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FINITE DIRECT SUMS CONTROLLED BY FINITELY MANY PERMUTATIONS

NICOLA GIRARDI

ABSTRACT. The classes of uniserial modules, biuniform modules, cyclically presented modules over a local ring, more generally, couniformly presented modules, and kernels of morphisms between indecomposable injective modules, are some among the classes of modules which are characterized by a pair of invariants. These invariants also completely describe when finite direct sums of such modules are isomorphic. In this paper, we are interested in modules characterized by finitely many invariants and in their finite direct sums. We give a general criterion to produce classes $\mathcal{S}$ of such modules, and we completely describe how modules satisfying said criterion can be grouped together to form isomorphic finite direct sums. The connection between the regularity of finite direct sums of modules in $\mathcal{S}$ and a certain associated hypergraph $H(\mathcal{S})$ is also investigated.

1. Introduction. Let us first introduce the behavior of finite direct sums of modules studied in this paper.

Definition 1.1. Let R be a ring and $\mathcal{S}$ a class of right R-modules. Let $n \geq 1$ be an integer. We say that *finite direct sums of modules in the family $\mathcal{S}$ are controlled by n permutations* if the following holds: There exist equivalence relations $\equiv_1, \dots, \equiv_n$ on $\mathcal{S}$ such that, given modules $X_1, \dots, X_r, Y_1, \dots, Y_s \in \mathcal{S}$, the direct sum $X_1 \oplus \cdots \oplus X_r$ is isomorphic to the direct sum $Y_1 \oplus \cdots \oplus Y_s$ if and only if $r = s$, and there are n permutations $\sigma_1, \dots, \sigma_n$ of $\{1, \dots, r\}$ such that $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for $1 \leq i \leq n$ and $1 \leq \mu \leq r$.

In this paper, we find sufficient conditions (Setting 3.4) for the finite direct sums of modules of a class $\mathcal{S}$ to be controlled by n permutations (Theorem 3.16). Actually, we almost always work in a slightly less demanding setting (Setting 3.2) and obtain a slightly more general,

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though less elegant, result (Theorem 3.14). It is crucial that these conditions ensure that the modules in $\mathcal{S}$ be of *finite type* (Theorem 3.5), i.e., their endomorphism rings have finitely many maximal right ideals, all of which are two-sided. We take advantage of the techniques introduced in [**12**] for the study of categories of modules of finite type. In Section 4 we give examples of classes of modules $\mathcal{S}$ for which Setting 3.2 or Setting 3.4 holds.

In Section 5 we give a necessary and sufficient condition for the finite direct sums of modules in $\mathcal{S}$ to be controlled by n permutations, in terms of a certain hypergraph associated to $\mathcal{S}$ (Proposition 5.3 and Theorem 5.4).

Minor results include a slightly improved version of the classical Krull-Schmidt theorem (Theorem 2.2).

Now let us take a glance at the bigger picture in which our results fit.

The Krull-Schmidt-Remak-Azumaya theorem [**8**, Theorem 2.12] implies that (even infinite) direct sums of modules with local endomorphism ring are controlled by one permutation, where the only equivalence relation is given by isomorphism, $X \equiv_1 Y$ if and only if $X \cong Y$.

Recall that a module U is a *uniserial module* if its lattice of submodules is linearly ordered. When R is a *serial ring*, i.e., both R_R and ${}_RR$ are direct sums of uniserial modules, finite direct sums of uniserial modules arise naturally as direct-sum decompositions of finitely presented R-modules [**16**]. Although these direct-sum decompositions are not unique, i.e., not controlled by only one permutation, it is true that finite direct sums of uniserial modules (over any ring) are controlled by two permutations [**7**]. More generally, finite direct sums of *biuniform modules* are controlled by two permutations [**8**, Theorem 9.13]. Other classes of modules whose finite direct sums are controlled by two permutations have been studied: *cyclically presented modules over a local ring* [**2**], *kernels of morphisms between indecomposable injective modules* [**9**] and *couniformly presented modules* [**10**].

Let $\mathcal{S}$ be a class of modules X such that $\mathrm{End}_R(X)/J(\mathrm{End}_R(X))$ is a product of two division rings. The question whether finite direct sums of modules in $\mathcal{S}$ are controlled by two permutations has been thoroughly investigated in [**13**], which is a major source of inspiration for this paper.

Let $Q = (Q_0, Q_1)$ be a finite quiver on $n \geq 1$ vertices. Let $\mathcal{T}$ be the class of representations M of Q by right R-modules (which are modules over the path ring $R[Q]$) such that M_i has local endomorphism ring for each $i \in Q_0$. Finite direct sums of representations in $\mathcal{T}$ are controlled by n permutations [**14**, Theorem 4.7]. For $n = 2$, via additive functors induced by the kernel, the cokernel and the image of a morphism, it is possible to view the aforementioned classes as special cases [**14**, Section 5]. In this paper, we proceed a step further in the generalization.

For any category $\mathbf{C}$, we denote by $|\mathbf{C}|$ its class of objects. For each pair of objects X, Y of $\mathbf{C}$, we denote by $\mathbf{C}(X, Y)$ the set of morphisms $X \to Y$, and we write $\mathbf{C}_X$ for $\mathbf{C}\,(X, X)$. All of our rings R are associative rings with identity $1 \neq 0$, $U(R)$ denotes the group of units of R and $J(R)$ denotes the Jacobson radical of R. All of our R-modules are unitary right R-modules unless otherwise stated.

2. Preliminaries. Recall that the *Jacobson radical of a preadditive category* $\mathbf{C}$ is the two-sided ideal $\mathbf{J}$ of $\mathbf{C}$ defined by, for all $X, Y \in |\mathbf{C}|$,

$$\begin{aligned}&\mathbf{J}(X,Y)\\ &= \{f\colon X \to Y : 1_X - gf \text{ has a left inverse for all } g\colon Y \to X\}\\ &= \{f\colon X \to Y : 1_X - gf \text{ has a two-sided inverse for all } g\colon Y \to X\}.\end{aligned}$$

Thus, $\mathbf{J}(X, X)$ is the Jacobson radical of the endomorphism ring $\mathbf{C}_X$. Compare with [**15**, page 21].

The following generalizes [**14**, Lemmas 3.1 and 3.2].

Lemma 2.1. *Let* $\mathbf{C}$ *be a preadditive category. Then*:

(i) *Let* $A, B \in |\mathbf{C}|$ *be such that* $B \neq 0$ *and* $\mathbf{C}_A$ *has only the trivial idempotents. Then a morphism* $f \in \mathbf{C}(A, B)$ *is an isomorphism if and only if it has a right inverse.*

(ii) *Let* $f = f_1 \cdots f_n$ *be a composition of morphisms in* $\mathbf{C}$ *between non-zero objects whose endomorphism rings have only the trivial idempotents. Then* f *is an isomorphism if and only if* $f_1, \dots, f_n$ *are all isomorphisms.*

(iii) *If* X, Y *are objects of* $\mathbf{C}$ *such that* $\mathbf{C}_X$ *is a local ring and* $\mathbf{C}_Y$ *has only the trivial idempotents, then* $\mathbf{J}(X, Y)$ *is the set of non-isomorphisms.*

Note that, if $\mathbf{C}$ is an additive category in which the idempotents split, the condition that the endomorphism ring of a non-zero object X of $\mathbf{C}$ has only the trivial idempotents amounts to the condition that X be an indecomposable object. In general, for a non-zero object X of $\mathbf{C}$, we only have the implication that, if $\mathbf{C}_X$ has only the trivial idempotents, then X is indecomposable.

Proof. (i) Let $g\colon B \to A$ be a right inverse for f, so that $fg = 1_B$. Then gf is an idempotent endomorphism of A. Since $gf = 0_A$ implies $1_B = fg = (fg)(fg) = 0_B$, which is false because $B \neq 0$, we must have $gf \neq 0_A$. Then $A \neq 0$ and, as $\mathbf{C}_A$ has only the trivial idempotents, $gf = 1_A$, so that g is a two-sided inverse for f.

(ii) A composition of isomorphisms is an isomorphism, so that if all $f_1, \dots, f_n$ are isomorphisms, then $f_1 \cdots f_n$ is an isomorphism. Conversely, suppose that $f_1 \cdots f_n$ is an isomorphism. To prove that $f_1, \dots, f_n$ are all isomorphisms, it suffices to prove the case $n = 2$ and use induction. From $1 = f_1 f_2 (f_1 f_2)^{-1}$, we obtain that f_1 has a right inverse, and hence is an isomorphism by (i). It follows that $f_2 = f_1^{-1}(f_1 f_2)$ is also an isomorphism.

(iii) If $f\colon X \to Y$ is an isomorphism, then $1_X - f^{-1}f = 0_X$ is not invertible in $\mathbf{C}_X$ because $X \neq 0$; thus, $f \notin \mathbf{J}$. If $f\colon X \to Y$ is not in the Jacobson radical, let $g\colon Y \to X$ be such that $1_X - gf$ is not invertible. Since $\mathbf{C}_X$ is a local ring, gf is an automorphism of X. In particular, g has a right inverse. As $X \neq 0$ and $\mathbf{C}_Y$ has only trivial idempotents, (i) applies to show that g is an isomorphism. Then $f = g^{-1}(gf)$ is also an isomorphism. $\square$

We will need the following version of the classical Krull-Schmidt theorem, which generalizes [**13**, Theorem 3.3].

Theorem 2.2. *Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be objects with local endomorphism ring of an additive category $\mathbf{A}$. Suppose that $g\colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ is an isomorphism. Then $n = m$ and there exists a permutation σ of $\{1, \dots, n\}$ such that each $g_{\sigma(i),i}\colon X_i \to Y_{\sigma(i)}$ is an isomorphism.*

Proof. Let $\mathbf{J}$ be the Jacobson radical of $\mathbf{A}$ and $Q\colon \mathbf{A} \to \mathbf{A}/\mathbf{J}$ be the canonical reduction functor. Let $C\colon \mathbf{A}/\mathbf{J} \to \widehat{\mathbf{A}/\mathbf{J}}$ be the canonical additive fully faithful functor into the idempotent completion $\widehat{\mathbf{A}/\mathbf{J}}$

of $\mathbf{A}/\mathbf{J}$ [**6**]. For all morphisms α in $\mathbf{A}$ between objects with local endomorphism ring, α is an isomorphism if and only if $Q(\alpha) \neq 0$ (Lemma 2.1 (iii)) if and only if $CQ(\alpha) \neq 0$.

Recall that the Krull-Schmidt theorem holds for additive categories in which idempotents split [**6**, Theorem 2.2]; thus, $n = m$.

View g as an $n \times n$ matrix, where $g_{ji}\colon X_i \to Y_j$. For each $j = 1, \dots, n$ let I_j be the set of indices $i = 1, \dots, n$ such that g_{ij} is an isomorphism. Let $S = \{I_1, \dots, I_n\}$. We need to pick an element $\sigma(j)$ from I_j for each $j = 1, \dots, n$ in such a way that $\sigma(j) \neq \sigma(k)$ if $j \neq k$. By Hall's theorem, this can be done if and only if $|T| \leq |\cup_{I \in T} I|$ for all $T \subseteq S$. Assume by contradiction there is a subset $\overline{T}$ of S such that $|\overline{T}| > |\cup_{I \in \overline{T}} I|$. Without loss of generality, we may assume that $\overline{T} = \{I_1, \dots, I_r\}$ and that $\{1, \dots, n\} \setminus (I_1 \cup \cdots \cup I_r) = \{s+1, \dots, n\}$ with $0 \leq s < r \leq n$. Thus, we can write g in block matrix form as

$$g = \begin{pmatrix} \alpha & * \\ \alpha' & * \end{pmatrix},$$

where $\alpha\colon X_1 \oplus \cdots \oplus X_r \to Y_1 \oplus \cdots \oplus Y_s$ and $\alpha'\colon X_1 \oplus \cdots \oplus X_r \to Y_{s+1} \oplus \cdots \oplus Y_n$ is such that $CQ(\alpha') = 0$, and similarly, we write

$$g^{-1} = \begin{pmatrix} \beta & \beta' \\ * & * \end{pmatrix},$$

where $\beta\colon Y_1 \oplus \cdots \oplus Y_s \to X_1 \oplus \cdots \oplus X_r$ and $\beta'\colon Y_{s+1} \oplus \cdots \oplus Y_n \to X_1 \oplus \cdots \oplus X_r$. Computing the top left entry of $g^{-1}g$ we have $1 = \beta\alpha + \beta'\alpha'$, from which $1 = CQ(\beta\alpha)$. Hence, $CQ(\beta\alpha)$ is an automorphism of $CQ(X_1) \oplus \cdots \oplus CQ(X_r)$ which factors through $CQ(Y_1) \oplus \cdots \oplus CQ(Y_s)$. Since idempotents split in $\widehat{\mathbf{A}/\mathbf{J}}$, we conclude that $CQ(X_1) \oplus \cdots \oplus CQ(X_r)$ is a direct summand of $CQ(Y_1) \oplus \cdots \oplus CQ(Y_s)$ [**6**, Lemma 2.1]. By [**6**, Theorem 2.2], it follows that $r \leq s$, a contradiction. $\square$

Let n be a positive integer. A ring S is *of type n* if and only if $S/J(S)$ is the product of n division rings, if and only if S has n maximal right ideals, all of which are two-sided. A module, or more generally, an object in a preadditive category, is *of type n* if its endomorphism ring is of type n [**12**]. For instance, a module with local endomorphism ring is a module of type 1. It is convenient to let the zero module be the unique module of type 0.

Proposition 2.3. *The class of modules of finite type is closed by direct summands and the type is additive, i.e., if $X \cong A_1 \oplus A_2$ is of finite type, then the type of X is the sum of the type of A_1 and that of A_2.*

Proof. Case $X = 0$ is trivially true; thus, assume the type of X is $n \geq 1$. Let $\{e_1, e_2\}$ be the complete set of orthogonal idempotents of $S = \mathrm{End}_R X$ corresponding to the given decomposition of X. Then $\mathrm{End}_R A_i \cong e_i S e_i$. Recall that $(e_i S e_i)/J(e_i S e_i) \cong \overline{e}_i \overline{S} \overline{e}_i$, where $\overline{e}_i = e_i + J(S)$ and $\overline{S} = S/J(S)$.

Let $\overline{s}_1, \dots, \overline{s}_n$ be the complete set of centrally primitive orthogonal idempotents of $\overline{S}$ associated with the direct-product decomposition of $\overline{S}$ into n division rings. Now, $\overline{e}_1 \overline{s}_i$ is an idempotent of the division ring $\overline{s}_i \overline{S}$ (whose identity is $\overline{s}_i$). Thus, either $\overline{e}_1 \overline{s}_i = 0$ or $\overline{e}_1 \overline{s}_i = \overline{s}_i$. As $\overline{e}_1 = \overline{e}_1 \overline{s}_1 + \cdots + \overline{e}_1 \overline{s}_n$, there exists a subset I_1 of $\{1, \dots, n\}$ such that $\overline{e}_1 = \sum_{i \in I_1} \overline{s}_i$. Inasmuch as $\overline{1} = \overline{e}_1 + \overline{e}_2$, we have that $\overline{e}_2 = \sum_{i \in I_2} \overline{s}_i$, where I_2 is the complement of I_1 in $\{1, \dots, n\}$. This shows that $\overline{e}_i \overline{S} \overline{e}_i$ is a direct product of $|I_i|$ division rings, and $|I_1| + |I_2| = n$. $\square$

Corollary 2.4. *Let $X \neq 0$ be a module of finite type. Then X has a decomposition $X = X_1 \oplus \cdots \oplus X_n$ with $n \geq 1$, and $X_1, \dots, X_n$ indecomposable modules of finite type.*

Proof. By induction on the type of X. $\square$

3. A sufficient condition. Recall that a ring morphism $f\colon R \to S$ is *local* if $f(r) \in U(S)$ implies $r \in U(R)$, for every $r \in R$.

Lemma 3.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be preadditive categories, and let $F\colon \mathbf{A} \to \mathbf{B}$ be an additive functor. The following conditions are equivalent*:

(i) *If $f\colon M \to N$ and $g\colon N \to M$ are morphisms in $\mathbf{A}$ such that $F(f)$ and $F(g)$ are isomorphisms, then f and g are isomorphisms.*

(ii) *For each $M \in |\mathbf{A}|$, the ring morphism $\mathbf{A}_M \to \mathbf{B}_{F(M)}$ is a local morphism.*

Proof. It is trivial that (i) implies (ii). Assume (ii) holds, and suppose the hypotheses of (i) hold. Then $F(fg)$ and $F(gf)$ are automorphisms of $F(N)$ and $F(M)$, respectively. Thus, fg and gf are automorphisms of N and M, respectively. It follows that f and g are both right and left invertible, hence isomorphisms. $\square$

Recall that an additive functor F is *local* if, when $F(f)$ is an isomorphism, then so is f [**6**]. For instance, if $\mathbf{A}$ is a preadditive category and $\mathbf{J}$ is its Jacobson radical, then $\mathbf{A} \to \mathbf{A}/\mathbf{J}$ is a local functor. More generally, we say that F is *almost local* if it satisfies the equivalent conditions of Lemma 3.1.

If $\mathbf{A}$ is a full subcategory of an additive category $\mathbf{C}$, by $\mathrm{Sums}\,(\mathbf{A})$ we mean the full subcategory of $\mathbf{C}$ whose objects are all the objects of $\mathbf{C}$ isomorphic to finite direct sums of objects of $\mathbf{A}$.

From now on, we work in the following setting.

Setting 3.2. Let R be a ring, $\mathbf{A}$ a full subcategory of Mod-R without a zero module, and n a positive integer. We assume that we have an additive functor $T\colon \mathrm{Sums}\,(\mathbf{A}) \to \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, where $\mathbf{A}_1, \dots, \mathbf{A}_n$ are additive categories, such that:

(S1) For each $1 \leq i \leq n$ and each $X \in |\mathbf{A}|$, $T_i(X)$ is an object of type ≤ 1 of $\mathbf{A}_i$, where $T_i = P_i T$ and $P_i\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{A}_i$ is the canonical projection functor;

(S2) The restriction of T to $\mathbf{A}$ yields an almost local functor.

We note that we may consider the inverse image of an ideal along an additive functor, just as we consider the inverse image of an ideal along a ring morphism. Namely, suppose $G\colon \mathbf{A}_1 \to \mathbf{A}_2$ is an additive functor and $\mathbf{I}$ is an ideal of $\mathbf{A}_2$. Then we define the *inverse image ideal* $G^{-1}\mathbf{I}$ of $\mathbf{I}$ as

$$(G^{-1}\mathbf{I})(M,N) = \{f \in \mathbf{A}_1(M,N) : G(f) \in \mathbf{I}(G(M), G(N))\},$$

for all pairs of objects M, N of $\mathbf{A}_1$. In short, $f \in G^{-1}(\mathbf{I})$ if and only if $G(f) \in \mathbf{I}$, for any morphism f in the category $\mathbf{A}_1$.

For each $1 \leq i \leq n$, let $\mathbf{P}_i$ be the inverse image of the Jacobson radical $\mathbf{J}_i$ of $\mathbf{A}_i$ along the additive functor T_i.

Following [**13**], a *completely prime ideal* $\mathbf{I}$ of a preadditive category $\mathbf{C}$ is a collection of subgroups $\mathbf{I}(X,Y)$ of $\mathbf{C}(X,Y)$ for all pairs of objects X,Y of $\mathbf{C}$, subject to the conditions

(C1) $\mathbf{I}(X,X)$ is properly contained in $\mathbf{C}_X$ for each non-zero object X of $\mathbf{C}$ and

(C2) for any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, $gf \in \mathbf{I}(X,Z)$ if and only if either $f \in \mathbf{I}(X,Y)$ or $g \in \mathbf{I}(Y,Z)$.

This definition extends that of a completely prime ideal I of a ring R, which is a proper ideal I satisfying $ab \in I$ if and only if $a \in I$ or $b \in I$ for all $a,b \in R$.

The restriction of $\mathbf{P}_i$ to $\mathbf{A}$ fails to be a completely prime ideal because it may happen for some object X of $\mathbf{A}$ that $T_i(X) = 0$, and in that case $\mathbf{P}_i(X,X)$ is not a proper ideal of $\mathbf{A}_X$. Nevertheless, we have the following:

Lemma 3.3. *Condition* (C2) *holds for the restriction of* $\mathbf{P}_i$ *to* $\mathbf{A}$.

Proof. Let $f \in \mathbf{A}(X,Y)$ and $g \in \mathbf{A}(Y,Z)$. If $T_i(X)$, $T_i(Y)$, $T_i(Z)$ are all non-zero, by Lemma 2.1, we have $gf \in \mathbf{P}_i$ if and only if $T_i(gf) \in \mathbf{J}_i$, if and only if $T_i(gf)$ is not an isomorphism, if and only if either $T_i(g)$ or $T_i(f)$ is not an isomorphism, if and only if either $T_i(g) \in \mathbf{J}_i$ or $T_i(f) \in \mathbf{J}_i$, if and only if either $g \in \mathbf{P}_i$ or $f \in \mathbf{P}_i$. The case in which one of $T_i(X)$, $T_i(Y)$, $T_i(Z)$ is zero is trivial. $\square$

If we substitute requirement (S1) with:

(S1′) For each $1 \le i \le n$ and each $X \in |\mathbf{A}|$, $T_i(X)$ is an object of type 1 of $\mathbf{A}_i$,

we obtain that the restriction of $\mathbf{P}_i$ to $\mathbf{A}$ is a completely prime ideal, for in that case $1_A \notin \mathbf{P}_i$ for all objects A of $\mathbf{A}$; hence, also condition (C1) is satisfied.

Setting 3.4. This is the same as Setting 3.2, except for the fact that we replace condition (S1) with condition (S1′).

As a side note, another way to see that the restriction of $\mathbf{P}_i$ to $\mathbf{A}$ is completely prime under the conditions (S1′) and (S2) is the following.

By Lemma 2.1, if a preadditive category $\mathbf{B}$ consists of objects of type 1, then its Jacobson radical is a completely prime ideal. Then observe that the inverse image of a completely prime ideal along an additive functor is a completely prime ideal.

Theorem 3.5. *Let $M \in |\mathbf{A}|$, and let $E = \mathrm{End}_R(M)$. Then*:

(i) *For each $1 \leq i \leq n$, either $\mathbf{P}_i(M,M) = E$ or $\mathbf{P}_i(M,M)$ is a completely prime two-sided ideal of E.*

(ii) *There exist indices $1 \leq i_1, \dots, i_t \leq n$ such that $\mathbf{P}_{i_1}(M,M), \dots, \mathbf{P}_{i_t}(M,M)$ are the maximal right ideals of E. Since they are all two-sided ideals, E is a ring of type $t \leq n$.*

(iii) *The canonical injective ring morphism $p\colon E/J(E) \to \prod_{\ell=1}^t E/\mathbf{P}_{i_\ell}(M,M)$ is an isomorphism.*

Proof. (i) When $\mathbf{P}_i(M,M)$ is proper, it is completely prime by (C2).

(ii) By (S2) we have a local morphism $E \to \prod_{i=1}^n \mathrm{End}_{\mathbf{A}_i}(T_i(M))$ induced by T, from which we obtain the local morphism

$$E \longrightarrow \prod_{i|T_i(M)\neq 0} \mathrm{End}_{\mathbf{A}_i}(T_i(M))/J(\mathrm{End}_{\mathbf{A}_i}(T_i(M))),$$

whose codomain is a direct product of division rings. Note that the product is non-empty, because $M \neq 0$. Then the set of non-units of E is the union $\cup_{i:T_i(M)\neq 0}\mathbf{P}_i(M,M)$. Insofar as every ideal of this union is completely prime by (i), any proper right or left ideal of E is contained in some $\mathbf{P}_i(M,M)$, cf. [**4**, Proposition 1.11 (i)]. This is in particular true for the maximal right ideals; thus, (ii) follows.

(iii) By the Chinese remainder theorem, p is an isomorphism. $\square$

We now recall some results from [**6**] which are crucial for our proof of Theorem 3.14.

In any preadditive category $\mathbf{D}$, for any ideal I of $\mathbf{D}_X$, for any object X of $\mathbf{D}$, we construct *the ideal* $\mathbf{I}$ *of* $\mathbf{D}$ *associated to* I as follows. For any morphism $f \in \mathbf{D}(M,N)$, we let $f \in \mathbf{I}(M,N)$ if and only if $\beta f \alpha \in I$ for all $\alpha \in \mathbf{D}(X,M)$ and all $\beta \in \mathbf{D}(N,X)$. This is the largest among the ideals $\mathbf{I}'$ such that $\mathbf{I}'(X,X) \subseteq I$, and in fact, $\mathbf{I}(X,X) = I$ [**12**].

Remark 3.6. Note that $\mathbf{I}(M, N)$ depends only on the objects M, N, X and on the morphisms between them. Therefore, if we consider any *full* subcategory $\mathbf{E}$ of $\mathbf{D}$, the ideal of $\mathbf{E}$ associated to I is a restriction of that of $\mathbf{D}$ associated to I. Thus, we may say that *M is isomorphic to N modulo* $\mathbf{I}$ if M and N are isomorphic as objects of $\mathbf{E}/\mathbf{I}$, where $\mathbf{E}$ is any full subcategory of $\mathbf{D}$ containing the objects M, N, X.

Notation 3.7. For each module M of finite type that is an object of a full subcategory $\mathbf{D}$ of right R-modules, let $V(\mathbf{D}, M)$ be the collection of ideals of $\mathbf{D}$ associated to maximal ideals of $\mathrm{End}_R(M)$. We will write $V(M)$ for $V(\mathbf{D}, M)$ if the category is understood. Moreover, we let $V(\mathbf{D}) = \cup V(\mathbf{D}, M)$, where the union is taken over all modules M of finite type which are objects of $\mathbf{D}$.

Remark 3.8. The construction of $V(\mathbf{D})$ needs some special attention. An ideal $\mathbf{I}$ of $\mathbf{D}$ is a proper class if $\mathbf{D}$ is large. Indeed, this follows from the observation that we have an injective function $|\mathbf{D}| \to \mathbf{I}$ sending any object X to the pair $((X, X), \mathbf{I}(X, X))$. When $\mathbf{I}$ is a proper class, it cannot be a member of a collection $V(\mathbf{D})$. The problem can be worked around, as in [**11**]. Let $\mathcal{S}$ be the class of all pairs (M, P) where M is a module of finite type which is an object of $\mathbf{D}$, and P is a maximal ideal of $\mathbf{D}_M$. For each (M, P) and (N, Q) in $\mathcal{S}$, let $(M, P) \sim (N, Q)$ when the ideal of $\mathbf{D}$ associated to P coincides with that associated to Q. Thus, we may identify $V(\mathbf{D})$ with a class of representatives of $\mathcal{S}$ modulo $\sim$. Consequently, we may identify $V(\mathbf{D}, M)$ with the subclass of $V(\mathbf{D})$ of those elements (N, Q) such that $(N, Q) \sim (M, P)$ for some maximal ideal P of $\mathbf{D}_M$. Thus $V(\mathbf{D}, M)$ is a finite set.

Theorem 3.9. *Let* $\mathbf{B}$ *be a full subcategory of Mod-R whose objects are modules of finite type. Let M and N be objects of Sums*$(\mathbf{B})$. *The following are equivalent*:

(i) *M and N are isomorphic.*

(ii) *M and N are isomorphic in* $\mathrm{Sums}(\mathbf{B})/\mathbf{P}$ *for each* $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{B}))$.

(iii) *M and N are isomorphic in* $\mathrm{Sums}(\mathbf{B})/\mathbf{P}$ *for each* $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{B}), X)$ *for any* $X \in |\mathbf{B}|$.

Proof. The equivalence of (i) and (ii) is a special case of [**12**, Theorem 4.10], obtained by plugging in $\mathbf{C} = \mathrm{Sums}\,(\mathbf{B})$. The cited result actually deals with the larger category $\mathrm{add}\,(\mathbf{C})$, but we can restrict our attention to $\mathrm{Sums}\,(\mathbf{B})$ and its factors in view of Remark 3.6.

It is trivial that (ii) implies (iii). To show that (iii) implies (ii), let $X \in \mathrm{Sums}\,(\mathbf{B})$ be a module of finite type, and write $X = B_1 \oplus \cdots \oplus B_t$ for some $t \geq 1$ and $B_1, \ldots, B_t \in |\mathbf{B}|$. It suffices to prove (by induction on t) that

$$V(\mathrm{Sums}\,(\mathbf{B}), X) = V(\mathrm{Sums}\,(\mathbf{B}), B_1) \,\dot{\cup} \cdots \dot{\cup}\, V(\mathrm{Sums}\,(\mathbf{B}), B_t). \tag{3.10}$$

The case $t = 1$ is trivial; thus, let $t \geq 2$. By Proposition 2.3, $B_1 \oplus \cdots \oplus B_{t-1}$ is of finite type. Therefore, by [**12**, Corollary 3.5], we have

$$V(\mathrm{Sums}\,(\mathbf{B}), X) = V(\mathrm{Sums}\,(\mathbf{B}), B_1 \oplus \cdots \oplus B_{t-1}) \,\dot{\cup}\, V(\mathrm{Sums}\,(\mathbf{B}), B_t),$$

and (3.10) follows by the inductive hypothesis. $\square$

Lemma 3.11. *Let* $\mathbf{B}$ *be a full subcategory of* Mod-R *whose objects are modules of finite type. Fix* $X \in |\mathbf{B}|$ *and* $\mathbf{P} \in V(\mathbf{B}, X)$*, and let* $F \colon \mathbf{B} \to \mathbf{B}/\mathbf{P}$ *be the canonical functor. Then* $\mathbf{B}/\mathbf{P}$ *has only one non-zero object up to isomorphism, and for any object* N *of* $\mathbf{B}$*, the following are equivalent*:

(i) $\mathbf{P} \in V(\mathbf{B}, N)$;

(ii) $\mathbf{P}(N, N)$ *is maximal*;

(iii) $\mathbf{P}(N, N)$ *is proper.*

(iv) $F(X) \cong F(N)$.

Proof. That $\mathbf{B}/\mathbf{P}$ has only one non-zero object up to isomorphism follows from [**12**, Lemma 4.5]. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are thus all trivial. If (iv) holds, then $\mathrm{End}_R(X)/\mathbf{P}(X, X) \cong \mathrm{End}_R(N)/\mathbf{P}(N, N)$, so that $\mathbf{P}(N, N)$ is maximal. By [**12**, Remarks 4.6 (2)], $\mathbf{P}$ is associated to $\mathbf{P}(N, N)$; hence, (i) holds. $\square$

For each $1 \leq i \leq n$ and each $M \in |\mathbf{A}|$, let $\mathbf{Q}_{i,M}$ be the ideal of $\mathrm{Sums}\,(\mathbf{A})$ associated to $\mathbf{P}_i(M, M)$.

Next we define the equivalence relations that will control finite direct sums of modules in $|\mathbf{A}|$. For each $1 \le i \le n$, we define a preorder $\preceq_i$ on $|\mathbf{A}|$. For each pair of objects $M, N \in |\mathbf{A}|$, we let $M \preceq_i N$ if there exists an $f\colon M \to N$ such that $T_i(f)\colon T_i(M) \to T_i(N)$ is an isomorphism. In view of Lemma 2.1, this amounts to $f \notin \mathbf{P}_i$ when $T_i(M)$ and $T_i(N)$ are non-zero. We let $\equiv_i$ be the equivalence relation on $|\mathbf{A}|$ defined by $M \equiv_i N$ if and only if $M \preceq_i N$ and $N \preceq_i M$.

Let us seek the connection between these equivalence relations and the ideals $\mathbf{Q}_{i,M}$.

Lemma 3.12. *Let $1 \le i \le n$ and $M, N \in |\mathbf{A}|$. Then $M \equiv_i N$ if and only if $\mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$. When this is the case, $\mathbf{P}_i(M, M)$ is maximal if and only if $\mathbf{P}_i(N, N)$ is maximal.*

Proof. Suppose $M \equiv_i N$. Then let $f\colon M \to N$ and $g\colon N \to M$ be morphisms in $\mathbf{A}$ such that $T_i(f)$ and $T_i(g)$ are isomorphisms. If $T_i(M) = 0$, then also $T_i(N) = 0$; hence, $\mathbf{P}_i(M, M)$ and $\mathbf{P}_i(N, N)$ are both improper, and $\mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$ is the improper ideal of Sums $(\mathbf{A})$. Thus, we can assume that $T_i(M)$ and $T_i(N)$ are non-zero. As a consequence, by Lemma 2.1, f and g are not in $\mathbf{P}_i$. Suppose $b\colon B_1 \to B_2$ is a morphism in Sums $(\mathbf{A})$ such that $b \in \mathbf{Q}_{i,M}(B_1, B_2)$. To prove that $b \in \mathbf{Q}_{i,N}$, we need to show that, for each $\alpha\colon N \to B_1$ and each $\beta\colon B_2 \to N$, we have $\beta b \alpha \in \mathbf{P}_i(N, N)$. We have $g(\beta b \alpha) f \in \mathbf{P}_i(M, M)$ because $b \in \mathbf{Q}_{i,M}$. In view of (C2) and of the fact that $f, g \notin \mathbf{P}_i$, it follows that $\beta b \alpha \in \mathbf{P}_i(N, N)$, as required. This proves that $\mathbf{Q}_{i,M} \subseteq \mathbf{Q}_{i,N}$ and the reverse inclusion follows by symmetry.

Now assume that $\mathbf{Q} = \mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$. If this is the improper ideal of Sums $(\mathbf{A})$, then $\mathbf{P}_i(M, M)$ and $\mathbf{P}_i(N, N)$ are improper. This implies that $T_i(M) = T_i(N) = 0$, so that $T_i(0\colon M \to N)$ and $T_i(0\colon N \to M)$ are isomorphisms, and $M \equiv_i N$. We can now suppose that $\mathbf{Q}$ is proper. This implies that $1_N \notin \mathbf{P}_i(N, N) = \mathbf{Q}_{i,M}(N, N)$; therefore, there exist morphisms $f\colon M \to N$ and $g\colon N \to M$ in $\mathbf{A}$ such that $gf \notin \mathbf{P}_i(M, M)$. Thus, both g and f are not in $\mathbf{P}_i$ and, by Lemma 2.1, both $T_i(f)$ and $T_i(g)$ are isomorphisms, so that $M \equiv_i N$.

Suppose that $M \equiv_i N$ and that $\mathbf{P}_i(M, M)$ is maximal. We have that $\mathbf{P}_i(N, N) = \mathbf{Q}_{i,M}(N, N)$, and we know that $\mathbf{Q}_{i,M}(N, N)$ is improper or maximal by [**12**, Lemma 4.4]. Since $\mathbf{P}_i(N, N)$ is proper, it is maximal. $\square$

Lemma 3.13. *Let X be an object of $\mathbf{A}$ such that $\mathbf{P}_i(X,X)$ is maximal, and let F be the reduction functor modulo $\mathbf{Q}_{i,X}$. Let N be any object of $\mathbf{A}$. If $X \equiv_i N$, then $F(X) \cong F(N)$, while if $X \not\equiv_i N$, then $F(N) = 0$.*

Proof. If $X \equiv_i N$, we have that $\mathbf{P}_i(N,N) = \mathbf{Q}_{i,X}(N,N)$ is maximal; hence, $F(X) \cong F(N)$ by Lemma 3.11. Suppose now that $X \not\equiv_i N$. If $T_i(N) = 0$, then $\mathbf{P}_i(N,N)$ is improper and, since it is contained in $\mathbf{Q}_{i,X}(N,N)$, $F(N) = 0$. Thus, assume $T_i(N) \neq 0$. Note that also $T_i(X) \neq 0$, so that we may apply Lemma 2.1 as follows: For any pair of morphisms $f\colon X \to N$ and $g\colon N \to X$, either $T_i(f)$ or $T_i(g)$ is not an isomorphism, so that $T_i(gf)$ is not an isomorphism, i.e., $T_i(gf) \in \mathbf{J}_i$; hence, $gf \in \mathbf{P}_i(X,X)$. This shows that $1_N \in \mathbf{Q}_{i,X}(N,N)$; thus, $F(N) = 0$. $\square$

Theorem 3.14. *Let $X = X_1 \oplus \cdots \oplus X_r$ and $Y = Y_1 \oplus \cdots \oplus Y_s$, where $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ are objects of $\mathbf{A}$. For each $1 \leq i \leq n$, define $\mathcal{X}_i = \{\mu : T_i(X_\mu) \neq 0\}$ and $\mathcal{Y}_i = \{\mu : T_i(Y_\mu) \neq 0\}$. Then $X \cong Y$ if and only if there exist bijections $\{\sigma_i\colon \mathcal{X}_i \to \mathcal{Y}_i\}_{i=1,\ldots,n}$ such that $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $1 \leq i \leq n$ and each $\mu \in \mathcal{X}_i$.*

Proof. Assume that the bijections exist. To show $X \cong Y$, by Theorem 3.9, we must show that X and Y are isomorphic in $\mathrm{Sums}\,(\mathbf{A})/\mathbf{Q}$ for each $\mathbf{Q} \in V(\mathrm{Sum}\,(\mathbf{A}), M)$ for every $M \in |\mathbf{A}|$. By Theorem 3.9, we then have $\mathbf{Q} = \mathbf{Q}_{i,M}$ for some $1 \leq i \leq n$ such that $\mathbf{P}_i(M,M)$ is maximal. The mapping σ_i induces a bijection

$$\{\mu \in \mathcal{X}_i : X_\mu \equiv_i M\} \longrightarrow \{\mu \in \mathcal{Y}_i : Y_\mu \equiv_i M\}.$$

Let $k \geq 0$ be the common cardinality of the two sets. Note that, if $1 \leq \mu \leq r$ is not in $\mathcal{X}_i$, i.e., $T_i(X_\mu) = 0$, then $\mathbf{P}_i(X_\mu, X_\mu)$ is the improper ideal. Since $\mathbf{P}_i(X_\mu, X_\mu) \subseteq \mathbf{Q}_{i,M}(X_\mu, X_\mu)$, it follows that $F(X_\mu) = 0$. Therefore, $F(X) \cong \oplus_{\mu \in \mathcal{X}_i} F(X_\mu) \cong F(M)^k$, where the last isomorphism holds by Lemma 3.13. Since the same holds for Y, it follows that $F(X) \cong F(Y)$.

For the converse implication, assume that $f\colon X \to Y$ and $g\colon Y \to X$ are mutually inverse isomorphisms. Then $T_i(f)\colon T_i(X) \to T_i(Y)$ and $T_i(g)\colon T_i(Y) \to T_i(X)$ are mutually inverse isomorphisms in $\mathbf{A}_i$. By

Theorem 2.2, we obtain a bijection $\sigma_i: \mathcal{X}_i \to \mathcal{Y}_i$ such that $T_i(f_{\sigma_i(\mu),\mu}) = (T_i(f))_{\sigma_i(\mu),\mu}: T_i(X_\mu) \to T_i(Y_{\sigma_i(\mu)})$ is an isomorphism for all $\mu \in \mathcal{X}_i$. Therefore $X_\mu \preceq_i Y_{\sigma_i(\mu)}$ for all $\mu \in \mathcal{X}_i$.

Reasoning in the same way with g, we obtain a bijection $\tau_i: \mathcal{Y}_i \to \mathcal{X}_i$ such that $Y_\mu \preceq_i X_{\tau_i(\mu)}$ for all $\mu \in \mathcal{Y}_i$.

Therefore, $X_\mu \preceq_i Y_{\sigma_i(\mu)} \preceq_i X_{\tau_i\sigma_i(\mu)}$. Continuing inductively, we have $X_\mu \preceq_i Y_{\sigma_i(\mu)} \preceq_i X_{(\tau_i\sigma_i)^k(\mu)}$ for all integers $k \geq 1$. Since there exist some $k \geq 1$ such that $(\tau_i\sigma_i)^k = 1$ (the symmetric group of the finite set $\mathcal{X}_i$ is finite, hence its elements have finite order), we have $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $\mu \in \mathcal{X}_i$, as required. $\square$

Corollary 3.15. *Let $X, Y \in |\mathbf{A}|$. Then $X \cong Y$ if and only if $X \equiv_i Y$ for all $1 \leq i \leq n$.*

It is easy to see that the stronger condition in Setting 3.4 allows us to conclude that finite direct sums of modules in $|\mathbf{A}|$ are controlled by n permutations, namely:

Theorem 3.16. *Suppose the conditions of Setting 3.4 hold. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_s$ be objects of $\mathbf{A}$. Then $X_1 \oplus \cdots \oplus X_r \cong Y_1 \oplus \cdots \oplus Y_s$ if, and only if, $r = s$ and there exist permutations $\sigma_1, \dots, \sigma_n \in S_r$ such that $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $1 \leq i \leq n$ and each $1 \leq \mu \leq r$.*

4. Examples.

4.1. DCP modules over rings of finite type. Let R be a ring. A DCP module is a direct summand of a cyclically presented module, i.e., a direct summand of a module isomorphic to R/xR for some $x \in R$. The DCP modules over rings R of finite type have been studied in [**1**]. Via a suitable duality, the kernels of morphisms between heterogeneous injective modules of finite Goldie dimension, i.e., between finite direct sums of pairwise non-isomorphic indecomposable injective modules, were also studied in that paper.

The setting of [**1**] is a particular instance of our Setting 3.2. Namely, let R be a ring of finite type, with maximal ideals $M_1, \dots, M_n$. We denote R/M_i by K_i when we view it as a division ring, and by S_i when we view it as a simple right (or simple left) R-module. Inasmuch as S_i is an R-K_i-bimodule, we have the additive functors $T_{2i-1} := \mathrm{Tor}_1^R(-, S_i)$ and $T_{2i} := - \otimes S_i$, both Mod-$R \to$ Mod-K_i. Let $T = T_1 \times \cdots \times T_{2n}$ and $\mathbf{A}$ be the full subcategory of Mod-R whose objects are the non-zero DCP right R-modules. At the end of [**1**, Section 2], it is proved that $T_i(A_R)$ is of type ≤ 1 for any DCP module A_R; hence, (S1) is satisfied. Moreover, (S2) is satisfied by the proof of [**1**, Theorem 3.2]. Thus, [**1**, Theorem 5.3] follows from Theorem 3.14.

4.2. Artinian modules with heterogeneous socle. An Artinian module whose socle is heterogeneous, i.e., is a finite direct sum of pairwise non-isomorphic simple modules, is known to be a module of finite type [**12**]. We show that finite direct sums of certain Artinian modules are controlled by finitely many permutations.

Fix a ring R and a finite set $\{S_i : i \in I\}$ of pairwise non-isomorphic simple right R-modules. Let $i \in I$. For each right R-module M, let $T_i(M)$ be the *trace of* S_i *in* M, i.e., the largest submodule of M generated by S_i [**3**, page 109]. For each morphism $f\colon M \to N$ in Mod-R, let $T_i(f)\colon T_i(M) \to T_i(N)$ be the restriction of f. Let $\mathbf{A}$ be the full subcategory of Mod-R whose objects are the non-zero Artinian right R-modules whose socle is isomorphic to a submodule of $\oplus_{i\in I} S_i$. Consider the product functor

$$T\colon \mathrm{Sums}\,(\mathbf{A}) \longrightarrow \prod_{i\in I} \text{Mod-}R.$$

Let $i \in I$ and $A \in |\mathbf{A}|$, and assume $T_i(A) \neq 0$. Inasmuch as $T_i(A)$ is generated by S_i, it is isomorphic to $S_i^{(r)}$ for some $r \geq 1$. From $T_i(A) \leq \mathrm{Soc}\, A \leq \oplus_{i\in I} S_i$, we obtain that $T_i(A)$ is heterogeneous. Therefore, $T_i(A) \cong S_i$ is a simple module, in particular, it is of type 1. Thus, (S1) holds.

Suppose now that f is an endomorphism of $A \in |\mathbf{A}|$ and that $T_i(f)$ is an automorphism for each $i \in I$. Let $K = \ker f \cap \mathrm{Soc}\, A$. Then K is isomorphic to a submodule of $\oplus_{i\in I} S_i$, say $K \cong \oplus_{i\in F} S_i$ for some $F \subseteq I$. Let $i \in F$. Then $\mathrm{Tr}_K(S_i) \neq 0$, and this is a submodule of $\mathrm{Tr}_A(S_i) = T_i(A)$. As this is either zero or simple, we have $T_i(A) \neq 0$ and $\mathrm{Tr}_K(S_i) = T_i(A)$. This implies that $f(T_i(A)) = 0$, which is impossible, because $T_i(f)$ is an isomorphism and $T_i(A) \neq 0$. This shows that F is empty; thus, $K = 0$. Since A is an Artinian module, $\operatorname{Soc} A \leq_e A$; thus, $\ker f = 0$. An injective endomorphism of an Artinian module is an automorphism; therefore, f is an automorphism. This proves that our functor T satisfies also (S2); hence, Theorem 3.14 holds for $\mathbf{A}$.

4.3. Noetherian modules with heterogeneous top. This class of modules is dual to the previous one. Let $\{S_i : i \in I\}$ be a finite set of pairwise non-isomorphic simple right R-modules. Let $\mathbf{N}$ be the full subcategory of Mod-R whose objects are the non-zero Noetherian modules N such that $N/\operatorname{Rad}(N)$ is isomorphic to a submodule of $\oplus_{i\in I} S_i$.

For each $i \in I$, define $T_i\colon \text{Mod-}R \to \text{Mod-}R$ as follows. Recall that, if $\mathcal{U}$ is a family of right R-modules, for each right R-module X, the *reject of $\mathcal{U}$ in X* is the smallest submodule X' of X such that X/X' is cogenerated by $\mathcal{U}$, and it is denoted by $\operatorname{Rej}_X(\mathcal{U})$ [**3**, page 109]. Any morphism $f\colon X \to Y$ preserves the reject, i.e., $f(\operatorname{Rej}_X(\mathcal{U})) \subseteq \operatorname{Rej}_Y(\mathcal{U})$. Now let $T_i(X) = X/\operatorname{Rej}_X(S_i)$, and for each morphism $f\colon X \to Y$, let $T_i(f)$ be the induced morphism $X/\operatorname{Rej}_X(S_i) \to Y/\operatorname{Rej}_Y(S_i)$. Then we consider the product functor

$$T\colon \operatorname{Sums}(\mathbf{N}) \longrightarrow \prod_{i\in I} \text{Mod-}R.$$

Fix $i \in I$ and $N \in |\mathbf{N}|$. Let us show that either $\operatorname{Rej}_N(S_i) = N$ or $\operatorname{Rej}_N(S_i)$ is a maximal submodule of N, so that (S1) is satisfied. Suppose $\operatorname{Rej}_N(S_i)$ is a proper submodule of N. On the one hand, $N/\operatorname{Rej}_N(S_i)$ is a quotient of the heterogeneous semisimple module $N/\operatorname{Rad}(N)$, which is isomorphic to a submodule of $\oplus_{\ell\in I} S_\ell$; hence, $N/\operatorname{Rej}_N(S_i)$ is isomorphic to $\oplus_{\ell\in F} S_\ell$, for some non-empty $F \subseteq I$. On the other hand, $N/\operatorname{Rej}_N(S_i)$ is cogenerated by S_i; thus, F can only be the singleton of i. Therefore, $N/\operatorname{Rej}_N(S_i) \cong S_i$ and $\operatorname{Rej}_N(S_i)$ is a maximal submodule of N.

Note also that, if M is a maximal submodule of N, it is necessarily equal to $\operatorname{Rej}_N(S_i)$ for some $i \in I$. Indeed, N/M is a simple quotient of $\oplus_{i\in I} S_i$; hence, $N/M \cong S_i$ for some $i \in I$. In particular, N/M is cogenerated by S_i; hence, $\operatorname{Rej}_N(S_i) \subseteq M$. We already know that $\operatorname{Rej}_N(S_i)$ is maximal when it is proper; therefore, equality holds.

Suppose now that f is an endomorphism of N and that $T_i(f)$ is an isomorphism for each $i \in I$. In particular, $f(N) + \operatorname{Rej}_N(S_i) = N$ for each $i \in I$. Thus, $f(N) = N$ for, if $f(N)$ were proper, then it would be contained in some maximal submodule $\mathrm{Rej}_N(S_i)$ of N. In view of the fact that a surjective endomorphism of a Noetherian module is an automorphism, we conclude that f is an automorphism. We have therefore proved that the product functor T also satisfies condition (S2). Therefore, Theorem 3.14 applies to the category $\mathbf{N}$.

4.4. Representations of type 1 pointwise. Let R be a ring, and let $Q = (Q_0, Q_1)$ be a finite quiver. Then let $\mathrm{Rep}_R(Q)$ be the category of representations of Q by right R-modules and R-homomorphisms. Let $\mathbf{P}$ be the full subcategory of $\mathrm{Rep}_R(Q)$ whose objects are the quiver representations $X \neq 0$ such that X_i is a module of type ≤ 1 for each $i \in Q_0$. For each $i \in Q_0$, we consider the functor $T_i\colon \mathrm{Rep}_R(Q) \to \text{Mod-}R$ defined by $T_i(X) = X_i$ and $T_i(f) = f_i$, where X is an object and f is a morphism of $\mathrm{Rep}_R(Q)$. Let $T\colon \mathrm{Rep}_R(Q) \to \prod_{i\in Q_0} \text{Mod-}R$ be the product of these functors. It is easy to see that the conditions of Setting 3.2 are satisfied, except for the fact that $\mathbf{P}$ is not a category of modules. This can be worked around by the canonical category equivalence of $\mathrm{Rep}_R(Q)$ with Mod-$R[Q]$, where $R[Q]$ is the path ring of Q. Hence, Theorem 3.14 applies to $\mathbf{P}$, cf. [**14**].

5. The associated hypergraph. The aim of this section is to establish when finite direct sums of modules that belong to a fixed class of modules $\mathcal{A}$ are controlled by finitely many permutations, in terms of a hypergraph associated to $\mathcal{A}$. The results of this section are thus a generalization of some results of [**13**].

By a *hypergraph* we mean a class of vertices V together with a class E of non-empty finite subsets of V, which are the edges of the hypergraph such that the union of the class E is V. The original definition of hypergraph as given in [**5**] is way too restrictive for our purposes, in that it allows only finite sets of vertices and edges, although it allows edges to be repeated.

We denote by $H = (V, E)$ a hypergraph whose class of vertices is V and whose class of edges is E. We say that H is *n-uniform* if all its edges have n elements, and it is called *simple* if there are no inclusion relations between its edges. Also recall that a *partial hypergraph* is obtained from H by selecting a subclass F of the class of edges E and is denoted $H[F]$. The class of vertices of $H[F]$ is necessarily the union of the class F.

Let $H = (V, E)$ be a hypergraph. Let $\mathbf{N}^{(V)}$ be the free commutative monoid with free basis V. Thus, the element v of V, when seen as an element of $\mathbf{N}^{(V)}$, is the function $V \to \mathbf{N}$ which maps v to 1 and everything else to 0. If $e \in E$, denote by $\chi(e)$ the characteristic function of e, i.e., $\chi(e) = \sum_{v \in e} v$. To the hypergraph H, we associate the submonoid $M(H)$ of $\mathbf{N}^{(V)}$ generated by the characteristic functions of edges.

Remark 5.1. When V is a proper class, the construction of (the underlying class of) $\mathbf{N}^{(V)}$ needs to be handled with care. It is certainly possible to consider functions $V \to \mathbf{N}$, but even the zero element of $\mathbf{N}^{(V)}$ is a proper class, thus it cannot be a member. Indeed, the zero element of $\mathbf{N}^{(V)}$ is $\{(v, 0) : v \in V\}$, which is a proper class. To work around the issue, consider the class $\mathcal{S}$ of functions from finite subsets of V to $\mathbf{N} \setminus \{0\}$. We are going to identify (that which should be) an element of $\mathbf{N}^{(V)}$ with its restriction to its support. If $f_1, f_2 \in \mathcal{S}$, say $f_i\colon S_i \to \mathbf{N} \setminus \{0\}$, we define $f_1 + f_2\colon S_1 \cup S_2 \to \mathbf{N} \setminus \{0\}$ by letting $(f_1 + f_2)(s) = f_1(s) + f_2(s)$ for $s \in S_1 \cap S_2$, and $(f_1 + f_2)(s) = f_i(s)$ when s belongs only to S_i. Note in particular that the zero element is the empty set.

Definition 5.2. Let $H = (V, E)$ be a hypergraph and n a positive integer. We say that *the relations of $M(H)$ are controlled by n permutations* if there exist n equivalence relations $\sim_1, \dots, \sim_n$ on the class of edges E such that, given $e_1, \dots, e_r, f_1, \dots, f_s \in E$, the equality

$$\chi(e_1) + \cdots + \chi(e_r) = \chi(f_1) + \cdots + \chi(f_s)$$

holds in $M(H)$ if, and only if, $r = s$ and there exist n permutations $\sigma_1, \dots, \sigma_n \in S_r$ such that $e_\mu \sim_i f_{\sigma_i(\mu)}$ for all $1 \le i \le n$ and $1 \le \mu \le r$.

If $\mathbf{C}$ is a full subcategory of Mod-R whose objects are modules of finite type, the hypergraph $H(\mathbf{C})$ associated to $\mathbf{C}$ is the hypergraph which has $V(\mathbf{C})$ as its class of vertices and whose edges are the finite sets $V(X) = V(\mathbf{C}, X)$ where X is an object of $\mathbf{C}$. The following dictionary between $\mathbf{C}$ and its hypergraph $H(\mathbf{C})$ justifies turning our attention to hypergraphs and to the relations in their associated monoids.

Proposition 5.3. *Let* $\mathbf{C}$ *be a full subcategory of modules of finite type, and let* n *be a positive integer. The finite direct sums of modules in* $\mathbf{C}$ *are controlled by* n *permutations if and only if the relations of* $M(H(\mathbf{C}))$ *are controlled by* n *permutations.*

Proof. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_s$ be objects of $\mathbf{C}$. We claim that

$$\chi(V(X_1)) + \cdots + \chi(V(X_r)) = \chi(V(Y_1)) + \cdots + \chi(V(Y_s))$$

holds in $M(H(\mathbf{C}))$ if and only if

$$X_1 \oplus \cdots \oplus X_r \cong Y_1 \oplus \cdots \oplus Y_s.$$

Indeed, the former equation holds if and only if, for each $\mathbf{P} \in V(\mathbf{C})$, the number of indices i such that $\mathbf{P} \in V(X_i)$ is equal to the corresponding number of indices j such that $\mathbf{P} \in V(Y_j)$. By an application of Lemma 3.11, this is equivalent to $X_1 \oplus \cdots \oplus X_r$ and $Y_1 \oplus \cdots \oplus Y_s$ being isomorphic in $\mathrm{Sums}(\mathbf{C})/\mathbf{P}$, for all $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{C}), M)$ and for all $M \in |\mathbf{C}|$, which is equivalent to the latter equation by Theorem 3.9. This proves the claim.

Suppose that the finite direct sums of modules in $\mathbf{C}$ are controlled by the equivalence relations $\equiv_1, \dots, \equiv_n$ on the class of objects of $\mathbf{C}$. Let $V(X)$ and $V(Y)$ be edges of $H(\mathbf{C})$, where X and Y are objects of $\mathbf{C}$. Let $V(X) \sim_i V(Y)$ if and only if $X \equiv_i Y$. The definition of $\sim_i$ does not depend on the choice of X and Y, because $X \cong Y$ if and only if $V(X) = V(Y)$ [**11**, Theorem 4.2]. It is easy to see that the relations of $M(H(\mathbf{C}))$ are controlled by the equivalence relations $\sim_1, \dots, \sim_n$. A similar argument shows the converse. ◻

From now on, let $n \geq 2$ be a fixed integer, and let $\mathbf{C}$ be a fixed full subcategory of Mod-R whose objects are *indecomposable* right R-modules of finite type i, with $i \leq n$. As a consequence, the hypergraph $H(\mathbf{C})$ is *simple*, cf. [**12**, end of Section 4].

For each n-tuple of pairwise disjoint classes $X_1, \dots, X_n$, we define the *n-partite complete hypergraph on* $X_1 \dot\cup \cdots \dot\cup X_n$ to be the hypergraph $P(X_1, \dots, X_n)$ with class of vertices $X_1 \dot\cup \cdots \dot\cup X_n$ and whose class of edges $E(X_1, \dots, X_n)$ consists of all n-element subsets of vertices which have exactly one vertex from each X_i [**5**, page 19]. Thus, $P(X_1, \dots, X_n)$ is simple and n-uniform.

A hypergraph $H = (V, E)$ is n-partite if V is a disjoint union $V = U_1 \dot\cup \cdots \dot\cup U_n$ with each U_i not empty and such that, for each $e \in E$, $e \cap U_i$ has at most one element. Clearly, a partial hypergraph of an n-partite complete hypergraph is n-partite.

Recall that, in a commutative monoid M, an element x is an *atom* if it is non-zero and, for all $a, b \in M$, the equality $x = a + b$ implies $a = 0$ or $b = 0$. The following generalizes [**13**, Proposition 3.5].

Theorem 5.4. *Let $H = (V, E)$ be a simple hypergraph. The following are equivalent*:

(i) *The relations of $M(H)$ are controlled by n permutations.*

(ii) *There exists an injective morphism $\phi: M(H) \to M(P(X_1, \ldots, X_n))$ of monoids which sends atoms to atoms, where $X_1, \ldots, X_n$ are suitable pairwise disjoint classes.*

(iii) *There exists an injective mapping $\eta: E \to E(X_1, \ldots, X_n)$ such that*

$$\chi(e_1) + \cdots + \chi(e_r) = \chi(f_1) + \cdots + \chi(f_s) \tag{5.5}$$

in $M(H)$ if and only if

$$\chi(\eta(e_1)) + \cdots + \chi(\eta(e_r)) = \chi(\eta(f_1)) + \cdots + \chi(\eta(f_s)) \tag{5.6}$$

in $M(P(X_1, \ldots, X_n))$, where $X_1, \ldots, X_n$ are suitable pairwise disjoint classes.

Proof. Suppose (i) holds, for a suitable choice of equivalence relations $\sim_1, \ldots, \sim_n$ on the class of edges E. Let $\pi_i: E \to E/{\sim_i}$ be the canonical projection. Technically, it may not be possible to form the quotient $E/{\sim_i}$, because the equivalence classes of $\sim_i$ may be proper classes; in that case we tacitly replace the quotient with a class of representatives.

Let $X_i = (E/{\sim_i}) \times \{i\}$, and let $p_i: E \to X_i$ be defined by $p_i(e) = (\pi_i(e), i)$. This makes $X_1, \ldots, X_n$ pairwise disjoint classes. The mapping p_i induces a monoid homomorphism $\widetilde{p}_i: M(H) \to \mathbf{N}^{(X_i)}$ as follows: If $\chi(e_1) + \cdots + \chi(e_r)$ is an arbitrary element of $M(H)$, let

$$\widetilde{p}_i(\chi(e_1) + \cdots + \chi(e_r)) = p_i(e_1) + \cdots + p_i(e_r).$$

To show that it is well-defined, suppose (5.5) holds in $M(H)$. Then $r = s$, and there exists a permutation $\sigma \in S_r$ such that $e_\mu \sim_i f_{\sigma(\mu)}$ for all $1 \leq \mu \leq r$. Therefore,

$$\begin{aligned} p_i(e_1) + \cdots + p_i(e_r) &= p_i(f_{\sigma(1)}) + \cdots + p_i(f_{\sigma(r)}) \\ &= p_i(f_1) + \cdots + p_i(f_r), \end{aligned}$$

and $\widetilde{p}_i$ is well-defined. Furthermore, we have an injective monoid homomorphism

$$p = \widetilde{p}_1 \times \cdots \times \widetilde{p}_n \colon M(H) \longrightarrow \mathbf{N}^{(X_1)} \times \cdots \times \mathbf{N}^{(X_n)}.$$

To show injectivity, suppose that

$$p_i(e_1) + \cdots + p_i(e_r) = p_i(f_1) + \cdots + p_i(f_s)$$

holds in $\mathbf{N}^{(X_i)}$ for each $1 \leq i \leq n$. Then $r = s$, and there exists a permutation $\sigma_i \in S_r$ such that $p_i(e_\mu) = p_i(f_{\sigma_i(\mu)})$, i.e., such that $e_\mu \sim_i f_{\sigma_i(\mu)}$ for each $1 \leq \mu \leq r$. This implies (5.5), hence p is injective.

In the following diagram, α is the isomorphism defined by $\alpha(g_1, \ldots, g_n) = g_1 + \cdots + g_n$, while the bottom morphism is set inclusion.

$$\begin{tikzcd} M(H) \arrow[r, "p"] \arrow[d, dashed, "\phi"'] & \mathbf{N}^{(X_1)} \times \cdots \times \mathbf{N}^{(X_n)} \arrow[d, "\alpha"] \\ M(P(X_1, \ldots, X_n)) \arrow[r] & \mathbf{N}^{(X_1 \,\dot{\cup} \cdots \dot{\cup}\, X_n)} \end{tikzcd}$$

For each $e \in E$, we have $\alpha p(\chi(e)) = p_1(e) + \cdots + p_n(e)$. Thus, $\varepsilon = \{p_1(e), \ldots, p_n(e)\}$ is an edge in $E(X_1, \ldots, X_n)$, and $\chi(\varepsilon) = \alpha p(\chi(e)) \in M(P(X_1, \ldots, X_n))$. Inasmuch as $M(H)$ is generated by $\{\chi(e) : e \in E\}$, we conclude that the image of αp is contained in $M(P(X_1, \ldots, X_n))$; thus, we can complete the diagram to a commutative square by adding ϕ, necessarily injective. Since the atoms of $M(H')$ are the characteristic functions of edges for any simple hypergraph H', the equality $\phi(\chi(e)) = \chi(\varepsilon)$ also implies that ϕ sends atoms to atoms. We have thus proved that (ii) holds.

Now assume (ii). For each $e \in E$, $\chi(e)$ is an atom of $M(H)$; thus, $\phi(\chi(e))$ is an atom of $M(P(X_1, \ldots, X_n))$. Hence, it is equal to $\chi(\eta(e))$

for some $\eta(e) \in E(X_1, \ldots, X_n)$. Since $\eta(e)$ is uniquely determined, we have a mapping $\eta: E \to E(X_1, \ldots, X_n)$. If (5.5) holds, applying ϕ to it we obtain that (5.6) holds. If the latter holds, the former holds by injectivity of ϕ. In particular, $\eta(e) = \eta(f)$ implies that $\chi(\eta(e)) = \chi(\eta(f))$, which is equivalent to $\chi(e) = \chi(f)$, which implies $e = f$. Hence, η is injective, and this completes the proof that (ii) implies (iii).

Eventually, assume (iii). For each $1 \le i \le n$ and each pair $e, f \in E$, define $e \sim_i f$ if $\eta(e) \cap X_i = \eta(f) \cap X_i$. This is an equivalence relation on E. Assume (5.5) holds, so that (5.6) holds. Inasmuch as $P(X_1, \ldots, X_n)$ is n-uniform, we must have $r = s$.

Write $\chi(\eta(e_\mu)) = e_{\mu,1} + \cdots + e_{\mu,n}$, where $e_{\mu,i} \in X_i$ for all $1 \le i \le n$, and write $\chi(\eta(f_\mu))$ accordingly. Thus,

$$\sum_{\mu=1}^{r} \sum_{i=1}^{n} e_{\mu,i} = \sum_{\mu=1}^{r} \sum_{i=1}^{n} f_{\mu,i},$$

and, in view of the fact that the classes $X_1, \ldots, X_n$ are pairwise disjoint, it follows that

$$\sum_{\mu=1}^{r} e_{\mu,i} = \sum_{\mu=1}^{r} f_{\mu,i}$$

for each $1 \le i \le n$. Thus, there exist $\sigma_1, \ldots, \sigma_n \in S_r$ such that $e_{\mu,i} = f_{\sigma_i(\mu),i}$, i.e., $\eta(e_\mu) \cap X_i = \eta(f_{\sigma_i(\mu)}) \cap X_i$. Hence, $e_\mu \sim_i f_{\sigma_i(\mu)}$, for $1 \le i \le n$ and $1 \le \mu \le r$. Conversely, if $r = s$ and such permutations exist, then (5.6) holds; hence, (5.5) also holds. This proves that (i) holds with respect to the equivalence relations $\sim_1, \ldots, \sim_n$. $\square$

Corollary 5.7. *If the relations of the monoid of a simple hypergraph $H = (V, E)$ are controlled by n permutations, then the same goes for any partial hypergraph of H.*

Proof. Let F be a subclass of E, and consider the partial hypergraph $H[F]$. There is a canonical injective monoid homomorphism $\iota: M(H[F]) \to M(H)$ which sends atoms to atoms. Thus, if ϕ is as in Theorem 5.4 (ii), then $\phi\iota$ shows that the relations of $M(H[F])$ are controlled by n permutations. $\square$

Consider the *intersection graph* G of the edges E of H, i.e., the simple graph having E as its class of vertices, and such that two elements of E are adjacent in G whenever their intersection is non-empty. Partition E as the disjoint union $E = \cup_{i\in I} E_i$ of the maximal connected subclasses of vertices of G. For each $i \in I$, let $H_i = H[E_i]$, i.e., let H_i be the partial hypergraph of H on the subclass of edges E_i, and denote by V_i its class of vertices. Note that V is the disjoint union $V = \cup_{i\in I} V_i$. We refer to the hypergraphs H_i as the *connected components* of H.

Corollary 5.8. *The relations of the monoid of a simple hypergraph $H = (V, E)$ are controlled by n permutations if and only the same goes for each connected component of H.*

Proof. There is a canonical isomorphism of monoids $\oplus_{i\in I}\mathbf{N}^{(V_i)} \to \mathbf{N}^{(V)}$, namely, the one which sends $(g_i\colon V_i \to \mathbf{N})_{i\in I}$ to the function $g\colon V \to \mathbf{N}$ obtained by $g(x) = g_i(x)$ for $x \in V_i$. It is easy to see that it induces an isomorphism $q\colon \oplus_{i\in I} M(H_i) \to M(H)$.

For the implication not already known by the previous corollary, assume that the relations of each $M(H_i)$ are controlled by n permutations, so that there is an injective monoid homomorphism $\phi_i\colon M(H_i) \to M(P(X_{i,1}, \dots, X_{i,n}))$ which sends atoms to atoms, for each $i \in I$. Without loss of generality, suppose that the classes $X_{i,j}$ are pairwise disjoint. Therefore, once we define $X_j = \cup_{i\in I} X_{i,j}$, we obtain that $X_1, \dots, X_n$ are pairwise disjoint. For each $i \in I$, let $\iota_i\colon M(P(X_{i,1}, \dots, X_{i,n})) \to M(P(X_1, \dots, X_n))$ be the canonical embedding of monoids. Define $\phi\colon \oplus_{i\in I} M(H_i) \to M(P(X_1, \dots, X_n))$ by $\phi((g_i)_{i\in I}) = \sum_{i\in I} \iota_i \phi_i(g_i)$. It is easy to check that ϕ is injective and sends atoms to atoms, as is required. $\square$

For integers r and n such that $1 \le r \le n$, let K_n^r denote the *r-uniform complete hypergraph of order n*, i.e., the hypergraph whose vertices are the elements of a set X of cardinality n and whose edges are all the r-element subsets of X [**5**, page 5]. Thus, the number of edges of K_n^r is $\binom{n}{r}$. Recall that, in a hypergraph, the *degree* of a vertex v, denoted by $d(v)$, is the number of edges e such that $v \in e$.

The following extends [**13**, Proposition 3.9].

Corollary 5.9. *Let $n \geq 2$ be an integer. If a simple hypergraph $H = (V, E)$ admits K_{2n}^n as a partial hypergraph, then the relations of $M(H)$ are not controlled by n permutations.*

Proof. In view of Corollary 5.7, we may assume $H = K_{2n}^n$. Assume, by contradiction, that the relations of $M(K_{2n}^n)$ are controlled by n permutations, and let ϕ, η, $X_1, \dots, X_n$ be as in Theorem 5.4. We will show that the partial hypergraph $C = P(X_1, \dots, X_n)[\eta(E)]$ is a copy of K_{2n}^n, and that the latter is not n-partite, which contradicts C being a partial hypergraph of an n-partite hypergraph.

By a construction by induction, it is possible to write E as a disjoint union

$$E = \{e_1, \dots, e_m\} \dot\cup \{V \setminus e_1, \dots, V \setminus e_m\}.$$

Necessarily, $m = |E|/2$. The element $s = \sum_{v \in V} v$ of $M(K_{2n}^n)$ can be written as $s = \chi(e_i) + \chi(V \setminus e_i)$ for any $i = 1, \dots, m$.

Let u be a vertex of C. Then $u \in \eta(e_i)$ or $u \in \eta(V \setminus e_i)$ for some i. Since $\phi(s) = \chi(\eta(e_i)) + \chi(\eta(V \setminus e_i))$, it follows that the coefficient of u in $\phi(s)$ is strictly positive. This implies that $u \in \eta(e_i)$ or $u \in \eta(V \setminus e_i)$, now for *all* indices $i = 1, \dots, m$. Since η is injective, it follows that the degree $d_C(u)$ of u in C is at least m. Let U be the set of vertices of C. Then

$$m|U| \leq \sum_{u \in U} d_C(u) = n|\eta(E)| = n|E| = 2mn,$$

from which $|U| \leq 2n$. Since C is n-uniform on $|U|$ vertices, we must have $|\eta(E)| \leq \binom{|U|}{n}$. But η is injective; hence, $|\eta(E)| = |E| = \binom{2n}{n}$, so that $|U| \geq 2n$. Thus, $|U| = 2n$, and it follows that C is the complete n-uniform hypergraph on $2n$ vertices.

To reach the required contradiction, let us finally show that C is not n-partite. Suppose it is n-partite. Then write U as a disjoint union $U = U_1 \dot\cup \cdots \dot\cup U_n$ in such a way that, for each $\varepsilon \in \eta(E)$, the set $\varepsilon \cap U_i$ has at most one element. Insofar as $2n = |U_1| + \cdots + |U_n|$, there exists $i = 1, \dots, n$ such that U_i has at least two elements. Pick an n-element subset ε of U with two elements from U_i. Then this is an edge of C by completeness, contradiction. $\square$

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Department of Pure and Applied Mathematics, University of Padova, 35121 Padova, Italy
Email address: girardi@math.unipd.it