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ON (D_{12})-MODULES

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ABSTRACT. It is known that a direct summand of a (D_{12})-module need not be a (D_{12})-module. In this paper we establish some properties of *completely* (D_{12})*-modules* (modules for which every direct summand is a (D_{12})-module). After giving some examples of completely (D_{12})-modules, it is proved that every finitely generated weakly supplemented completely (D_{12})-module is a finite direct sum of local modules. We also prove that a direct sum of (D_{12})-modules need not be a (D_{12})-module. Then we deal with some special cases of direct sums of (D_{12})-modules. We conclude this work by characterizing some rings in terms of (D_{12})-modules.

1. Introduction. Throughout this paper, we assume that all rings are associative with identity and all modules are unital right modules. Let R be a ring and M a right R-module. For undefined terms, see [**3, 9, 13**]. We write $E(M)$ for the injective hull of M. The notation $N \leq M$ means that N is a submodule of M. A submodule N of M is called a *small submodule* if, whenever $N + L = M$ for some submodule L of M, we have $L = M$; and in this case we write $N \ll M$. A module M is said to be $\oplus$*-supplemented* if, for every submodule N of M, there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K. Keskin and Xue (in [**8**]) investigated a proper generalization of $\oplus$-supplemented modules. The module M is said to have (D_{12}) (or is a (D_{12})-module) if, for every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $\operatorname{Ker}\alpha$ is small in K.

In this paper we continue the study of (D_{12})-modules. In Section 2, we introduce the notion of (D_{13})-modules. We prove that the class of (D_{12})-modules strictly contains the class of (D_{13})-modules. In Section 3, we will be concerned with direct summands of (D_{12})-modules. A module M is said to be a *completely* (D_{12})*-module* if every direct summand of M has (D_{12}). It is known that a direct summand of a

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(D_{12})-module need not be (D_{12}) (Example 3.1). We begin by exhibiting some examples of completely (D_{12})-modules. Then we provide a characterization of direct summands having (D_{12}) (Theorem 3.6). Section 4 deals with direct sums of (D_{12})-modules. We give an example showing that a direct sum of (D_{12})-modules need not be (D_{12}). We show that a direct sum of (D_{12})-modules has (D_{12}) if the direct sum is a duo module. The main result of Section 5 shows that every finitely generated weakly supplemented completely (D_{12})-module is a finite direct sum of local modules. The last section is devoted to the study of some rings whose modules have (D_{12}).

2. (D_{13})**-modules.** A module M is said to have (D_{13}) (or a (D_{13})-module) if every factor module of M is isomorphic to a direct summand of M. Note that every semisimple module has (D_{13}) and every (D_{13})-module has (D_{12}).

Proposition 2.1. *Let M be an indecomposable module. If M is a (D_{13})-module, then M is a hollow module.*

Proof. Let N be a proper submodule of M. Since M has (D_{13}), there exists a direct summand K of M such that $K \cong M/N$. But M is indecomposable implies $K = M$. Thus, M/N is indecomposable. Therefore, M is hollow by [**3**, 2.13]. $\square$

Proposition 2.2. *A local module L has (D_{13}) if and only if L is simple.*

Proof. If L has (D_{13}) and K is the maximal submodule of L, then $L \cong L/K$ is simple. $\square$

Corollary 2.3. *Let E be an indecomposable module such that* $\operatorname{Rad}(E) \neq E$. *The following are equivalent*:

(i) *E has (D_{13})*;

(ii) *E is simple.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 2.1, E is hollow. Since $\operatorname{Rad}(E) \neq E$, E is local. Thus E is simple by Proposition 2.2.

(ii) $\Rightarrow$ (i). This is obvious. $\square$

Example 2.4. (a) It is clear that, for any prime integer p, the $\mathbf{Z}$-module $\mathbf{Z}_{p^\infty}$ is a (D_{13})-module.

(b) The converse of Proposition 2.1 need not be true in general as shown by the following examples:

(1) If L is a local module which is not simple (e.g., $L = \mathbf{Z}/p^k\mathbf{Z}$ with $k \geq 2$ and p is prime), then L does not have (D_{13}). Note that L has (D_{12}).

(2) If R is a DVR with quotient field Q, then Q_R is a hollow module which is not a (D_{13})-module. In fact, $Q_R/R \ncong Q_R$ since Q_R/R is artinian and $\mathrm{Soc}\,(Q_R) = 0$. Note that Q_R has (D_{12}).

A module M is said to have (D_2) if M satisfies the condition that, if N is a submodule of M for which M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

Remark 2.5. (1) It is clear that every (D_{13})-module having (D_2) is semisimple. Thus, a quasi-projective module P has (D_{13}) if and only if P is semisimple by [**9**, Proposition 4.38]. This gives that, for a ring R, R_R has (D_{13}) if and only if R is semisimple.

(2) It is clear that, if M is a (D_{13})-module with $\mathrm{Rad}\,(M) \neq M$, then M has a simple direct summand.

Proposition 2.6. *Let M be a module with* $\mathrm{Rad}\,(M) = 0$*. Then M has (D_{12}) if and only if M has (D_{13}). In this case, M is a cosemisimple module.*

Proof. Suppose that M has (D_{12}). Let N be a submodule of M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $\mathrm{Ker}\,\alpha$ is small in K. But $\mathrm{Rad}\,(M) = 0$. Then $\mathrm{Ker}\,\alpha = 0$, and hence $K \cong M/N$. Thus, M has (D_{13}). Moreover, we have $\mathrm{Rad}\,(M/N) = 0$ since $\mathrm{Rad}\,(K) = 0$. So M is cosemisimple. $\square$

Example 2.7. Let F be a free $\mathbf{Z}$-module and p a prime. It is easily seen that F contains a submodule N such that $F/N \cong \mathbf{Z}/p^2\mathbf{Z}$. Thus, $\mathrm{Rad}\,(F/N) \neq 0$. Hence, F is not cosemisimple. It follows that F is not a (D_{12})-module by Proposition 2.6.

Corollary 2.8. *Let M be a module with* $\operatorname{Rad}(M) = 0$. *If M is a* (D_{12})*-module having* (D_2), *then M is semisimple.*

Proof. By Proposition 2.6 and Remark 2.5. □

Proposition 2.9. *If every injective R-module has* (D_{13}), *then R is right hereditary.*

Proof. By hypothesis, every factor module of an injective R-module is injective. The result follows from [**13**, 39.16]. □

A module M is said to have *finite hollow dimension* (or *finite dual Goldie dimension*) if, for some $n \in \mathbf{N}$, there exists an epimorphism with small kernel from M to a direct sum of n hollow modules. In this case, we say that M has hollow dimension n, and we denote this by $h.\dim(M) = n$. If $M = 0$, we put $h.\dim(M) = 0$, and if M does not have finite hollow dimension, we set $h.\dim(M) = \infty$ (see [**3**, 5.2]).

Example 2.10. Since $\mathbf{Q}/\mathbf{Z} \cong \oplus_{p\,\mathrm{prime}} \mathbf{Z}_{p^\infty}$, there is a submodule L of $\mathbf{Q}$ such that $\mathbf{Q}/L \cong \mathbf{Z}_{p^\infty}$ for some prime p. Suppose that the $\mathbf{Z}$-module $\mathbf{Q}$ is a (D_{12})-module. Then there exist a direct summand K of $\mathbf{Q}$ and an epimorphism $\alpha : K \to \mathbf{Q}/L$ such that $\operatorname{Ker}\alpha$ is small in K. But $\mathbf{Q}$ is indecomposable. Thus $K = \mathbf{Q}$. Therefore, $\mathbf{Q}/\operatorname{Ker}\alpha \cong \mathbf{Q}/L$. Hence, $h.\dim(\mathbf{Q}/\operatorname{Ker}\alpha) = h.\dim(\mathbf{Q}/L) = 1$. This contradicts the fact that $h.\dim(\mathbf{Q}/\operatorname{Ker}\alpha) = h.\dim(\mathbf{Q}) = \infty$ (see [**3**, 5.4]).

In particular, $\mathbf{Q}$ does not have (D_{13}). So the converse of Proposition 2.9 need not be true in general.

Proposition 2.11. *Suppose that R is Noetherian and* $\operatorname{Rad}(E) \neq E$ *for every indecomposable injective R-module E. The following are equivalent*:

(i) *Every indecomposable injective module has* (D_{13});

(ii) *R is semisimple.*

Proof. (i) $\Rightarrow$ (ii). Since R is Noetherian, every injective module is a direct sum of indecomposable modules. By Corollary 2.3, every injective module is semisimple. It follows that R is semisimple.

(ii) $\Rightarrow$ (i). This is clear. □

Corollary 2.12. *The following are equivalent for an artinian ring R:*

(i) *Every indecomposable injective module has (D_{13});*

(ii) *R is semisimple.*

Proof. This follows from Proposition 2.11 and the fact that every artinian ring is Noetherian and every module over an artinian ring has a small radical. □

3. Direct summands of (D_{12})-modules. The following example exhibits a (D_{12})-module which contains a direct summand which is not a (D_{12})-module. This example appeared in [**8**].

Example 3.1. Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $\dim({}_QW) = 2$ and $\dim(W_Q) = 1$. Consider the indecomposable injective right R-module $U = [(R \oplus R)/D]_R$ with $D = \{(ur, -vr) \mid r \in R\}$ where $W = Ru + Rv$ and the right R-module $M = U \oplus S$, where S is the simple right R-module R/W. By [**8**, Example 4.5], U does not have (D_{12}) but, by [**8**, Example 4.6], M has (D_{12}).

A module M is said to be *completely* (D_{12}) (or a *completely* (D_{12})-*module*) if every direct summand of M has (D_{12}).

The following proposition presents some examples of completely (D_{12})-modules.

Proposition 3.2. (1) *Let $M = U \oplus V$ be such that U and V are hollow. Then M is completely (D_{12}).*

(2) *Let $M = \oplus_{i\in I} M_i$ be such that, for every $i \in I$, M_i is local with local endomorphism ring. If* $\mathrm{Rad}(M) \ll M$, *then M is completely (D_{12}).*

(3) *Let R be a commutative or a right Noetherian ring and $M = \oplus_{i\in I} M_i$ such that all M_i are local modules. If* $\mathrm{Rad}(M) \ll M$, *then M is completely (D_{12}).*

Proof. By [**4**, Corollary 2, Propositions 6 and 8]. □

A module M is called *refinable* if, for any submodules U, $V \leq M$ with $U + V = M$, there exists a direct summand U' of M with $U' \leq U$ and $U' + V = M$ (see [**3**, 11.26]).

A submodule $N \leq M$ is called a *weak supplement* (*supplement*) of a submodule L of M if $N + L = M$ and $N \cap L \ll M$ ($N + L = M$ and $N \cap L \ll N$). The module M is called *weakly supplemented* (*supplemented*) if every submodule N of M has a weak supplement (supplement).

Proposition 3.3. *Let M be a weakly supplemented refinable module. Then M has (D_{12}).*

Proof. Let N be a proper submodule of M. By hypothesis, there exists a submodule L of M such that $M = N + L$ and $N \cap L \ll M$. Since M is refinable, there is a direct summand K of M such that $M = N + K$ and $K \leq L$. Consider the projection $\pi : K \to M/N$. We have $\operatorname{Ker} \pi = N \cap K \leq N \cap L \ll M$. This implies $\operatorname{Ker} \pi \ll K$. Therefore M is a (D_{12})-module. □

Corollary 3.4. *Every weakly supplemented refinable module is completely (D_{12}).*

Proof. This is a consequence of Proposition 3.3 and the fact that every direct summand of a weakly supplemented refinable module is weakly supplemented refinable. □

Let M be an R-module. By $P(M)$ we denote the sum of all radical submodules of M. If $P(M) = 0$, M is called *reduced*.

Proposition 3.5. *Let M be a (D_{12})-module. If $P(M)$ is a direct summand of M, then $P(M)$ is a (D_{12})-module.*

Proof. Let L be a submodule of M such that $M = P(M) \oplus L$. Let X be a submodule of $P(M)$. By hypothesis, there exist a direct summand K of M and an epimorphism $\alpha : K \to M/(X \oplus L)$ such that $\operatorname{Ker} \alpha$ is small in K. It is clear that $M/(X \oplus L) \cong P(M)/X$. Thus, $\operatorname{Rad}(K/\operatorname{Ker} \alpha) = K/\operatorname{Ker} \alpha$. Since $\operatorname{Ker} \alpha \ll K$, we have $\operatorname{Rad}(K) = K$. Therefore, $K \leq P(M)$. It follows that $P(M)$ is a (D_{12})-module. □

The following result gives a characterization of direct summands having (D_{12}).

Theorem 3.6. *Let $M = M_1 \oplus M_2$. Then M_2 is a (D_{12})-module if and only if, for every submodule N of M containing M_1, there exist a* *direct summand K of M_2 and an epimorphism $\varphi : M \to M/N$ such that K is a supplement of* $\operatorname{Ker}\varphi$ *in M.*

Proof. Suppose that M_2 is a (D_{12})-module. Let $N \leq M$ with $M_1 \leq N$. Consider the submodule $N \cap M_2$ of M_2. Then there exist a direct summand K of M_2 and an epimorphism $\alpha : K \to M_2/(N \cap M_2)$ such that $\operatorname{Ker}\alpha$ is small in K. Note that $M = N + M_2$ and K is a direct summand of M. Let $M = K \oplus K'$ for some submodule K' of M. Consider the projection map $\eta : M \to K$ and the isomorphism $\beta : M_2/(N \cap M_2) \to M/N$ defined by $\beta(x + N \cap M_2) = x + N$. Thus, $\beta\alpha\eta = \varphi : M \to M/N$ is an epimorphism. Clearly, we have $\operatorname{Ker}\varphi = \operatorname{Ker}\alpha \oplus K'$. Therefore, $M = K + \operatorname{Ker}\varphi$. Moreover, $K \cap \operatorname{Ker}\varphi = \operatorname{Ker}\alpha$ is small in K.

Conversely, suppose that every submodule of M containing M_1 has the stated property. Let H be a submodule of M_2. Consider the submodule $H \oplus M_1$ of M. By hypothesis, there exist a direct summand K of M_2 and an epimorphism $\varphi : M \to M/(H \oplus M_1)$ such that $M = K + \operatorname{Ker}\varphi$ and $K \cap \operatorname{Ker}\varphi$ is small in K. Let $f : K \to M/(H \oplus M_1)$ be the restriction of φ to K. Consider the isomorphism $\eta : M/(H \oplus M_1) \to M_2/H$ defined by $m_1 + m_2 + (H \oplus M_1) \mapsto m_2 + H$. Therefore, $\alpha = \eta f : K \to M_2/H$ is an epimorphism. Clearly, $\operatorname{Ker}\alpha = \operatorname{Ker} f = K \cap \operatorname{Ker}\varphi$. Thus, $\operatorname{Ker}\alpha$ is small in K. Hence, M_2 is a (D_{12})-module. $\square$

4. Direct sums of ((D_{13})-modules) (D_{12})-modules.

We begin this section by giving an example showing that the class of (D_{12})-modules is not closed under direct sums.

Example 4.1. Let R be a local ring which is not right perfect (e.g., $R = K[[X]]$, where K is any field or R is a DVR). Then the R-module R is a (D_{12})-module. By [**8**, Theorem 4.7], there is a countably generated free R-module F such that F is not a (D_{12})-module.

The following result provides a condition which ensures a direct sum of (D_{12})-modules is a (D_{12})-module.

Theorem 4.2. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of (D_{12})-modules M_i $(i \in I)$ such that, for every submodule N of M, we have $N = \oplus_{i\in I}(N \cap M_i)$. Then M has (D_{12}).*

Proof. Let $N \leq M$. By hypothesis, we have $N = \oplus_{i\in I}(N \cap M_i)$. Let $i \in I$, and consider the submodule $N \cap M_i$ of M_i. Then there exist a direct summand K_i of M_i and an epimorphism $\alpha_i : K_i \to M_i/(N \cap M_i)$ such that $\operatorname{Ker}\alpha_i$ is small in K_i. Define the epimorphism $\oplus_{i\in I}\alpha_i : \oplus_{i\in I}K_i \to \oplus_{i\in I}[M_i/(N \cap M_i)] \cong M/[\oplus_{i\in I}(N \cap M_i)] = M/N$ defined by $k_{i_1} + \cdots + k_{i_n} \mapsto \alpha_{i_1}(k_{i_1}) + \cdots + \alpha_{i_n}(k_{i_n})$ with $k_{i_j} \in K_{i_j}$ for every $j = 1, \dots, n$. Note that $\oplus_{i\in I}K_i$ is a direct summand of M. Clearly, $\operatorname{Ker}(\oplus_{i\in I}\alpha_i) = \oplus_{i\in I}\operatorname{Ker}\alpha_i$. So a trivial verification shows that $\operatorname{Ker}(\oplus_{i\in I}\alpha_i)$ is small in $\oplus_{i\in I}K_i$. This completes the proof. □

A module M is called *duo* if every submodule of M is fully invariant in M.

Corollary 4.3. *Let $M = \oplus_{i\in I}M_i$ be a direct sum of (D_{12})-modules M_i $(i \in I)$. If M is a duo module, then M has (D_{12}).*

Proof. By Theorem 4.2. □

In the remainder of this section we assume that all rings are commutative.

Let R be a ring. Let Ω be the set of all maximal ideals of R. If $m \in \Omega$ and M is an R-module, we denote as in [**14**, page 53] by $K_m(M) = \{x \in M \mid x = 0$ or the only maximal ideal over $\operatorname{Ann}_R(x)$ is $m\}$ as the m-local component of M. We call M *m-local* if $K_m(M) = M$. This is equivalent to the following condition: m is the only maximal ideal over p for every $p \in \operatorname{Ass}(M)$. In this case M is an R_m-module by the following operation: $(r/s)x = rx'$ with $x = sx'$ $(r \in R, s \in R - m)$ (x' exists because $\operatorname{Ann}_R(x) + Rs = R$). The submodules of M over R and R_m are identical.

Let M be an R-module. For $K(M) = \{x \in M \mid Rx \text{ is supplemented}\}$ we always have a decomposition $K(M) = \oplus_{m\in\Omega}K_m(M)$ (see [**14**, Proposition 2.3]).

Lemma 4.4. *Let $m \in \Omega$, and let M be an m-local module. If N is a proper submodule of M, then M/N is also an m-local module.*

Proof. Let x be an element of M such that $x \notin N$. Suppose that there is an $\alpha \in R - m$ with $x\alpha \in N$. Since M is m-local, the only maximal ideal over $\operatorname{Ann}_R(x)$ is m. Therefore, $R\alpha + \operatorname{Ann}_R(x) = R$. Thus, there exist $r \in R$ and $\beta \in \operatorname{Ann}_R(x)$ such that $1 = r\alpha + \beta$.

This gives $x = xr\alpha = x\alpha r$. Hence, $x \in N$, which is a contradiction. It follows that the only maximal ideal over $\mathrm{Ann}_R(x+N)$ is m. This completes the proof. $\square$

Proposition 4.5. *Let M be a module. Then $K(M)$ is a (D_{12})-module if and only if $K_m(M)$ has (D_{12}) for all $m \in \Omega$.*

Proof. Since $N = \oplus_{m\in\Omega} K_m(N)$ for any submodule N of $K(M)$, the sufficiency is clear by Theorem 4.2.

Conversely, suppose that $K(M)$ is a (D_{12})-module. Let $m_1 \in \Omega$. Let L_{m_1} be a submodule of $K_{m_1}(M)$. Then there exist a direct summand K of $K(M)$ and an epimorphism $\varphi : K \to K(M)/(L_{m_1} \oplus (\oplus_{m\neq m_1} K_m(M)) \cong K_{m_1}(M)/L_{m_1}$ such that $\mathrm{Ker}\,\varphi \ll K$. It is easily seen that $\mathrm{Ker}\,\varphi = \oplus_{m\in\Omega}(\mathrm{Ker}\,\varphi \cap K_m(M))$ and $K = \oplus_{m\in\Omega}(K \cap K_m(M))$. Since $K/\mathrm{Ker}\,\varphi \cong K_{m_1}(M)/L_{m_1}$, we have $K \cap K_m(M) = \mathrm{Ker}\,\varphi \cap K_m(M)$ for all $m \neq m_1$ (see Lemma 4.4). As $\mathrm{Ker}\,\varphi \ll K$, we have $K \cap K_m(M) = 0$ for all $m \neq m_1$. Thus, $K \leq K_{m_1}(M)$. Therefore, $K_{m_1}(M)$ is a (D_{12})-module. $\square$

Corollary 4.6. *A torsion $\mathbf{Z}$-module M has (D_{12}) if and only if the primary components of M have (D_{12}).*

Proof. By Proposition 4.5. $\square$

Remark 4.7. Note that all the results from 4.2 to 4.6 are true for (D_{13})-modules. Their proofs are similar to those of (D_{12})-modules.

Proposition 4.8. *Let p be a prime integer. Let $M = \oplus_{i\in I} M_i$ be a direct sum of $\mathbf{Z}$-modules M_i $(i \in I)$ with $M_i \cong \mathbf{Z}_{p^\infty}$ for each $i \in I$. Then M has (D_{12}) if and only if M has (D_{13}).*

Proof. Assume that M has (D_{12}). Let N be a submodule of M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $\mathrm{Ker}\,\alpha$ is small in K. It is easily seen that $K = \oplus_{\lambda\in\Lambda} K_\lambda$ with $K_\lambda \cong \mathbf{Z}_{p^\infty}$ for each $\lambda \in \Lambda$ for some $\Lambda \subset I$. Consider the projections $\pi_\lambda : K \to K_\lambda$ $(\lambda \in \Lambda)$. Since $\mathrm{Ker}\,\alpha \ll K$, we have $\pi_\lambda(\mathrm{Ker}\,\alpha) \ll K_\lambda$ for each $\lambda \in \Lambda$. Therefore, $\pi_\lambda(\mathrm{Ker}\,\alpha)$ is a cyclic submodule of K_λ for each $\lambda \in \Lambda$. It follows that $\mathrm{Ker}\,\alpha = \oplus_{j\in J}\mathbf{Z}x_j$ is a direct sum of nonzero cyclic submodules $\mathbf{Z}x_j$ $(j \in J)$ of K (see [**6**, Theorem 17]). Since the injective hull $E(\mathrm{Ker}\,\alpha)$ of $\mathrm{Ker}\,\alpha$ is a direct summand of K, there is a direct summand L of K such that $K = [\oplus_{j \in J} E(\mathbf{Z}x_j)] \oplus L$. Note that $E(\mathbf{Z}x_j) \cong \mathbf{Z}_{p^\infty}$ for all $j \in J$. Then $K/\mathrm{Ker}\,\alpha \cong [\oplus_{j \in J} E(\mathbf{Z}x_j)/\mathbf{Z}x_j] \oplus L \cong [\oplus_{j \in J} E(\mathbf{Z}x_j)] \oplus L = K$. This shows that M has (D_{13}). The converse is obvious. ◻

Example 4.9. Let p be a prime integer. If $M = \oplus_{i=1}^n M_i$ with $M_i \cong \mathbf{Z}_{p^\infty}$ for $1 \le i \le n$, then the $\mathbf{Z}$-module M has (D_{13}). In fact, for every proper submodule N of M, the module M/N is a p-primary torsion injective $\mathbf{Z}$-module. Since $h.\dim(M/N) \le h.\dim(M)$ (see [**3**, page 49]), $M/N = \oplus_{k=1}^m L_k$ with $L_k \cong \mathbf{Z}_{p^\infty}$ for $1 \le k \le m$ and $m \le n$. Therefore, M/N is isomorphic to a direct summand of M.

5. Decompositions of (D_{12})-modules.

Lemma 5.1. *Let M be a (D_{12})-module.*

(1) *If* $\mathrm{Rad}\,M \neq M$*, then M has a nonzero local direct summand.*

(2) *If M has a nonzero hollow factor module, then M has a nonzero hollow direct summand.*

(3) *If M contains a hollow submodule H such that H is not small in M, then M has a nonzero hollow direct summand.*

Proof. (1) Let N be a maximal submodule of M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $\mathrm{Ker}\,\alpha$ is small in K. Clearly, $K \neq 0$ and $\mathrm{Ker}\,\alpha$ is a maximal submodule of K. Therefore, K is local.

(2) Let L be a proper submodule of M such that M/L is hollow. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to M/L$ such that $\mathrm{Ker}\,\alpha \ll K$. Hence, $K/\mathrm{Ker}\,\alpha \cong M/L$ is hollow. Therefore, K is hollow since $\mathrm{Ker}\,\alpha \ll K$.

(3) Let H be a nonsmall hollow submodule of M. Then there exists a proper submodule L of M such that $M = H + L$. But $M/L \cong H/H \cap L$. Then M/L is a nonzero hollow module. The result follows from (2). ◻

Corollary 5.2. *Let M be a finitely generated (D_{12})-module. Then M has a nonzero local direct summand.*

Corollary 5.3. (See [**7**, Lemma 2.1].) *Let M be an indecomposable (D_{12})-module with* $\operatorname{Rad} M \neq M$. *Then M is local.*

Corollary 5.4. *Let M be an indecomposable module which satisfies one of the conditions*:

(i) *M has a nonzero hollow factor module,*

(ii) *M has finite hollow dimension.*

Then M has (D_{12}) if and only if M is hollow.

Proof. (i) By Lemma 5.1.

(ii) By (i) and the fact that every module with finite hollow dimension has a nonzero hollow factor module (see [**3**, 5.2]). $\square$

Proposition 5.5. *Let R be a commutative Noetherian local ring. Let M be an R-module with* $\operatorname{Rad}(M) = M$. *If M is a supplemented completely (D_{12})-module, then M is a finite direct sum of hollow modules.*

Proof. By [**11**, Corollary 2.5], M is a sum of finitely many hollow modules. By Lemma 5.1, M has a nonzero hollow direct summand. Moreover, M has finite hollow dimension by [**3**, 20.21]. Note that every direct summand of M is supplemented (see [**13**, 41.1 (7)]). Repeated application of Lemma 5.1 shows that M is a finite direct sum of hollow modules. $\square$

Corollary 5.6. *Let R be a commutative Noetherian local ring. Let M be an indecomposable R-module with* $\operatorname{Rad}(M) = M$. *Then M is a supplemented (D_{12})-module if and only if M is hollow.*

Proof. By Proposition 5.5. $\square$

We denote by $MOD-R$ the category of the right R-modules. We call a function $r : MOD-R \to \mathbf{N} \cup \{\infty\}$ a *rank function* on $MOD-R$ if, for all $M, N \in MOD-R$, the conditions (R1) $r(M) = 0 \Leftrightarrow M = 0$ and (R2) $r(M \oplus N) = r(M) + r(N)$ hold. Note that if r is a rank function and $r(M) = 1$ for a module M, then M is indecomposable. Clearly, $\operatorname{h.dim}$ is a rank function.

Theorem 5.7. *Let M be a nonzero completely (D_{12})-module with* $\operatorname{Rad} M \ll M$. *Then the following are equivalent*:

(i) *M is a finite direct sum of local modules*;

(ii) *M has finite hollow dimension*;

(iii) *M is finitely generated and weakly supplemented*;

(iv) *There exists a rank function r such that $r(M)$ is finite.*

Proof. (i)$\Rightarrow$ (ii)$\Rightarrow$(iv). These are obvious.

(iv)$\Rightarrow$ (i). Let M be a completely (D_{12})-module with $\operatorname{Rad} M \ll M$ such that $r(M)$ is finite for some rank function r. Corollary 5.3 shows that there is no loss of generality in assuming that M is not indecomposable. So there exist nonzero submodules A and B of M such that $M = A \oplus B$. Since r is a rank function, we have $r(A) \neq 0$, $r(B) \neq 0$ and $r(M) = r(A) + r(B)$. The proof is by induction on $n = r(M)$. Since M is not indecomposable, we have $n \geq 2$. If $n = 2$, then $r(A) = r(B) = 1$. Therefore, A and B are indecomposable. It follows that A and B are local by Corollary 5.3. Suppose that $n > 2$ and assume that, for every completely (D_{12})-module N with small radical such that $r(N) < n$ for some rank function r, N is a finite direct sum of local modules. Note that $r(A) < n$ and $r(B) < n$. Hence, A and B are finite direct sums of local modules. Thus M is a finite direct sum of local modules, as required.

(i) $\Rightarrow$ (iii). This is clear from the fact that any finite direct sum of weakly supplemented modules is weakly supplemented (see [**3**, 17.13]).

(iii) $\Rightarrow$ (ii). By [**3**, 18.6]. $\square$

Corollary 5.8. *Every finitely generated weakly supplemented completely (D_{12})-module is a finite direct sum of local submodules.*

Proof. By Theorem 5.7. $\square$

Corollary 5.9. *Let M be a nonzero module having a composition series. The following are equivalent*:

(i) *M is completely (D_{12})*;

(ii) *M is a (finite) direct sum of local modules.*

Proof. (i) $\Rightarrow$ (ii). Since M has a composition series, M is artinian. This implies that M is weakly supplemented. The result follows from Corollary 5.8.

(ii) $\Rightarrow$ (i). By Proposition 3.2 and [**1**, Lemma 12.8]. $\square$

Corollary 5.10 (See [**4**, Proposition 11].) *Every finitely generated completely $\oplus$-supplemented module is a finite direct sum of local modules.*

Proof. By Corollary 5.8. $\square$

Note that, if R is a commutative Noetherian ring, then $M = K(M)$ if and only if R/p is local for all $p \in \mathrm{Ass}\,(M)$ (see [**14**, Lemma 1.5]).

Proposition 5.11. *Let R be a commutative Noetherian ring. Let M be an R-module which is a direct sum of cyclic modules and such that $Rad\,(M) \ll M$. The following are equivalent*:

(i) *M is supplemented*;

(ii) *M is completely (D_{12}) and weakly supplemented*;

(iii) *M is a direct sum of local modules*;

(iv) *R/p is local for all $p \in \mathrm{Ass}\,(M)$.*

Proof. (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). By [**5**, Proposition 2.5].

(iii) $\Rightarrow$ (ii). By Proposition 3.2 and the fact that (iii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii). Assume that $M = \oplus_{i\in I} M_i$ such that M_i is cyclic for all $i \in I$. Let $i \in I$. By hypothesis, M_i is completely (D_{12}) and weakly supplemented (see [**3**, 17.13]). Therefore, M_i is a direct sum of local modules by Corollary 5.8. This completes the proof. $\square$

Theorem 5.12. *Let R be a commutative local ring, and let M be a finitely generated R-module. Then M is completely (D_{12}) if and only if $M \cong \oplus_{i=1}^{n} R/I_i$ where each I_i is an ideal of R.*

Proof. By [**3**, 18.10], M is weakly supplemented. Suppose that M is completely (D_{12}). By Corollary 5.8, $M = \oplus_{i=1}^{k} H_i$ where each H_i is local. Thus for every $i = 1, \ldots, k$, there exists an ideal I_i of R such that $H_i \cong R/I_i$. Conversely, if $M \cong \oplus_{i=1}^{k} R/I_i$ for ideals $I_1, \ldots, I_k$ of

R, then M is a direct sum of local modules. By Proposition 3.2 (3), M is completely (D_{12}). ◻

6. Some characterizations of rings over which every module in a class of modules has (D_{12}).

Proposition 6.1. *Let $R = R_1 \oplus \cdots \oplus R_n$ be such that R_i $(1 \leq i \leq n)$ are commutative rings. Let $1 = e_1 + \cdots + e_n$ with $e_i \in R_i$ for $1 \leq i \leq n$. Let M be an R-module. Then*:

(1) *We have $M = e_1M \oplus \cdots \oplus e_nM$.*

(2) *For every $1 \leq i \leq n$, e_iM can be regarded as an R_i-module such that the submodules of e_iM are the same whether it is regarded as an R_i-module or as an R-module.*

(3) *M_R has (D_{12}) if and only if the modules $(e_iM)_{R_i}$ $(1 \leq i \leq n)$ have (D_{12}).*

Proof. (1) This is obvious.

(2) This is clear from the fact that $(r_1 + r_2 + \cdots + r_n)e_ix = r_ie_ix$ with $r_j \in R_j$ and $x \in M$.

(3) It is easily seen that, for every submodule N of M, we have $N = (e_1M \cap N) \oplus \cdots \oplus (e_nM \cap N)$. If the modules $(e_iM)_{R_i}$ $(1 \leq i \leq n)$ have (D_{12}), then the modules $(e_iM)_R$ $(1 \leq i \leq n)$ have (D_{12}). Therefore, M has (D_{12}) (see Theorem 4.2). Conversely, suppose that M has (D_{12}). Let L_1 be a submodule of e_1M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to M/(L_1 \oplus e_2M \oplus \cdots \oplus e_nM) \cong e_1M/L_1$ such that $\operatorname{Ker} \alpha$ is small in K. Since $K = e_1K \oplus \cdots \oplus e_nK$ and $\alpha(e_iK) = 0$ for every $i \geq 2$ $(e_ie_j = 0$ for all $i \neq j)$, the restriction β of α on e_1K is an epimorphism. In addition, we have $\operatorname{Ker} \beta \ll e_1K$. Therefore, $(e_1M)_R$ has (D_{12}), and hence $(e_1M)_{R_1}$ has (D_{12}). ◻

Corollary 6.2. *Let $R = R_1 \oplus \cdots \oplus R_n$ be such that R_i $(1 \leq i \leq n)$ are commutative rings. Let $\mathcal{P}$ be a property of modules such that $\mathcal{P}$ is closed under direct summands. The following statements are equivalent*:

(i) *Every R-module satisfying $\mathcal{P}$ has (D_{12});*

(ii) *For every $1 \leq i \leq n$, every R_i-module satisfying $\mathcal{P}$ has (D_{12}).*

Proof. (i) $\Rightarrow$ (ii). Let $1 \leq i \leq n$, and consider an R_i-module M_i which satisfies $\mathcal{P}$. Then M_i can be regarded as an R-module by writing $(r_1 + r_2 + \cdots + r_n)x_i = r_i x_i$ with $r_j \in R_j$ and $x_i \in M_i$. Moreover, the submodules of M_i are the same whether it is regarded as an R_i-module or as an R-module. So $(M_i)_R$ satisfies $\mathcal{P}$ since $(M_i)_{R_i}$ satisfies $\mathcal{P}$. By hypothesis, $(M_i)_R$ has (D_{12}). Therefore, $(M_i)_{R_i}$ has (D_{12}).

(ii) $\Rightarrow$ (i). By Proposition 6.1. $\square$

Lemma 6.3. *The following are equivalent for a ring R:*

(i) *R is semiperfect*;

(ii) *R_R has (D_{12}).*

Proof. By [**8**, Proposition 4.3 and Theorem 4.4] and [**9**, Corollary 4.42]. $\square$

Proposition 6.4. *The following are equivalent for a commutative ring R*:

(i) *R is semiperfect*;

(ii) *R_R has (D_{12})*;

(iii) *Every cyclic R-module has (D_{12}).*

Proof. (i) $\Leftrightarrow$ (ii). By Lemma 6.3.

(iii) $\Rightarrow$ (ii). This is clear.

(i) $\Rightarrow$ (iii). Let L be a cyclic R-module. Then there is an ideal I of R such that $L \cong (R/I)_R$. By [**1**, Corollary 27.9], the ring R/I is semiperfect. By Lemma 6.3, the module $(R/I)_{R/I}$ has (D_{12}). Since the submodules of R/I are the same whether it is regarded as an (R/I)-module or as an R-module, $(R/I)_R$ has (D_{12}). That is, L has (D_{12}). $\square$

It would be desirable to determine the class of rings over which every right module has (D_{12}). But we have not been able to do this. The following result and its corollary give the answer to this question in the case of commutative rings. Example 6.7 shows that this result is not true, in general, for noncommutative rings.

Theorem 6.5. *Let R be a commutative local ring with maximal ideal m, and let $E = E(R/m)$. Then the following statements are equivalent*:

(i) *Every R-module has* (D_{12});

(ii) R *is perfect and every submodule of the module* $R_R \oplus E$ *has* (D_{12});

(iii) (a) R *is perfect such that every submodule of* R_R *has* (D_{12}), *and*

(b) *every finitely generated submodule of the module* E *has* (D_{12});

(iv) R *is an artinian principal ideal ring.*

Proof. (i) $\Rightarrow$ (ii). This is clear by [**8**, Theorem 4.7].

(ii) $\Rightarrow$ (iii). This is evident.

(iii) $\Rightarrow$ (iv). Let N be a nonzero finitely generated submodule of $E(R/m)$. By hypothesis, N has (D_{12}). Since $\mathrm{Rad}\,(N) \neq N$, N has a nonzero local direct summand L by Lemma 5.1. But L is essential in E. Thus, $N = L$ is cyclic. By [**12**, Corollary, page 161], the ideals of R are totally ordered. Now let I be any nonzero ideal of R. Since I_R has (D_{12}) and $\mathrm{Rad}\,(I_R) \ll I_R$ (see [**1**, Remark 28.5 (3)]), $I = A \oplus B$ where A_R is a local module and B is an ideal of R by Lemma 5.1. But $A \subseteq B$ or $B \subseteq A$. Then $B = 0$, and hence I is a principal ideal of R. Therefore, R is a principal ideal ring. Thus, R is artinian since it is a perfect ring.

(iv) $\Rightarrow$ (i). By [**5**, Theorem 1.1]. $\square$

Corollary 6.6. *The following are equivalent for a commutative ring* R:

(i) *Every R-module has* (D_{12});

(ii) R *is an artinian principal ideal ring.*

Proof. (i) $\Rightarrow$ (ii). By [**8**, Theorem 4.7], R is perfect. Thus, R is a direct sum of local rings. On account of Corollary 6.2 and Theorem 6.5, R is an artinian principal ideal ring.

(ii) $\Rightarrow$ (i). By [**5**, Theorem 1.1]. $\square$

Example 6.7. Let $f : F \to F'$ be an isomorphism of fields such that F is an infinite extension of F'. The group $R = F \times F$ can be converted into a ring by the following operation: $(x,y)(x',y') = (xx', xy' + yf(x'))$, where $(x,y),(x',y') \in R$. The ring R is a left artinian ring and $\{0\} \times F$ is the unique proper left ideal of R. Let E be the injective hull of the simple left R-module $R/(\{0\} \times F)$. Suppose that E has (D_{12}). By Corollary 5.3 and [**1**, Corollary 15.21], E is a local R-module. Thus, E is an artinian R-module, which is impossible (see [**10**]).

This example also shows that, in general, the injective hull of a (D_{12})-module does not have (D_{12}).

A family of sets is said to have the *finite intersection property* if the intersection of every finite subfamily is nonempty. Let R be a commutative ring, and let M be an R-module. The module M is called *linearly compact* if, whenever $\{x_\alpha + M_\alpha\}_{\alpha\in X}$ is a family of cosets of submodules of M ($x_\alpha \in M$ and $M_\alpha \leq M$) with the finite intersection property, then $\cap_{\alpha\in X} x_\alpha + M_\alpha \neq \varnothing$. The ring R is said to be a *maximal* ring if R is linearly compact as R-module. The ring R is called *almost maximal* if R/I is a linearly compact R-module for all nonzero ideals I of R.

A commutative ring R is said to be a *valuation* ring if it satisfies one of the following three equivalent conditions:

(i) For any two elements a and b, either a divides b or b divides a;

(ii) The ideals of R are linearly ordered by inclusion;

(iii) R is a local ring and every finitely generated ideal is principal.

Proposition 6.8. *The following statements are equivalent for a commutative local ring R*:

(i) *Every finitely generated R-module has* (D_{12});

(ii) *R is an almost maximal valuation ring.*

Proof. (i) $\Rightarrow$ (ii). Since R is a local ring, every finitely generated module is weakly supplemented by [**3**, 18.10]. Note that every finitely generated R-module is completely (D_{12}) by assumption. Therefore, every finitely generated R-module is a direct sum of cyclic submodules by Corollary 5.8. From [**2**, Theorem 4.5] it follows that R is an almost maximal valuation ring.

(ii) $\Rightarrow$ (i). By [**2**, Theorem 4.5] and Theorem 5.12. $\square$

Proposition 6.9. *Let R be a commutative ring. The following statements are equivalent*:

(i) *Every finitely generated R-module has* (D_{12});

(ii) *R is a finite product of almost maximal valuation rings.*

Proof. (i) $\Rightarrow$ (ii). Since R_R has (D_{12}), the ring R is semiperfect by Lemma 6.3. So R is a direct sum of local rings. The result follows from Corollary 6.2 and Proposition 6.8.

(ii) $\Rightarrow$ (i). By Corollary 6.2 and Proposition 6.8. □

A module M is said to be *finitely presented* if $M \cong F/K$ for some finitely generated free module F and finitely generated submodule K of F.

Proposition 6.10. *The following assertions are equivalent for a commutative local ring R:*

(i) *Every finitely presented R-module has* (D_{12});

(ii) *R is a valuation ring.*

Proof. By [**7**, Theorem 2.5]. □

Corollary 6.11. *The following assertions are equivalent for a commutative local Noetherian ring R:*

(i) *Every finitely generated R-module has* (D_{12});

(ii) *R is a principal ideal ring.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 6.10, R is a valuation ring. Thus, every finitely generated ideal of R is principal. But R is Noetherian. Then, R is a principal ideal ring.

(ii) $\Rightarrow$ (i). Note that every finitely generated module is finitely presented. So the result follows from Proposition 6.10. □

Proposition 6.12. *The following assertions are equivalent for a commutative Noetherian ring R:*

(i) *Every finitely generated R-module has* (D_{12});

(ii) *R is a finite product of local principal ideal rings.*

Proof. (i) $\Rightarrow$ (ii). By Corollaries 6.2, 6.11 and Proposition 6.9.

(ii) $\Rightarrow$ (i). By Corollaries 6.2 and 6.11. □

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