ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

PERIODIC SOLUTIONS WITH SINGULARITIES IN TWO DIMENSIONS IN THE n-BODY PROBLEM

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ABSTRACT. Analytical methods are used to prove the existence of a periodic, symmetric solution with singularities in the planar 4-body problem. Numerical calculation and simulation are used to generate the orbit. The analytical method easily extends to any even number of bodies. Multiple simultaneous binary collisions are a key feature of the orbits generated.

1. Introduction. The n-body problem of celestial mechanics is one of the most important problems in the field of dynamical systems. The following differential equation

$$m_i\ddot{\rho}_i = \sum_{j\neq i} -\frac{m_i m_j(\rho_i-\rho_j)}{|\rho_i-\rho_j|^3} \tag{1}$$

gives a mathematical description of the planar n-body problem, where $\rho_i \in \mathbf{R}^2$ denotes the position of the ith body having mass m_i. All derivatives are taken with respect to time t. The potential energy of the system is given by

$$U = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{|\rho_i-\rho_j|}, \tag{2}$$

and the kinetic energy is given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i|\dot{\rho}_i|^2. \tag{3}$$

2010 AMS *Mathematics subject classification.* Primary 37N05, 65P99, 70F15, 70F16.
Keywords and phrases. Simultaneous binary collision, 4-body problem, regularization, singularity, $2n$-body problem, collision singularity.
Received by the editors on July 17, 2009, and in revised form on February 8, 2010.
DOI:10.1216/RMJ-2012-42-5-1601

Linearly stable symmetric periodic orbits are one aspect of the n-body problem. The elliptic Lagrangian triangular periodic orbits are linearly stable for certain values of eccentricity and the three masses [**8, 12**]. The Montgomery-Chenciner figure-eight orbit for three equal masses [**3, 9**] has been shown by Roberts [**11**] to be linearly stable by an innovative symmetry reduction technique he developed.

Singularities are another particular aspect of the n-body problem. Binary collisions, triple collisions, etc., are discussed at length in [**13**]. The Simultaneous Binary Collision (SBC) problem has been widely studied as well, both analytically and numerically. Simó [**7**] showed that the block regularization in the cases of the n-body problem which reduce to one-dimensional problems is differentiable, but the map passing from initial to final conditions (in suitable choices of transversal sections) is exactly $C^{8/3}$. Ouyang and Yan [**10**] give another approach for the regularization and analyze some properties of SBC solutions in the collinear four-body problem. Elbialy [**4**] studied the nature of the collision-ejection orbits associated to SBC.

Schubart [**13**] combined these two aspects to produce a singular linearly stable periodic orbit in the three-body equal mass collinear problem. The motion of the middle mass regularly alternates between binary collisions with each of the outer two masses. His work was subsequently extended to the unequal mass case by both Hénon [**5**] and Hietarinta and Mikkola [**6**]. Sweatman [**15**] later extended this work to a four-body periodic solution in one dimension, with bodies alternating between SBC of the outer mass pairs and binary collision of the inner two masses.

In this paper, we present the analytic existence of a family of singular symmetric periodic planar orbits in the four-body equal mass problem. The initial conditions of these orbits are symmetric in both positions and velocities, which lead to periodic simultaneous binary collisions with each of the four masses alternating between collisions with its two nearest neighbors. Due to the abundance of symmetries present in the initial conditions, we can reduce the number of variables needed to just four: two for representing position and two for representing momentum. In contrast to its one-dimensional counterparts, the proof for existence of this orbit is surprisingly simple. We begin in subsection 2.1 by giving a description of the proposed orbit and prove its existence. In subsection 2.2 we present the numerical methods used to produce the initial conditions that will lead to this orbit. In Section 3, we consider variants on this orbit, giving a family of orbits with singularities for an even number of equal masses.

Since the initial submission of this paper, we have been doing additional work with Dr. Lennard Bakker (Brigham Young University) and Dr. Gareth Roberts (College of the Holy Cross) implementing Robert's linear stability technique as presented in [**11**]. After precisely defining the symmetries that are present in the regularized coordinates, it is shown that the group of symmetries in the orbit is isomorphic to the dihedral group D_4. Further, as a consequence of Robert's technique, we have shown that the four-body planar orbit presented in this paper is linearly stable [**2**]. Further work has also been done on orbits in this family with alternating unequal masses [**1**]. Rather than a single mass parameter, the bodies have masses m_1, m_2, m_1, m_2 as numbered moving counterclockwise through the plane. Since some symmetries have been lost by this change in masses, it is necessary to choose two initial condition parameters as well as two initial velocities. Although numerically this is not a difficult problem, an analytical technique will require much more work.

2. The proposed orbit.

2.1. Analytical description. Initially we focused on finding a symmetric, periodic SBC orbit for four equal masses in two dimensions. Without loss of generality, we assume that the orbit begins with the four bodies lying at $(\pm 1, 0)$ and $(0, \pm 1)$ with initial velocities $(0, \pm v)$ and $(\pm v, 0)$, respectively, where $v \in (0, +\infty)$. For convenience, throughout the rest of the paper, we number the bodies 1 to 4 as in Figure 1.

The singularity of SBC in this problem is not essential. For a better understanding of the behavior of the motion of the bodies in a neighborhood of a collision, the standard technique is to make a change of coordinates and rescale time. In the new coordinates, the orbits which approach collision can be extended across the collision in a smooth manner with respect to the new time variable. This technique is called *regularization*. In our problem, regularization describes the behavior of the bodies approaching and escaping collisions, similar to the collisions of billiard balls.

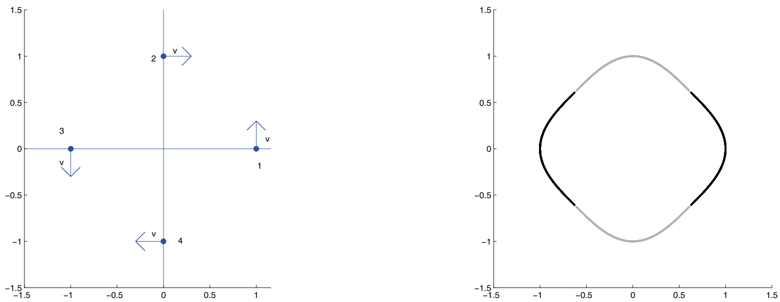

FIGURE 1. On the left, we illustrate the initial conditions leading to the four-body two-dimensional periodic SBC oribt. On the right, the orbit is shown.

Due to the symmetry of the initial conditions and the equations governing the motion of the bodies, the symmetry that is present in the initial conditions is maintained in the regularized sense.

Main theorem. *Let $E = T - U$ be the total energy and m the mass for each of the four bodies. For any $E < 0$ and $m > 0$, there exists a symmetric, periodic, four-body orbit with* SBC *in* $\mathbf{R}^2$.

Without loss of generality, we can assume $m = 1$ and the initial positions are as illustrated in Figure 1. The proof will be given at the end of this section.

Let t_0 be the time of first SBC. For $t \in [0, t_0)$, let the coordinate of body 1 be (x_1, x_2). By symmetry, the coordinates of bodies 2, 3, and 4 are (x_2, x_1), $(-x_1, -x_2)$ and $(-x_2, -x_1)$, respectively. Using equation (1), the acceleration of a body at point (x_1, x_2) is given by:

(4)
$$(\ddot{x_1}, \ddot{x_2}) = -\left[\frac{(x_1 - x_2, x_2 - x_1)}{(2(x_1 - x_2)^2)^{3/2}} + \frac{(2x_1, 2x_2)}{(4x_1^2 + 4x_2^2)^{3/2}} + \frac{(x_1 + x_2, x_1 + x_2)}{(2(x_1 + x_2)^2)^{3/2}}\right].$$

We now perform the regularization of the system. The system has the Hamiltonian:

$$H = \frac{1}{8}(w_1^2 + w_2^2) - \frac{\sqrt{2}}{x_1 - x_2} - \frac{\sqrt{2}}{x_1 + x_2} - \frac{1}{\sqrt{x_1^2 + x_2^2}} \tag{5}$$

where $w_1 = 4\dot{x}_1$ and $w_2 = 4\dot{x}_2$ are the conjugate momenta to x_1 and x_2. Note that SBC happens when $x_1 = \pm x_2$. We introduce a new set of coordinates:

$$q_1 = x_1 - x_2, \qquad q_2 = x_1 + x_2.$$

Their conjugate momenta p_i are given by using a generating function $F = (x_1 - x_2)p_1 + (x_1 + x_2)p_2$:

$$w_1 = p_1 + p_2, \qquad w_2 = p_2 - p_1.$$

The Hamiltonian corresponding to the new coordinate system is

$$H = \frac{1}{4}(p_1^2 + p_2^2) - \frac{\sqrt{2}}{q_1} - \frac{\sqrt{2}}{q_2} - \frac{\sqrt{2}}{\sqrt{q_1^2 + q_2^2}}. \tag{6}$$

Following the work of Sweatman [**15**], we introduce another canonical transformation:

$$q_i = Q_i^2, \qquad P_i = 2Q_i p_i, \quad i = 1, 2,$$

with $Q_i > 0$. We also introduce a new time variable s, which satisfies $dt/ds = q_1 q_2$. This produces a regularized Hamiltonian in extended phase space:

$$\begin{aligned} \Gamma &= \frac{dt}{ds}(H - E) \\ &= \frac{1}{16}(P_1^2 Q_2^2 + P_2^2 Q_1^2) - \sqrt{2}(Q_1^2 + Q_2^2) - \frac{\sqrt{2}Q_1^2 Q_2^2}{\sqrt{Q_1^4 + Q_2^4}} - Q_1^2 Q_2^2 E \end{aligned} \tag{7}$$

where E is the total energy of the Hamiltonian H.

The regularized Hamiltonian gives the following differential equations of motion:

(8)
$$Q_1' = \frac{1}{8}P_1Q_2^2$$
(9)
$$Q_2' = \frac{1}{8}P_2Q_1^2$$
(10)
$$P_1' = -\frac{1}{8}P_2^2Q_1 + 2\sqrt{2}Q_1 + \frac{2\sqrt{2}Q_1Q_2^2}{\sqrt{Q_1^4+Q_2^4}} - \frac{2\sqrt{2}Q_1^5Q_2^2}{(Q_1^4+Q_2^4)^{3/2}} + 2EQ_1Q_2^2$$
(11)
$$P_2' = -\frac{1}{8}P_1^2Q_2 + 2\sqrt{2}Q_2 + \frac{2\sqrt{2}Q_2Q_1^2}{\sqrt{Q_1^4+Q_2^4}} - \frac{2\sqrt{2}Q_2^5Q_1^2}{(Q_1^4+Q_2^4)^{3/2}} + 2EQ_2Q_1^2$$

with initial conditions

(12) $\quad Q_1(0) = 1, \qquad Q_2(0) = 1, \qquad P_1(0) = -4v, \qquad P_2(0) = 4v$

where derivatives are with respect to s, and E is the total energy of the Hamiltonian H.

Theorem 1. *Let s_0 be the time of the first* SBC *in the regularized system. Then s_0 is a continuous function with respect to the initial velocity v. Furthermore,*

$$p_2(t_0) = \frac{P_2(s_0,v)}{2Q_2(s_0,v)}$$

is also continuous with respect to v.

Proof. At the first SBC, $Q_1(s_0) = 0$, and $Q_2(s_0) = \sqrt{q_2} = \sqrt{x_1+x_2} > 0$. Our goal is to show that $p_2(t_0)$ is a continuous function with respect to v.

Because $\Gamma = 0$ at $s = s_0$, $P_1(s_0) = -4\sqrt[4]{2}$ from (7). Since Γ is regularized, the solutions $P_i = P_i(s,v)$ and $Q_i = Q_i(s,v)$ are continuous functions with respect to the two variables s and v. At time $s = s_0$,

$$0 = Q_1(s_0(v), v).$$

To apply the implicit function theorem, we need to show that

$$\frac{\partial Q_1}{\partial s}(s_0, v) \neq 0.$$

From (8),

$$\frac{\partial Q_1}{\partial s}(s_0, v) = \frac{1}{8} P_1 Q_2^2 \mid_{(s_0,v)} = -\frac{1}{2}\sqrt[4]{2} Q_2(s_0)^2 < 0.$$

So $s_0 = s_0(v)$ is a continuous function of v. Therefore, both $P_2(s_0, v)$ and $Q_2(s_0, v)$ are continuous functions of v. Further, since $Q_2(s_0, v) > 0$, $p_2(t_0)$ is also a continuous function of v. □

Theorem 2. *A* $v = v_0$ *exists such that* $\dot{x}_1(t_0) + \dot{x}_2(t_0) = (1/2)p_2(t_0) = 0$, *where* t_0 *is the time of the first* SBC, *i.e., the net momentum of bodies* 1 *and* 2 *at the first* SBC *is* 0.

The outline of this proof is as follows. We will show that v_1 and v_2 exist such that $\dot{x}_1 + \dot{x}_2$ is negative at SBC for $v = v_1$ and positive at SBC for $v = v_2$. The result then follows by Theorem 1.

Proof. Consider Newton's equation before the time of the first SBC:

(13)
$$\ddot{x}_1 = \frac{x_2 - x_1}{2\sqrt{2}(x_1 - x_2)^3} - \frac{2x_1}{8(x_1^2 + x_2^2)^{3/2}} - \frac{x_1 + x_2}{2\sqrt{2}(x_1 + x_2)^3},$$

(14)
$$\ddot{x}_2 = \frac{x_1 - x_2}{2\sqrt{2}(x_1 - x_2)^3} - \frac{2x_2}{8(x_1^2 + x_2^2)^{3/2}} - \frac{x_1 + x_2}{2\sqrt{2}(x_1 + x_2)^3}.$$

Therefore,

$$\ddot{x}_1 + \ddot{x}_2 = -\frac{x_1 + x_2}{4(x_1^2 + x_2^2)^{3/2}} - \frac{1}{\sqrt{2}(x_1 + x_2)^2} < 0, \tag{15}$$

which means $\dot{x}_1 + \dot{x}_2$ is decreasing with respect to t.

At the initial time $t = 0$, $x_1 = 1$, $x_2 = 0$, $\dot{x}_1 = 0$, and $\dot{x}_2 = v$. Note that, for $v \in (0, \infty)$, there is no triple collision or total collision for $t \in [0, t_0]$, where t_0 is the time of the first SBC, as a triple collision implies total collapse by symmetry. Also, from the initial time to t_0, $0 \leq x_2 \leq x_1 \leq 1$, $0 < x_1 + x_2 < 2$, and $x_1^2 + x_2^2 < 4$.

Let $y(t) = x_1(t) + x_2(t)$. Then, for any choice of v, $\ddot{y}(t) < 0$ and $0 < y(t) < 2$ hold for any $t \in [0, t_0]$. In other words, $\dot{y}(t)$ is decreasing with respect to t.

First, we will show that v_1 exists such that $\dot{y}(t_0) < 0$. When $v = 0$ the four bodies form a central configuration and, as a consequence, the motion of the four bodies leads to total collapse. Consider the time interval $t \in [0, t_0/2)$. In this interval, the differential equations (13) and (14) have no singularity, and $\ddot{y}(t_0/2) < 0$. By continuous dependence on the initial conditions, $\dot{y}(t_0/2) = \dot{x}_1(t_0/2) + \dot{x}_2(t_0/2)$ is a continuous function with respect to the initial velocity v. When $v = 0$, $\dot{x}_1(t_0/2) < 0$, $\dot{x}_2(t_0/2) = 0$, which gives $\dot{y}(t_0/2) < 0$. Therefore, a $\delta > 0$ exists such that $\dot{y}(t_0/2) < 0$ holds for any $v \in (-\delta, \delta)$.

Choose $v_1 = \delta/2$; then $\dot{y}(t_0/2) < 0$. Because $\dot{y}(t)$ is decreasing with respect to t, $\dot{y}(t_0) \leq \dot{y}(t_0/2) < 0$.

Next we will show that a v_2 exists such that $\dot{y}(t_0) > 0$. Note that, as $v \to \infty$,

$$\lim_{v\to\infty} y(t_0) = \lim_{v\to\infty} x_1(t_0) + x_2(t_0) = 2$$

and

$$\lim_{v\to\infty} \dot{y}(t_0) = \infty.$$

Therefore some positive value v_2 exists such that $\dot{y}(t_0) > 0$. $\square$

Proof of the main theorem. From Theorem 2, we know that an initial velocity $v = v_0$ exists such that $\dot{x}_1(t_0) + \dot{x}_2(t_0) = 0$. Let $\{P_1, P_2, Q_1, Q_2\}$ for $s \in [0, s_0]$ be the solution in the regularized system corresponding to the orbit from $t = 0$ to $t = t_0$. Following collision, consider the behavior of the first and second bodies. Assume their velocity was reflected about the $y = x$ line in the plane. In the new coordinate system, this corresponds to a new set of functions

$$\{-P_1(2s_0 - s),\ -P_2(2s_0 - s),\ -Q_1(2s_0 - s),\ -Q_2(2s_0 - s)\}$$

for $s \in [s_0, 2s_0]$. We can easily check that

$$\{-P_1(2s_0 - s),\ -P_2(2s_0 - s),\ -Q_1(2s_0 - s),\ -Q_2(2s_0 - s)\}$$

for $s \in [s_0, 2s_0]$ is also a set of solutions for equations (8)–(11) with initial conditions at $s = s_0$. Also, $\{P_1(s), P_2(s), Q_1(s), Q_2(s)\}$ for $s \in [s_0, 2s_0]$ satisfies equations (8)–(11) with the same initial conditions at $s = s_0$. Note that equations (8)–(11) with initial conditions at $s = s_0$ have a unique solution for any choice of $v \in (0, \infty)$. Then, by uniqueness, the orbit for $s \in [s_0, 2s_0]$ must be the same as the orbit for $s \in [0, s_0]$ in reverse, i.e.,

$$P_i(s) = -P_i(2s_0 - s), Q_i(s) = -Q_i(2s_0 - s)$$

for $s \in [0, s_0]$. Therefore, at time $s = 2s_0$, bodies 1 and 2 will have returned to their initial positions with velocities $(0, -v)$ and $(-v, 0)$, respectively. Similarly, at time $s = 2s_0$, bodies 3 and 4 will have also returned to their initial positions with velocities $(0, v)$ and $(v, 0)$ respectively.

Next, we use symmetry and uniqueness to show the orbit from $s = 2s_0$ to $s = 4s_0$ and the orbit from $s = 0$ to $s = 2s_0$ will be symmetric with respect to the $y-$axis. Compare the motion of body 2 and body 3 from $s = 2s_0$ to $s = 4s_0$ with the motion of body 2 and body 1 from time $s = 0$ to $s = 2s_0$. The initial conditions of body 3 at $s = 2s_0$ and the initial conditions of body 1 at $s = 0$ are symmetric with respect to the $y-$axis. Also the initial conditions of body 2 at $s = 2s_0$ and the initial conditions of body 4 at $s = 0$ are symmetric with respect to the $x-$axis. Therefore, by uniqueness, the orbit of bodies 2 and 3 from $s = 2s_0$ to $s = 4s_0$ and the orbit of bodies 1 and 2 from $s = 0$ to $s = 2s_0$ must be symmetric with respect to $y-$axis. Therefore, the orbit of bodies 1 and 4 from $s = 2s_0$ to $s = 4s_0$ and the orbit of bodies 3 and 4 from $s = 0$ to $s = 2s_0$ are symmetric with respect to the $y-$axis. Hence, at $s = 4s_0$, the positions and velocities of the four bodies are exactly the same as at $s = 0$. Therefore, the orbit is periodic with period $s = 4s_0$. □

It is worth noting here that the previous proof implies a time-reversing symmetry for the periodic orbit. This provides further evidence for the conjecture made by Roberts [**11**], stating that linearly stable periodic orbits in the equal mass n-body problem must have a time-reversing symmetry. (Linear stability of this orbit is shown in [**2**].)

2.2. Numerical method. As we are searching for a periodic orbit of the n-body problem, we assume the value of the Hamiltonian needs to be negative. Using the initial positions of the four bodies described earlier, it is not hard to find the potential energy at $t = 0$:

$$U = 2\sqrt{2} + 1.$$

Then, acting under the negative Hamiltonian assumption:

$$2\sqrt{2} + 1 \geq \sum_{i=1}^{n} \frac{m_i |v_i|^2}{2}.$$

Since all masses are equal, if we require that the velocities of each body be equal in magnitude, we obtain:

$$v_{\max} = \sqrt{\frac{2\sqrt{2}+1}{2}} \tag{16}$$

with $v_{\max}$ defined to be the value of v such that the value of the Hamiltonian is zero. Define $\theta = v/v_{\max}$. This parameter is used in the numerical algorithm.

At this point it becomes necessary to find out just how much kinetic energy is required to obtain the periodic orbit. Since we know suitable bounds on the velocity parameter ($\theta \in (0, 1)$), we can search the interval numerically. We use an n-body simulator with the initial positions previously described. The simulation is run until one SBC occurs. For simplicity, we consider only the collision between the first and second bodies in the first quadrant. Summing their velocities immediately before the collision gives a vector running along line $y = x$ (due to symmetry), with both components having the same sign. The magnitude of this vector is given in Figure 2. Negative magnitudes represent vectors with both components less than zero.

Next, a standard bisection method is used to find the amount of energy required to cause the net velocity at collision to be zero. Using the initial interval $\theta \in [0, 1]$ and iterating to a tolerance of 10^{-14}, the correct value of θ was found to be $\theta = 0.46449539554694$.

It is worth noting that both the proof of existence and the numerical method do not guarantee the uniqueness of this orbit. Numerical

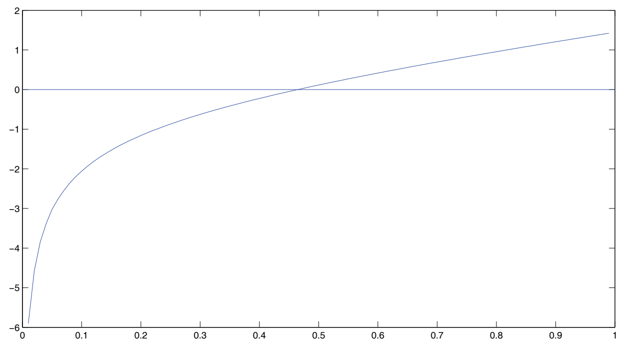

FIGURE 2. The magnitude of the net velocity of the first two bodies (vertical axis) at the time of collision for various values of θ (horizontal axis).

simulations demonstrate that, for values of θ near the correct value, the orbit remains for a significant length of time with the paths of the bodies lying in a "fattened" annular region roughly the shape of the original orbit. Near the extreme ends, the orbit experiences near total-collapse and fall apart rapidly. Although we do not focus on these questions just yet, a more thorough study of the dynamics could prove to be quite interesting.

3. Variants.

3.1. Orbits of more than four bodies. The same technique can be adopted to find similar orbits for any arbitrary even number n. A key feature of these orbits will be higher numbers of simultaneous binary collisions. For a given value of n, initial positions are given by spacing the bodies evenly about the unit circle. The potential energy (and the value of $v_{\max}$) is found numerically by iterating over each pair of planets and summing the reciprocal of the distances between them. (Recall that all $m_i = 1$.) Velocities are then assigned to the bodies in alternating counter-clockwise and clockwise directions, initially tangent to the circle. Again we consider the collision between the first and the second bodies. Although the net velocity of the two at collision will not lie along the $y = x$ line, the components of this vector will both have the same sign. The magnitudes of the net velocity between the first

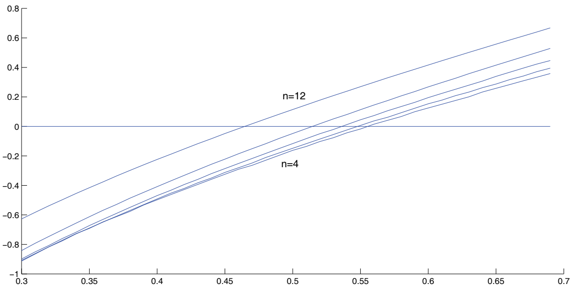

FIGURE 3. Curves showing the magnitude of the net velocity of the first two bodies (vertical axis) at the time of collision for various values of θ (horizontal axis) for $n = 4, 6, 8, 10, 12$.

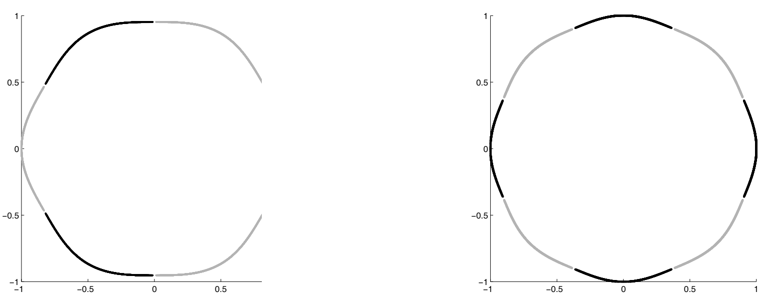

FIGURE 4. The six- and eight-body two-dimensional periodic SBC orbits.

two bodies at initial collision are shown in Figure 3 for various values of n. Lower curves in the graph correspond to higher values of n. Again, negative magnitudes correspond to both components being negative.

Pictures of the orbit for $n = 6$ and $n = 8$ are shown in Figure 4. It is readily seen that, as n increases, the shape of the orbit more closely approximates a circle.

Acknowledgments. Skyler Simmons would like to give thanks to Dr. Kening Lu, who recommended me for this project and special thanks to Jason Grout for hours of discussion that helped point my mathematical career in the right direction. We are also all indebted to the advice given by Dr. Lennard Bakker and all the referees during the revision process.

REFERENCES

1. Lennard F. Bakker, Tiancheng Ouyang, Duokui Yan and Skyler Simmons, *Existence and stability of symmetric periodic simultaneous binary collision orbits in the planar pairwise symmetric four-body problem*, Celestial Mech. Dynam. Astronom. **110** (2011), 271–290.

2. Lennard F. Bakker, Tiancheng Ouyang, Duokui Yan, Skyler Simmons and Gareth E. Roberts, *Linear stability for some symmetric periodic simultaneous binary collision orbits in the four-body problem*, Celestial Mech. Dynam. Astronom. **108** (2010), 147–164.

3. A. Chenciner and R. Montgomery, *A Remarkable periodic solution of the three-body problem in the case of equal masses*, Ann. Math. **152** (2000), 881–901.

4. Mohamed Sami ElBialy, *On simultaneous binary collisions in the planar n-body problem*, Z. Angew. Math. Phys. **44** (1993), 880–890.

5. M. Hénon, *Stability and interplay motions*, Celest. Mech. Dynam. Astron. **15** (1997), 243–261.

6. J. Hietarinta and S. Mikkola, *Chaos in the one-dimensional gravitational three-body problem*, Chaos **3** (1993), 183–203.

7. R. Martínez and C. and Simó, *The degree of differentiability of the regularization of simulaneous binary collisions in some N-body problems*, Nonlinearity **13** (2000), 2107–2130.

8. K.R. Meyer and D.S. Schmidt, *Elliptic relative equilibria in the N-body problem*, D. Diff. Eqn. **214** (2005), 256–298.

9. C. Moore, *Braids in classical dynamics*, Phys. Rev. Lett. **70** (1993), 3675–3679.

10. T. Ouyang and D. Yan, *Simultaneous binary collision for the collinear four body problem*, preprint.

11. G. Roberts, *Linear stability analysis of the figure-eight orbit in the three-body problem*, Ergod. Th. Dynam. Sys. **27** (2007), 1947–1963.

12. ———, *Linear stability of the elliptic Lagrangian triangle solutions in the three-body problem*, J. Diff. Eqn. **182** (2002), 191–218.

13. J. Schubart, *Numerische Aufsuchung periodischer Lösungen im Dreikörperproblem*, Astron. Narchr. **283** (1956), 17–22.

14. C.L. Siegel and J.K. Moser, *Lectures on celestial mechanics*, in *Classics in mathematics*, Springer-Verlag, Berlin, 1995 (translated from German by C.I. Kalme); reprint of 1971 translation.

15. W.L. Sweatman, *Symmetrical one-dimensional four-body problem*, Celest. Mech. Dynam. Astron. **82** (2002), 179–201.

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