(C^*_{\mathbf{R}}(G))$ is a dense subset of $A \rtimes_\alpha G$.*

(2) *Let B be a real C*-algebra. For every nondegenerate covariant homomorphism j from (A, G, α) to $M(B)$ such that $j(A)j(C^*_{\mathbf{R}}(G)) \subseteq B$, there exists a unique nondegenerate real homomorphism $\phi\colon A \rtimes_\alpha G \to B$ such that $\widetilde{\phi} \circ i = j$.*

(3) *$A \rtimes_\alpha G$ is the unique real C*-algebra up to isomorphism satisfying* (1) *and* (2).

We conclude this section with two theorems needed in the following section. The first deals with the interaction of tensor products and crossed products for real C*-algebras, extending Lemma 2.75 of [**14**]. The second deals with the crossed product obtained by the left regular representation of a group G on $C_0(G)$.

For $z \in C_c(G, \mathbf{C})$ and $a \in A$, we let $z \otimes a$ denote the element of $C_c(G, A)$ defined by $(z \otimes a)(s) = z(s)a$. By [**14**, Theorem 1.87], such elements span a dense subset of $A \rtimes_\alpha G$. Similarly, elements of the form $z \otimes c \otimes d$ span a dense subset of $(C \rtimes_\gamma G) \otimes D$ and also of $(C \otimes D) \rtimes_{\gamma \otimes \mathrm{id}} G$ as in the theorem below.

Note that if two C*-algebras A and B have real structures, then the tensor product $A \otimes_{\max} B$ has a real structure given by $\overline{a \otimes b} = \overline{a} \otimes \overline{b}$.

Theorem 6. *Let C and D be C*-algebras, and let G be a group with a group action γ on C. Then there is an isomorphism*

$$\phi\colon (C \rtimes_\gamma G) \otimes_{\max} D \longrightarrow (C \otimes_{\max} D) \rtimes_{\gamma\otimes\mathrm{id}} G.$$

Furthermore, if C, D and the dynamical system (C, G, γ) have real structures, then the isomorphism ϕ preserves the real structures on the corresponding algebras.

Proof. Throughout this proof, we suppress the notation indicating that tensor products carry the max norm. We will reproduce the proof given in [**14**], modifying it slightly to make the isomorphism more explicit.

For $f \in C_c(G, C)$ and $d \in D$ we define $\phi(f \otimes d) \in C_c(G, C \otimes D)$ by $\phi(f \otimes d)(s) = f(s) \otimes d$. This homomorphism extends to the desired homomorphism of C*-algebras.

Let i be the covariant homomorphism from $(C \otimes D, G, \gamma \otimes \mathrm{id})$ to $(C \otimes D) \rtimes_{\gamma\otimes\mathrm{id}} G$ given by Theorem 1, part (1). Let j be the covariant homomorphism from $(C \otimes D, G, \gamma \otimes \mathrm{id})$ to $(C \rtimes_\gamma G) \otimes D$ described in the proof of Lemma 2.75 of [**14**]. Then j is given by the formulas

$$\begin{aligned} j_{C\otimes D}\colon C \otimes D &\longrightarrow M((C \rtimes_\gamma G) \otimes D) \\ j_{C\otimes D}(c \otimes d)(f \otimes d') &= i_C(c) f \otimes dd' \\ j_G\colon G &\longrightarrow M((C \rtimes_\gamma G) \otimes D) \\ j_G(s)(f \otimes d') &= i_G(s) f \otimes d' \end{aligned}$$

for $c \in C$; $d, d' \in D$; $s \in G$; and $f \in C_0(G, C)$.

By Theorem 1, part (2), there is a homomorphism ψ from $(C \otimes D) \rtimes_{\gamma\otimes\mathrm{id}} G$ to $(C \rtimes_\gamma G) \otimes D$ such that $j = \widetilde{\psi} \circ i$ (that is, both $j_G = \widetilde{\psi} \circ i_G$ and $j_{C\otimes D} = \widetilde{\psi} \circ i_{C\otimes D}$).

Now we show that $i = \widetilde{\phi} \circ j$. If suffices to show that $i_{C\otimes D}(c \otimes d)(g)(s) = (\widetilde{\phi} \circ j_{C\otimes D})(c \otimes d)(g)(s)$ and $i_G(t)(g)(s) = (\widetilde{\phi} \circ j_G)(t)(g)(s)$ for $c \in C$, $d \in D$, $g \in C_0(G, C \otimes D)$, and $t, s \in G$. It also suffices to consider only functions g of the form $g = z \otimes (c' \otimes d') = \phi((z \otimes c') \otimes d')$ for $c' \in C$, $d' \in D$. Then

$$\begin{aligned}
(\widetilde{\phi} \circ j_{C\otimes D})(c \otimes d)(g)(s) &= \widetilde{\phi}(j_{C\otimes D}(c \otimes d))\phi((z \otimes c') \otimes d')(s) \\
&= \phi(j_{C\otimes D}(c \otimes d)((z \otimes c') \otimes d'))(s) \\
&= \phi(i_C(c)(z \otimes c') \otimes dd')(s) \\
&= cz(s)c' \otimes dd' \\
&= (c \otimes d)(z(s)c' \otimes d') \\
&= i_{C\otimes D}(g)(s)
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{\phi} \circ j_G)(t)(g)(s) &= \phi(j_G(t)((z \otimes c') \otimes d'))(s) \\
&= i_G(t)(z \otimes c')(s) \otimes d' \\
&= z(t^{-1}s)c' \otimes d' \\
&= ((z \otimes c) \otimes d)(t^{-1}s) \\
&= i_G(t)(g)(s).
\end{aligned}$$

Therefore, the relations $\widetilde{\phi} \circ \widetilde{\psi} = \mathrm{id}$ and $\widetilde{\psi} \circ \widetilde{\phi} = \mathrm{id}$ hold on the image of i and of j, respectively. Since $i_{C\otimes D}(C \otimes D) \cdot i_G(C_0(G))$ is dense in $(C \otimes D) \rtimes_{\gamma\otimes \mathrm{id}} G$ and since $j_{C\otimes D}(C \otimes D) \cdot j_G(C_0(G))$ is dense in $(C \rtimes_\gamma G) \otimes D$ (see the proof of Lemma 2.75 [**14**]) it follows that ϕ and ψ are inverses.

It remains only to show that when C, D and G have real structures, then ϕ respects the conjugate-linear involutions on the respective C*-algebras. Indeed for $f \in C_c(G)$ and $d \in D$, we have

$$\begin{aligned}
\phi(\overline{f \otimes d})(s) &= \phi(\overline{f} \otimes \overline{d})(s) = \overline{f}(s) \otimes \overline{d} \\
&= \overline{f(\overline{s}) \otimes d} = \overline{\phi(f \otimes d)(\overline{s})} = \overline{\phi(f \otimes d)}(s). \qquad \square
\end{aligned}$$

Let G be a locally compact group with real structure. As in [**14**, page 45], let $(C_0(G), G, \mathrm{lt})$ denote the C*-dynamical system given by left translation. That is, $\mathrm{lt}_s(f)(r) = f(s^{-1}r)$. If G has a real structure, then $C_0(G)$ also has a real structure given by $\overline{f}(s) = \overline{f(\overline{s})}$. The real structures on G and $C_0(G)$ are compatible with the action lt, making

$(C_0(G), G, \mathrm{lt})$ a C*-dynamical system with real structure. Indeed, for $s, t \in G$ and $f \in C_0(G)$, we have

$$\begin{aligned}\mathrm{lt}_{\overline{s}}(f)(r) &= f(\overline{s}^{-1}r) = \overline{\overline{f}(s^{-1}\overline{r})} \\ &= \overline{\mathrm{lt}_s(\overline{f})(\overline{r})} = \overline{\mathrm{lt}_s(\overline{f})}(r),\end{aligned}$$

showing that $\mathrm{lt}_{\overline{s}}(f) = \overline{\mathrm{lt}_s(\overline{f})}$.

The conjugate-linear involution on $C_0(G)$ passes to one on $L^2(G)$. Then the algebra of bounded operators $B(L^2(G))$ and the algebra of compact operators $\mathcal{K}(L^2(G))$ have real structures given by $\overline{T}(h) = \overline{T(\overline{h})}$ for $T \in \mathcal{K}(L^2(G))$ and $h \in L^2(G)$.

Theorem 7. *Let G be a locally compact group with real structure, then there is an isomorphism*

$$C_0(G) \rtimes_{\mathrm{lt}} G \cong \mathcal{K}(L^2(G))$$

respecting the real structures described above.

Proof. Define a homomorphism $\Theta\colon C_0(G) \rtimes_{\mathrm{lt}} G \to \mathcal{K}(L^2(G))$ defined by

$$\Theta(f)(h)(r) = \int_G f(s,r) h(s^{-1}r)\, d\mu(s)$$

for $f \in C_c(G, C_0(G)), h \in L^2(G)$, and $r \in G$. Then Θ is an isomorphism by [**14**, Lemma 7.5]. We only need to show that the real structures are preserved. We may identify f with a function in $C_c(G \times G)$ with real structure given by $\overline{f}(s,r) = \overline{f(\overline{s}, \overline{r})}$. Then we have

$$\begin{aligned}\Theta(\overline{f})(h)(r) &= \int_G \overline{f}(s,r) h(s^{-1}r)\, d\mu(s) = \int_G \overline{f(\overline{s}, \overline{r})} h(s^{-1}r)\, d\mu(s) \\ &= \overline{\int_G f(s, \overline{r}) \overline{h(\overline{s}^{-1}r)}\, d\mu(s)} = \overline{\int_G f(s, \overline{r}) \overline{h}(s^{-1}\overline{r})\, d\mu(s)} \\ &= \overline{\Theta(f)(\overline{h})(\overline{r})} = \overline{\Theta(f)}(h)(r). \qquad \square\end{aligned}$$

3. Dual groups with involution. Let G be a locally compact group, and let $\widehat{G}$ be the dual group consisting of continuous group homomorphisms from G to the circle group $\mathbf{T}$. We take the canonical involution on $\mathbf{T} \subset \mathbf{C}$ to be complex conjugation. For any involution $s \mapsto \overline{s}$ on G, there is an involution on $\widehat{G}$ defined by $\overline{\gamma}(s) = \overline{\gamma(\overline{s})}$. The group $\mathbf{Z}$ with trivial involution and the group $\mathbf{T}$ with involution given by complex conjugation are mutually dual to each other. We will always assume that these are the given involutions on $\mathbf{Z}$ and $\mathbf{T}$, particularly in Theorem 10 below.

If (A, G, α) is a real C*-dynamical system, then the dual action of $\widehat{G}$ on the crossed product is compatible with the involution, making $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$ a real C*-dynamical system. Indeed, if $\gamma \in \widehat{G}$ and $f \in C_c(G, A)$, then we check that $\widehat{\alpha}_{\overline{\gamma}}(\overline{f}) = \overline{\widehat{\alpha}_\gamma(f)}$ using the formula $\widehat{\alpha}_\gamma(f)(s) = \overline{\gamma(s)} f(s)$ for the dual action.

Theorem 8. *Let (A, G, α) be a C*-dynamical system with real structure. If G is abelian, then there is an isomorphism*

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong C_0(G, A) \rtimes_{\mathrm{lt} \otimes \mathrm{id}} G$$

of C^*-algebras with real structure.

Proof. The homomorphism $\Phi \colon (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \widehat{G} \to C_0(G, A) \rtimes_{\mathrm{lt} \otimes \mathrm{id}} G$ for $F \in C_c(\widehat{G}, C_0(G, A))$ and $s, r \in G$ defined by

$$\Phi(F)(s, r) = \int_{\widehat{G}} \alpha_r^{-1}(F(\gamma, s)) \overline{\gamma(s^{-1} r)} \, d\widehat{\mu}(\gamma)$$

is proven to be an isomorphism in [**14**, Lemmas 7.2, 7.3, 7.4]. To show that Φ respects the real structures, we compute

$$\begin{aligned}
\Phi(\overline{F})(s, r) &= \int_{\widehat{G}} \alpha_r^{-1}(\overline{F}(\gamma, s)) \overline{\gamma(s^{-1} r)} \, d\widehat{\mu}(\gamma) \\
&= \int_{\widehat{G}} \overline{\alpha_{\overline{r}}^{-1}(F(\overline{\gamma}, \overline{s}))} \overline{\gamma}(\overline{s}^{-1} \overline{r}) \, d\widehat{\mu}(\gamma) \\
&= \int_{\widehat{G}} \overline{\alpha_{\overline{r}}^{-1}(F(\gamma, \overline{s}))} \gamma(\overline{s}^{-1} \overline{r}) \, d\widehat{\mu}(\gamma) \\
&= \overline{\int_{\widehat{G}} \alpha_{\overline{r}}^{-1}(F(\gamma, \overline{s})) \overline{\gamma(\overline{s}^{-1} \overline{r})} \, d\widehat{\mu}(\gamma)}
\end{aligned}$$

$$\begin{aligned} &= \overline{\Phi(F)(\overline{s}, \overline{r})} \\ &= \overline{\Phi(F)}(s, r). \qquad \square \end{aligned}$$

Theorem 9 (Takai duality). *Let (A, G, α) be a C^*-dynamical system with real structure. If G is abelian, then there is an isomorphism*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \widehat{G} \cong \mathcal{K} \otimes A$$

of C^*-algebras with real structure.

Proof. By Theorems 4, 5 and 6, we have

$$\begin{aligned} (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \widehat{G} &\cong C_0(G, A) \rtimes_{\mathrm{lt} \otimes \mathrm{id}} G \\ &\cong (C_0(G) \otimes A) \rtimes_{\mathrm{lt} \otimes \mathrm{id}} G \\ &\cong (C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A \\ &\cong \mathcal{K} \otimes A, \end{aligned}$$

where all the isomorphisms respect the real structures. $\square$

Let $(A, \overline{\,\cdot\,})$ be a complex C*-algebra with conjugate-linear involution. If the involution is understood, let $A^\#$ be the real C*-algebra consisting of fixed points. Recall that the K-theory of a real C*-algebra is a graded group with period 8 and is a module over the ring $K_*(\mathbf{R})$.

Theorem 10. (1) *Let $(A, \overline{\,\cdot\,}, \mathbf{Z}, \overline{\,\cdot\,}, \alpha)$ be a real C^*-dynamical system. Then there is a long exact sequence in real K-theory*:

$$\begin{aligned} \cdots \longrightarrow K_*(A^\#) \xrightarrow{1-\alpha_*} K_*(A^\#) \xrightarrow{i} K_*((A \rtimes_\alpha \mathbf{Z})^\#) \\ \xrightarrow{\partial} K_*(A^\#) \xrightarrow{1-\alpha_*} K_*(A^\#) \longrightarrow \cdots, \end{aligned}$$

where ∂ has degree -1.

(2) *Let $(A, \overline{\,\cdot\,}, \mathbf{T}, \overline{\,\cdot\,}, \alpha)$ be a real C^*-dynamical system. Then there is a long exact sequence in real K-theory*:

$$\begin{aligned} \cdots \longrightarrow K_*(A^\#) \xrightarrow{\partial} K_*((A \rtimes_\gamma \mathbf{T})^\#) \xrightarrow{1-\hat{\gamma}_*} K_*((A \rtimes_\gamma \mathbf{T})^\#) \\ \longrightarrow K_*(A^\#) \longrightarrow \cdots \end{aligned}$$

where ∂ has degree -1.

Proof. The sequence in part (1) is the real Pimsner-Voiculescu exact sequence, found in [**13**, Theorem 1.5.5]. The sequence in part (2) is the dual Pimsner-Voiculescu exact sequence, obtained by constructing the Pimsner-Voiculescu sequence from the dynamical system $(A \rtimes_\gamma \mathbf{T}, \overline{\cdot}, \mathbf{Z}, \overline{\cdot}, \widehat{\gamma})$ and then using Takai duality (Theorem 9) and the stability of K-theory. ◻

Recall that united K-theory is an invariant defined for real C*-algebras, consisting of a triple of graded abelian groups $K^{\mathrm{CRT}}(A) = \{K_*(A), KU_*(A) = K_*(A_{\mathbf{C}}), KT_*(A) = K_*(T \otimes A)\}$ (real, complex and self-conjugate K-theory), as well as the natural transformations among the three. Then we have the same exact sequences for united K-theory.

Theorem 11. (1) *Let $(A, \overline{\cdot}, \mathbf{Z}, \overline{\cdot}, \alpha)$ be a real C*-dynamical system. Then there is a long exact sequence of* CRT*-modules*:

$$\cdots \longrightarrow K_*^{\mathrm{CRT}}(A^\#) \xrightarrow{1-\alpha_*} K_*^{\mathrm{CRT}}(A^\#) \xrightarrow{i} K_*^{\mathrm{CRT}}((A \rtimes_\alpha \mathbf{Z})^\#) \\ \xrightarrow{\partial} K_*^{\mathrm{CRT}}(A^\#) \xrightarrow{1-\alpha_*} K_*^{\mathrm{CRT}}(A^\#) \longrightarrow \cdots,$$

where ∂ has degree -1.

(2) *Let $(A, \overline{\cdot}, \mathbf{T}, \overline{\cdot}, \alpha)$ be a real C*-dynamical system. Then there is a long exact sequence of* CRT*-modules*

$$\cdots \longrightarrow K_*^{\mathrm{CRT}}(A^\#) \xrightarrow{\partial} K_*^{\mathrm{CRT}}((A \rtimes_\gamma \mathbf{T})^\#) \xrightarrow{1-\hat{\gamma}_*} K_*^{\mathrm{CRT}}((A \rtimes_\gamma \mathbf{T})^\#) \\ \longrightarrow K_*^{\mathrm{CRT}}(A^\#) \longrightarrow \cdots,$$

where ∂ has degree -1.

Proof. Let $(A, \overline{\cdot}, \mathbf{Z}, \overline{\cdot}, \alpha)$ be a real C*-dynamical system. We can tensor this system with any nuclear real C*-algebra B to get a real C*-dynamical system, $(A \otimes B, \overline{\cdot} \otimes 1, \mathbf{Z}, \overline{\cdot} \otimes 1, \alpha \otimes 1)$. Applying Theorem 10, we obtain a long exact sequence

$$\cdots \longrightarrow K_*((A \otimes B)^\#) \xrightarrow{1-\alpha_*} K_*((A \otimes B)^\#) \\ \xrightarrow{i} K_*(((A \otimes B) \rtimes_{\alpha \otimes 1} \mathbf{Z})^\#) \xrightarrow{\partial} K_*((A \otimes B)^\#) \\ \xrightarrow{1-\alpha_*} K_*((A \otimes B)^\#) \longrightarrow \cdots.$$

Using Theorem 6, we can replace the middle group with $K_*(((A \rtimes_\alpha \mathbf{Z}) \otimes B)^{\#})$. Taking B in turn to be $\mathbf{R}$, $\mathbf{C}$ and T, we obtain the desired sequences on real, complex and self-conjugate K-theory. Furthermore, the homomorphisms commute with the natural transformations of united K-theory by the naturalness of the Pimsner-Voiculescu exact sequence with respect to homomorphisms induced by products with KK-elements. This proves (1), and the proof of (2) is similar. $\square$

4. Real graph algebras and K-theory. We will use the notation and conventions of [**10**]. Let $E = (E_0, E_1, r, s)$ be a directed graph. We say that E is row-finite if each vertex only receives finitely many edges. Equivalently, E is row-finite if each row of the vertex matrix

$$(A_E)_{ij} = \{\text{the number of edges from vertex } j \text{ to vertex } i\}$$

has finite sum.

For a graph E, the algebra $C^*(E)$ is the universal C*-algebra generated by a set of non-zero orthogonal projections p_v indexed by E_0 and a set of partial isometries s_e indexed by E_1, subject to a certain set of relations. If E is row-finite, the relations are:

(1) $s_e^* s_e = p_{s(e)}$ for each edge e,

(2) $\sum_{e \in r^{-1}(v)} s_e s_e^* = p_v$ for each vertex v such that $r^{-1}(v)$ is non-zero.

For a graph E that is not row-finite, the relations are:

(1) $s_e^* s_e = p_{s(e)}$ for each edge e,

(2) $s_e s_e^* \leq p_{r(e)}$ for each edge e,

(3) $\sum_{e \in r^{-1}(v)} s_e s_e^* = p_v$ for each vertex v such that $r^{-1}(v)$ is non-zero and finite.

In either case, the real graph algebra $C^*_{\mathbf{R}}(E)$ is the closed algebra over $\mathbf{R}$ generated by the elements p_v and s_e, considered as a real subalgebra of $C^*(E)$. The associated conjugate-linear involution on $C^*(E)$ is given by $\lambda p_v \mapsto \overline{\lambda} p_v$ and $\lambda s_e \mapsto \overline{\lambda} s_e$.

Theorem 12. *Suppose that E is a graph in which every cycle has an entry. Then there is, up to isomorphism, a unique real C*-algebra $C^*_{\mathbf{R}}(E)$ generated by non-zero orthogonal projects p_v and partial isometries s_e subject to the relations above.*

Proof. Suppose that A and B are two real C*-algebras generated by sets of elements $\{p_v, s_e\}$ and $\{p'_v, s'_e\}$, respectively, satisfying the relations above. Then $A_{\mathbf{C}}$ and $B_{\mathbf{C}}$ are complex C*-algebras generated by the same sets of elements. So, by the Cuntz-Krieger uniqueness theorem ([**11**, Theorem 1.5]), there is an isomorphism ϕ from $A_{\mathbf{C}}$ to $B_{\mathbf{C}}$ mapping p_v to p'_v and s_e to s'_e. Then ϕ restricts to an isomorphism from A to B. $\square$

The next result can be used to compute the K-theory of $C^*_{\mathbf{R}}(E)$.

Theorem 13. *For a row-finite directed graph E with no sources there are long exact sequences for real K-theory*

$$\cdots \longrightarrow K_*(\mathbf{R})^{E_0} \xrightarrow{1-A_E^t} K_*(\mathbf{R})^{E_0} \longrightarrow K_*(C^*_{\mathbf{R}}(E)) \\ \xrightarrow{\partial} K_*(\mathbf{R})^{E_0} \xrightarrow{1-A_E^t} K_*(\mathbf{R})^{E_0} \longrightarrow \cdots$$

and united K-theory

$$\cdots \longrightarrow K^{\mathrm{CRT}}(\mathbf{R})^{E_0} \xrightarrow{1-A_E^t} K^{\mathrm{CRT}}(\mathbf{R})^{E_0} \longrightarrow K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E)) \\ \xrightarrow{\partial} K^{\mathrm{CRT}}(\mathbf{R})^{E_0} \xrightarrow{1-A_E^t} K^{\mathrm{CRT}}(\mathbf{R})^{E_0} \longrightarrow \cdots,$$

where ∂ has degree -1.

Proof. It suffices to develop the second sequence, since the first sequence is just the real part of the second.

Let γ be the gauge action of $\mathbf{T}$ on $C^*(E)$ defined by $\gamma_z(s_e) = zs_e$. It is easy to see that $\gamma_{\bar{z}}(a) = \overline{\gamma_z(\overline{a})}$ for all $a \in C^*(E)$, showing that $(C^*(E), \mathbf{T}, \gamma)$ is a real dynamical system. Then we obtain a long exact sequence

$$\cdots \longrightarrow K^{\mathrm{CRT}}(C^*(E)) \longrightarrow K^{\mathrm{CRT}}(C^*(E) \rtimes_\gamma \mathbf{T}) \\ \xrightarrow{1-\hat{\gamma}_*} K^{\mathrm{CRT}}(C^*(E) \rtimes_\gamma \mathbf{T}) \longrightarrow K^{\mathrm{CRT}}(C^*(E)) \longrightarrow \cdots$$

from the dual Pimsner-Voiculescu exact sequence, Theorem 11.

Let $E \times_1 \mathbf{Z}$ be the graph obtained from E with vertex set $E_0 \times \mathbf{Z}$ and edge set $E_1 \times \mathbf{Z}$; and where $s(e,n) = (s(e),n)$ and $r(e,n) = (r(e),n-1)$

(as in [**10**, page 64]). There is (by [**10**, Lemma 7.10]) an isomorphism ϕ from $C^*(E \times_1 \mathbf{Z})$ to $C^*(E) \rtimes_\gamma \mathbf{T}$ defined by $\phi(s_{(e,n)}) = t_{(e,n)}$ where $t_{(e,n)} \in C(\mathbf{T}, C^*(E))$ in turn is defined by $t_{(e,n)}(z) = z^n s_e$. Since $\overline{s_{(e,n)}} = s_{(e,n)}$ and $\overline{t_{(e,n)}} = t_{(e,n)}$ it follows that ϕ is a homomorphism of C*-algebras with real structures. Let β be the automorphism on $E \times_1 \mathbf{Z}$ defined by $\beta(s_{(e,n)}) = s_{(e,n-1)}$. Then, the relation $\phi \circ \beta = \widehat{\gamma} \circ \phi$ holds (as homomorphisms of complex C*-algebras) by [**10**, Lemma 7.10], so it must hold when restricted to the real C*-algebras. Hence, applying united K-theory, the square below commutes.

$$\begin{array}{ccc} K^{\mathrm{CRT}}(C^*(E) \rtimes_\gamma \mathbf{T}) & \xrightarrow{1-\hat{\gamma}_*} & K^{\mathrm{CRT}}(C^*(E) \rtimes_\gamma \mathbf{T}) \\ \Big\downarrow{\scriptstyle \phi_*} & & \Big\downarrow{\scriptstyle \phi_*} \\ K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E \times_1 \mathbf{Z})) & \xrightarrow{1-\beta_*} & K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E \times_1 \mathbf{Z})). \end{array}$$

Thus, in Sequence 1, the middle homomorphism can be replaced by $1 - \beta_*$. Using the relation $1 - \beta_*^{-1} = (1 - \beta_*)(-\beta_*^{-1}) = (-\beta_*^{-1})(1 - \beta_*)$, it is easy to see that $\ker(1 - \beta_*) = \ker(1 - \beta_*^{-1})$ and $\operatorname{coker}(1 - \beta_*) = \operatorname{coker}(1 - \beta_*^{-1})$. So we can further replace the middle homomorphism in the exact sequence by $1 - \beta_*^{-1}$.

As in [**10**, Lemma 7.13 and Corollary 7.14], we have

$$K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E \times_1 \mathbf{Z})) \cong \lim(K^{\mathrm{CRT}}(\mathbf{R})^{E_0}, A_E^t).$$

Furthermore, the same argument as in [**10**, Lemma 7.15] shows that the kernel and cokernel of the homomorphism

$$K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E \times_1 \mathbf{Z})) \xrightarrow{1-\beta_*^{-1}} K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E \times_1 \mathbf{Z}))$$

are isomorphic to that of

$$K^{\mathrm{CRT}}(\mathbf{R})^{E_0} \xrightarrow{1-A_E^t} K^{\mathrm{CRT}}(\mathbf{R})^{E_0},$$

completing the proof. $\square$

Let $n = |E_0|$ and $A = A_E$. Then the long exact sequence of Theorem 13 unsplices into

$$0 \longrightarrow \operatorname{coker}(1 - A^t) \longrightarrow K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E)) \xrightarrow{\partial} \ker(1 - A^t) \longrightarrow 0,$$

where $1 - A^t \colon K^{\mathrm{CRT}}(\mathbf{R})^n \to K^{\mathrm{CRT}}(\mathbf{R})^n$. But, unlike in the complex case, this does not immediately give us the K-theory groups. Focusing on the real part of united K-theory, the groups of $KO_*(\mathbf{R})$ in degrees 0–7 are:

$$KO_*(\mathbf{R}) = \mathbf{Z} \quad \mathbf{Z}_2 \quad \mathbf{Z}_2 \quad 0 \quad \mathbf{Z} \quad 0 \quad 0 \quad 0.$$

Taking advantage of the placement of the 0's, we immediately obtain $KO_*(C^*_{\mathbf{R}}(E))$ (up to isomorphism) in all degrees except 1 and 2. In degree 1, we find that $KO_1(C^*_{\mathbf{R}}(E))$ is an extension of $\operatorname{coker}(\mathbf{Z}_2^n \xrightarrow{1-A^t} \mathbf{Z}_2^n)$ by $\ker(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n)$. The extension necessarily splits, since $\ker(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n)$ is a free group. Hence we have the following:

$$\begin{aligned}
KO_0(C^*_{\mathbf{R}}(E)) &\cong \operatorname{coker}(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n) \\
KO_1(C^*_{\mathbf{R}}(E)) &\cong \operatorname{coker}(\mathbf{Z}_2^n \xrightarrow{1-A^t} \mathbf{Z}_2^n) \oplus \ker(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n) \\
KO_2(C^*_{\mathbf{R}}(E)) &\cong \text{an extension of } \operatorname{coker}(Z_2^n \xrightarrow{1-A^t} Z_2^n) \\
&\qquad\qquad \text{by } \ker(Z_2^n \xrightarrow{1-A^t} Z_2^n) \\
KO_3(C^*_{\mathbf{R}}(E)) &\cong \ker(\mathbf{Z}_2^n \xrightarrow{1-A^t} \mathbf{Z}_2^n) \\
KO_4(C^*_{\mathbf{R}}(E)) &\cong \operatorname{coker}(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n) \\
KO_5(C^*_{\mathbf{R}}(E)) &\cong \ker(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n) \\
KO_6(C^*_{\mathbf{R}}(E)) &\cong 0 \\
KO_7(C^*_{\mathbf{R}}(E)) &\cong 0.
\end{aligned}$$

The extension for $KO_2(C^*_{\mathbf{R}}(E))$ is therefore the only component undetermined up to isomorphism (so far). This extension splits in some cases but not always (contrary to what is stated in [**13**]). For example, in the special case of the real Cuntz algebras $\mathcal{O}_n^{\mathbf{R}}$, we have found that the extension is non-trivial when $n \equiv 1 \pmod 4$ and splits otherwise (see [**1**, subsection 5.1]). In that computation, the extension problem was solved by taking advantage of the algebraic structure relating $KO_*(C^*_{\mathbf{R}}(E))$ and $K_*(C^*(E)) = KU_*(C^*_{\mathbf{R}}(E))$. In fact, the following theorem indicates that the extension is always determined by this algebraic information and does not require any additional information from the real C*-algebra or the original graph. Put negatively, neither the real K-theory nor even the united K-theory represents more information about the real C*-algebra or the original graph than just the complex K-theory.

Theorem 14. *Let E and F be row-finite graphs with no sources. Then $K_*(C^*(E)) \cong K_*(C^*(F))$ if and only if $K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E)) \cong K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(F))$.*

Before proving this theorem, we develop some preliminaries on the *core* of the united K-theory, from [**7**, Chapter 5]. For a real C*-algebra A, let $KU_*(A) = K_*(A_{\mathbf{C}})$ and

$$h_*(KU_*(A)) = \ker(1-\psi_U)/\mathrm{image}\,(1+\psi_U).$$

Then there are natural maps

$$\begin{aligned} c'\colon \eta K_*(A) &\longrightarrow h_*(KU_*(A)) \\ r'\colon h_*(KU_*(A)) &\longrightarrow \eta K_*(A) \end{aligned}$$

defined by $c'(\eta x) = [cx]$ and $r'[y] = r\beta_U^{-1} y$ of degrees -1 and -2, respectively. Furthermore, there is a long exact sequence

(2)
$$\cdots \longrightarrow \eta K_*(A) \xrightarrow{\eta} \eta K_*(A) \xrightarrow{c'} h_*(K_*(A)) \xrightarrow{r'} \eta K_*(A) \longrightarrow \cdots .$$

Note that all of the groups of this sequence are $\mathbf{Z}_2$-modules so any extension problems have a unique solution up to isomorphism. Also note that the groups $\eta K_*(A)$ repeat with period 8, the groups $KU_*(A)$ repeat with period 2 and the groups $h_*(KU_*(A))$ repeat with period 4.

Following [**7**], we define the *core* of a real C*-algebra A by

$$\mathrm{core}\,(A) = \{\eta K_*(A), KU_*(A), \psi, \eta, c', r'\}.$$

For two real C*-algebras A and B, it follows from [**7**, Theorem 4.2.1] that $K^{\mathrm{CRT}}(A) \cong K^{\mathrm{CRT}}(B)$ if and only if $\mathrm{core}\,(A) \cong \mathrm{core}\,(B)$.

Proof of Theorem 14. It is enough to show that $\mathrm{core}\,(C^*_{\mathbf{R}}(E))$ can be computed from the module $KU_*(C^*_{\mathbf{R}}(E)) = K_*(C^*(E))$, independent of the knowledge of graph E.

Let $A = A_E$ be the incidence matrix for a graph E. We know that $KU_1(C^*_{\mathbf{R}}(E)) \cong KU_3(C^*_{\mathbf{R}}(E)) = \ker(\mathbf{Z}^n \xrightarrow{1-A^t} \mathbf{Z}^n)$ is free. We also know that $\psi_U = 1$ in degrees 0 and 1 and $\psi_U = -1$ in degrees 2 and 3. So it follows that

$$\begin{aligned} h_3(KU_*(C^*_{\mathbf{R}}(E)) &= \frac{\ker(1-\psi_U)}{\operatorname{image}(1+\psi_U)} \\ &= \{x \in KU_*(C^*_{\mathbf{R}}(E)) \mid 2x = 0\} \\ &= 0. \end{aligned}$$

We can also discern that $\eta\colon K_4(C^*_{\mathbf{R}}(E)) \to K_5(C^*_{\mathbf{R}}(E))$ vanishes by making use of the diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(1-A^t) \arrow[r] \arrow[d, "\eta"] & K_4(C^*_{\mathbf{R}}(E)) \arrow[r] \arrow[d, "\eta"] & 0 \arrow[r] \arrow[d, "\eta"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & K_5(C^*_{\mathbf{R}}(E)) \arrow[r] & \ker(1-A^t) \arrow[r] & 0 \end{tikzcd}$$

obtained from Theorem 13.

We simplify the notation by setting $M_* = \eta K_*(C^*_{\mathbf{R}}(E))$ and $N_* = KU_*(C^*_{\mathbf{R}}(E))$ (so that $\operatorname{core}(C^*_{\mathbf{R}}(E)) = \{M, N\}$). We know that $M_0 = M_5 = M_6 = M_7 = 0$ and that $N_3 = 0$. The remaining groups M_* can be computed up to isomorphism, provided that the groups N_* are known. Indeed, using the long exact sequence, we have isomorphisms $r'\colon N_6 \to M_4$ and $c'\colon M_1 \to N_0$ allowing us to compute M_4 and M_1. Finally, M_2 and M_3 are obtained from the following segments of long exact sequence 2:

$$\begin{gathered} 0 \longrightarrow M_1 \xrightarrow{\eta} M_2 \xrightarrow{c'} h_1(N_*) \longrightarrow 0 \\ 0 \longrightarrow h_5(N_*) \xrightarrow{r'} M_3 \xrightarrow{\eta} M_4 \longrightarrow 0. \end{gathered} \qquad \square$$

Finally, we can extend this result to all graphs using desingularization.

Theorem 15. *Let E and F be arbitrary graphs. Then $K_*(C^*(E)) \cong K_*(C^*(F))$ if and only if $K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E)) \cong K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(F))$.*

Proof. Let E and F be an arbitrary graphs. Then by [**6**, Theorem 2.11], there is a nonsingular graph E' (that is, E' is row-finite with no sources) such that $C^*(E)$ is isomorphic to $pC^*(E')p$ where p is a full projection of $C^*(E')$. Since the projection p is a sum of projections associated with vertices, it is clear that p is in the real graph algebra $C^*_{\mathbf{R}}(E)$. Hence, the isomorphism restricts to the underlying real C*-algebras to give $C^*_{\mathbf{R}}(E) \cong pC^*_{\mathbf{R}}(E')p$. Just as in the complex case ([**5**, Corollary 2.6]), the full corner $pC^*(E')p$ is stably isomorphic to $C^*(E')$. Therefore, $K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E)) \cong K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(E'))$.

Similarly, $K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(F)) \cong K^{\mathrm{CRT}}(C^*_{\mathbf{R}}(F'))$ where F' is nonsingular. Then the result follows from Theorem 14. □

Recall ([**10**, page 33]) that a graph is cofinal if, for every vertex v and every infinite path ℓ, there is a path from a vertex of ℓ to v.

Corollary 16. *Let E_1 and E_2 graphs satisfy*:

(1) *E_i is cofinal,*

(2) *every cycle has an entry,*

(3) *there is a path to each vertex in E_i from a cycle,*

(4) *there is a path from each singular vertex in E_i to every other vertex in E_i.*

Then the real graph C-algebras $C^*_{\mathbf{R}}(E_1)$ and $C^*_{\mathbf{R}}(E_2)$ are isomorphic if and only if the complex graph C*-algebras $C^*(E_1)$ and $C^*(E_2)$ are isomorphic.*

Proof. The conditions on E_1 and E_2 ensure that the complex graph C*-algebras are simple and purely infinite, by [**6**, Corollaries 2.14, 2.15]. It follows that the real graph C*-algebras are also simple and purely infinite. The rest follows from Theorem 15 by using the classifications of simple purely infinite C*-algebras by Phillips in the complex case [**8**] and by Boersema, Ruiz and Stacey in the real case [**4**]. □

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