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ONE HALF OF A MAXIMAL EMBEDDING DIMENSION NUMERICAL SEMIGROUP

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ABSTRACT. Let S be a numerical semigroup, and let p be a positive integer. Then $S/p = \{x \in \mathbf{N} \mid px \in S\}$ is also a numerical semigroup and, when $p = 2$, we say that $S/2$ is the one half of the numerical semigroup S. We characterize the numerical semigroups that are one half of a numerical semigroup with maximal embedding dimension. This characterization allows us to algorithmically determine, whether or not a given numerical semigroup is one half of a numerical semigroup with maximal embedding dimension.

1. Introduction. A numerical semigroup is a subset of $\mathbf{N}$ (here $\mathbf{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero element and has finite complement in $\mathbf{N}$. Given $A \subseteq \mathbf{N}$, we will denote by $\langle A \rangle$ the submonoid of $(\mathbf{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbf{N} \setminus \{0\},\ \lambda_1, \ldots, \lambda_n \in \mathbf{N}, a_1, \ldots, a_n \in A\}.$$

If $S = \langle A \rangle$, then we say that A is a system of generators of S. We say that A is a minimal system of generators of S if no proper subset of A generates S. It is well known (see for instance [**10**]) that every numerical semigroup admits a unique minimal system of generators, which has finitely many elements.

If S is a numerical semigroup and $\{n_1 < n_2 < \cdots < n_p\}$ is its minimal system of generators, then n_1 is called the multiplicity of S and we denote it by $\mathrm{m}(S)$. The positive integer p is the embedding dimension of S, and we denote it by $\mathrm{e}(S)$ (see [**3**]). It is easy to prove

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(see [**10**]) that $\mathrm{e}(S) \leq \mathrm{m}(S)$. We say that a numerical semigroup S has maximal embedding dimension (MED-semigroup) if $\mathrm{e}(S) = \mathrm{m}(S)$. These semigroups have been widely studied in the literature, not only from the semigroup point of view but also by its applications in Commutative Algebra through the value semigroup associated to one-dimensional analytically irreducible local domains (see [**1, 3, 4, 5, 16**]).

Let S be a numerical semigroup, and let p be a positive integer. Then in [**14**] it is proved that $S/p = \{x \in \mathbf{N} \mid px \in S\}$ is also a numerical semigroup, called the quotient of S by p. When $p = 2$ we say that $S/2$ is the half of the numerical semigroup S. One of the principal problems in numerical semigroup theory is the belonging question, that is, to find alternative ways to represent a numerical semigroup that allow us to easily decide whether or not an integer is in the semigroup. It is well known (see for example [**10**]) that if S is an MED-semigroup with multiplicity m then the minimal set of generators of S is of the form $\{\mathrm{m}, w(1), \ldots, w(\mathrm{m}-1)\}$, where $w(i)$ is the smallest element of S that is congruent with i modulo m. It is clear that an integer x belongs to S if and only if $x \geq w(x \bmod \mathrm{m})$. So, for an MED-semigroup, the belonging question is trivial and consequently is also trivial for the quotients of a MED-semigroup by a positive integer. Nowadays it is an open problem to give an algorithm that allows us to decide whether or not a given numerical semigroup is a quotient of an MED-semigroup by a positive integer. Also conjectured is the possibility that every numerical semigroup is a quotient of this form. In this work we intend to give a modest contribution towards to the solution of the above-mentioned problem. Our principal aim is to characterize the numerical semigroups that are one half of an MED-semigroup. In this way we also continue the study, performed in previous papers, of quotients of special cases of numerical semigroups. In [**15**] we study the quotients of embedding dimension two numerical semigroups. In [**11, 12**] it is proved that every numerical semigroup is one half of infinitely many symmetric numerical semigroups and in [**8**] it is shown that an irreducible numerical semigroup is one half of a pseudo-symmetric numerical semigroup and, as a consequence, we obtain that every numerical semigroup can be obtained by dividing by four a pseudo-symmetric numerical semigroup. The contents of this paper are organized as follows.

In Section 2 we will see that

$$\begin{aligned}\{S \mid S \text{ is an MED-semigroup}\}\\ &= \left\{\frac{S}{2} \mid S \text{ is an MED-semigroup and } \mathrm{m}(S) \text{ is even}\right\}\\ &\subset \left\{\frac{S}{2} \mid S \text{ is an MED-semigroup}\right\}\\ &\subset \{S \mid S \text{ is a numerical semigroup}\}.\end{aligned}$$

We characterize in Section 3 the numerical semigroups that are one half of an MED-semigroup. This characterization allows us to algorithmically determine whether or not a given numerical semigroup is one half of an MED-semigroup.

The one half of an MED-semigroup with even multiplicity is also an MED-semigroup. In the last section, we study and characterize the numerical semigroups that are one half of an MED-semigroup with odd multiplicity. We also see that, given a numerical semigroup S, the set

$$\left\{S' \mid S' \text{ is an MED-semigroup, } \mathrm{m}(S') \text{ is odd and } S = \frac{S'}{2}\right\}$$

is finite and that, if it is not empty, then it has maximum with respect to the set inclusion order.

2. First results. In the bibliography several characterizations of MED-semigroups exist (see [**3**]). We find the following lemma in [**13**].

Lemma 1. *Let S be a numerical semigroup. Then S is an* MED-*semigroup if and only if $x, y \in S\backslash\{0\}$ implies $x + y - \mathrm{m}(S) \in S$.*

The following result tells us that the one half of an MED-semigroup with even multiplicity is also an MED-semigroup.

Proposition 2. *If S is an* MED-*semigroup and $\mathrm{m}(S)$ is even, then $S/2$ is an* MED-*semigroup.*

Proof. It is clear that $\mathrm{m}(S/2) = \mathrm{m}(S)/2$. If $x, y \in (S/2)\backslash\{0\}$, then $2x, 2y \in S\backslash\{0\}$. By applying Lemma 1, we have that $2x + 2y - \mathrm{m}(S) \in$

S. Thus $2(x+y-\mathrm{m}(S/2)) \in S$ and consequently $x+y-\mathrm{m}(S/2) \in S/2$. By again using Lemma 1, we obtain that $S/2$ is an MED-semigroup. ☐

The one half of an MED-semigroup with odd multiplicity is not, in general, an MED-semigroup. In fact, $S = \langle 5, 8, 11, 14, 17\rangle$ is an MED-semigroup and $S/2 = \langle 4, 5, 7\rangle$ is not an MED-semigroup.

If S is a numerical semigroup, then the greatest integer that does not belong to S is called the Frobenius number of S (see [**7**]) and it is denoted here by $\mathrm{F}(S)$. Clearly, $\mathrm{F}(\mathbf{N}) = -1$ and if $S \neq \mathbf{N}$, then $\mathrm{F}(S)$ is a positive integer. If $X \subseteq \mathbf{Z}$ we denote by $2X$ the set $\{2x \mid x \in X\}$. If $x_1 < x_2 < \cdots < x_k$ are integers, then we use $\{x_1, x_2, \ldots, x_k, \rightarrow\}$ to denote the set $\{x_1, x_2, \ldots, x_k\} \cup \{z \in \mathbf{Z} \mid z > x_k\}$.

Next we prove that every MED-semigroup is one half of an MED-semigroup with even multiplicity.

Proposition 3. *If S is an* MED-*semigroup and k is an odd integer greater than or equal to* $2\mathrm{F}(S)+3$ *and not equal to 1, then* $\overline{S} = 2S \cup \{k, \rightarrow\}$ *is an* MED-*semigroup with multiplicity* $2\mathrm{m}(S)$ *and* $S = \overline{S}/2$.

Proof. Clearly $\overline{S}$ is a numerical semigroup and $S = \overline{S}/2$.

If $S = \mathbf{N}$, then $\overline{S} = 2S \cup \{k, \rightarrow\} = \langle 2, k\rangle$ is an MED-semigroup with multiplicity 2.

If $S \neq \mathbf{N}$, then $\mathrm{F}(S) + 1 \in S\backslash\{0\}$ and so $\mathrm{m}(S) \leq \mathrm{F}(S) + 1$. From $2\mathrm{m}(S) \leq 2\mathrm{F}(S) + 2 < k$ we deduce that $\mathrm{m}(\overline{S}) = 2\mathrm{m}(S)$. To conclude the proof we have to show that $\overline{S}$ is an MED-semigroup. For that, by applying Lemma 1, it suffices to prove that if $x, y \in \overline{S}\backslash\{0\}$, then $x + y - \mathrm{m}(\overline{S}) \in \overline{S}$.

We distinguish two cases. If $\{x, y\} \cap \{k, \rightarrow\} \neq \varnothing$, then $x + y - \mathrm{m}(\overline{S}) \geq k$ because $\max\{x, y\} \geq k$ and $\min\{x, y\} \geq \mathrm{m}(\overline{S})$. Thus $x + y - \mathrm{m}(\overline{S}) \in \overline{S}$. If $\{x, y\} \cap \{k, \rightarrow\} = \varnothing$, then $x, y \in 2S$. Hence, $s_1, s_2 \in S\backslash\{0\}$ exist such that $x = 2s_1$ and $y = 2s_2$. Therefore, $x + y - \mathrm{m}(\overline{S}) = 2s_1 + 2s_2 - 2\mathrm{m}(S) = 2(s_1 + s_2 - \mathrm{m}(S)) \in \overline{S}$, because $s_1 + s_2 - \mathrm{m}(S) \in S$ in view of Lemma 1. ☐

Our next aim will be to show that there exist numerical semigroups that are not one half of an MED-semigroup. For that the following result is fundamental.

Lemma 4. *Let $\overline{S}$ be an* MED-*semigroup, and let $S = \overline{S}/2$. Then*

(1) $\mathrm{m}(\overline{S}) \in S$;

(2) *if $s_1, s_2, s_3 \in S \backslash \{0\}$, then $s_1 + s_2 + s_3 - \mathrm{m}(\overline{S}) \in S$.*

Proof. (1) It is obvious.

(2) As $2s_1, 2s_2 \in \overline{S}\backslash\{0\}$, by using Lemma 1 we have that $2s_1 + 2s_2 - \mathrm{m}(\overline{S}) \in \overline{S}$. Notice that $2s_1 + 2s_2 - \mathrm{m}(\overline{S}) \neq 0$, because $\mathrm{m}(\overline{S}) \leq 2s_1$. Since $2s_1 + 2s_2 - \mathrm{m}(\overline{S}), 2s_3 \in \overline{S}\backslash\{0\}$, by again applying Lemma 1, we obtain $2s_1 + 2s_2 - \mathrm{m}(\overline{S}) + 2s_3 - \mathrm{m}(\overline{S}) \in \overline{S}$, that is, $2(s_1 + s_2 + s_3 - \mathrm{m}(\overline{S})) \in \overline{S}$. Therefore $s_1 + s_2 + s_3 - \mathrm{m}(\overline{S}) \in S$. □

Let S be a numerical semigroup, and let $n \in S\backslash\{0\}$. We define the Apéry set (see [**2**]) of n in S as

$$\mathrm{Ap}\,(S, n) = \{s \in S \mid s - n \notin S\}.$$

It is easy to prove (see for instance [**10**]) that $\mathrm{Ap}\,(S, n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$, where $w(i)$ is the smallest element of S that is congruent with i modulo n. Therefore $\mathrm{Ap}\,(S, n)$ has cardinality n.

Next we give a necessary condition for a numerical semigroup to be one half of an MED-semigroup.

Proposition 5. *Let $\overline{S}$ be an* MED-*semigroup, and let $S = \overline{S}/2$. Then*

$$\mathrm{m}(\overline{S}) \leq \frac{(\mathrm{e}(S)+1)(\mathrm{e}(S)+2)}{2}.$$

Proof. From Lemma 4 we easily deduce that if $\{n_1 < n_2 < \cdots < n_p\}$ is the minimal system of generators of S, then

$$\begin{aligned}\mathrm{Ap}\,(S, \mathrm{m}(\overline{S})) \subseteq \{0, n_1, \ldots, n_p, n_1 + n_1, \ldots,\\ n_1 + n_p, n_2 + n_2, \ldots, n_2 + n_p, \ldots,\\ n_{p-1} + n_{p-1}, n_{p-1} + n_p, n_p + n_p\}\end{aligned}$$

and therefore the cardinality of $\mathrm{Ap}\,(S, \mathrm{m}(\overline{S}))$ is less than or equal to $[(\mathrm{e}(S)+1)(\mathrm{e}(S)+2)]/2$. □

As a consequence of the previous proposition, we have that the numerical semigroup $S = \langle 4, 7\rangle$ is not one half of an MED-semigroup. In fact, if an MED-semigroup $\overline{S}$ exists such that $S = \overline{S}/2$, then by Proposition 2 we have that $\mathrm{m}(\overline{S})$ is odd, by Lemma 4 that $\mathrm{m}(\overline{S}) \in S$ and by Proposition 5 that $\mathrm{m}(\overline{S}) \leq 6$. This is a contradiction because $S = \langle 4, 7\rangle$ does not contain any odd integer less than or equal to 6.

We can resume the content of this section in the following result.

Theorem 6.

$$
\begin{aligned}
\{S \mid S & \text{ is an MED-semigroup}\} \\
&= \left\{\frac{S}{2} \mid S \text{ is an MED-semigroup and } \mathrm{m}(S) \text{ is even}\right\} \\
&\subset \left\{\frac{S}{2} \mid S \text{ is an MED-semigroup}\right\} \\
&\subset \left\{S \mid S \text{ is a numerical semigroup}\right\}.
\end{aligned}
$$

3. The characterization. In this section we propose to characterize the numerical semigroups that are one half of some MED-semigroup. From Proposition 3 we know that every MED-semigroup is of this form. Moreover, by Proposition 2, we have that the one half of an MED-semigroup with even multiplicity is also an MED-semigroup. Therefore, we can focus our study in the numerical semigroups that are one half of an MED-semigroup with odd multiplicity. The following result will be used in the proof of other results.

Lemma 7. *Let $X \subseteq \mathbf{N}$ be such that $0 \in X$ and $\mathbf{N}\backslash X$ is finite. Let $m = \min(X\backslash\{0\})$. Then X is an MED-semigroup if and only if $2m \in X$ and if $x, y \in X\backslash\{0\}$ implies $x + y - m \in X$.*

Proof. The necessary condition is deduced from Lemma 1. Let us see the sufficient condition. As $0 \in X$ and $\mathbf{N}\backslash X$ is finite by hypothesis, in view of Lemma 1, we only have to prove that $x, y \in X$ implies $x+y \in X$, that is, X is a numerical semigroup. If we have either $x = 0$ or $y = 0$, the result follows. Otherwise, if $x, y \in X\backslash\{0\}$, then $x + y - m \in X$.

Since $x+y-m, 2m \in X \backslash \{0\}$, we have $x+y-m+2m-m \in X$, and therefore $x+y \in X$. □

Let S be a numerical semigroup. The pseudo-Frobenius numbers of S (see [**9**]) are the elements of the set

$$\mathrm{PF}(S) = \{x \in \mathbf{Z} \backslash S \mid x+s \in S \text{ for every } s \in S \backslash \{0\}\}.$$

The cardinality of the previous set is an important invariant of S (see [**3**]) called the type of S and denoted by $\mathrm{t}(S)$. If $S = \mathbf{N}$, then $\mathrm{PF}(S) = \{-1\}$ and, if $S \neq \mathbf{N}$, then $\{\mathrm{F}(S)\} \subseteq \mathrm{PF}(S) \subseteq \mathbf{N} \backslash \{0\}$. The following result is easy to prove.

Lemma 8. *Let S be a numerical semigroup different from $\mathbf{N}$. Then $S \cup \mathrm{PF}(S)$ is also a numerical semigroup.*

The following result gives us a necessary condition for a numerical semigroup to be one half of an MED-semigroup with odd multiplicity.

Lemma 9. *If a numerical semigroup S is one half of an* MED-*semigroup with odd multiplicity, then S contains an odd integer smaller than* $2\mathrm{m}(S)$.

Proof. Let $\overline{S}$ be an MED-semigroup such that $S = \overline{S}/2$ and $\mathrm{m}(\overline{S})$ is an odd integer. In view of Proposition 4, we have that $\mathrm{m}(\overline{S}) \in S$. Besides $2\mathrm{m}(S) \in \overline{S}$, and therefore $\mathrm{m}(\overline{S}) < 2\mathrm{m}(S)$. □

In view of the previous lemma and Proposition 2, we obtain that $S = \langle 4, 9, 11 \rangle$ is another example of a numerical semigroup that is not one half of an MED-semigroup.

Given A and B subsets of $\mathbf{Z}$, we denote by $A+B$ the set $\{a+b \mid a \in A \text{ and } b \in B\}$.

Theorem 10. *A numerical semigroup S is one half of an* MED-*semigroup if and only if either S is an* MED-*semigroup or there exists an $m \in S$ such that m is an odd integer, $m < 2\mathrm{m}(S)$ and $x+y+z-m \in S$ for every x, y and z minimal generators of S.*

Proof. If S is not an MED-semigroup, then from Proposition 2 we know that $S = \overline{S}/2$, where $\overline{S}$ is an MED-semigroup such that $\mathrm{m}(\overline{S})$ is an odd integer. By the proof of Lemma 9, we have that $\mathrm{m}(\overline{S}) \in S$ and that $\mathrm{m}(\overline{S}) < 2\mathrm{m}(S)$. By now applying Lemma 4, we have that if x, y and z are minimal generators of S, then $x + y + z - \mathrm{m}(\overline{S}) \in S$.

If S is an MED-semigroup, then the result follows by Proposition 3. Suppose that S is not an MED-semigroup, and so $S \neq \mathbf{N}$. Let us show that

$$\overline{S} = (2S) \cup (\{m\} + 2(S \cup \mathrm{PF}\,(S)))$$

is an MED-semigroup and $S = \overline{S}/2$. Observe that if $s_1, s_2, s_3 \in S\backslash\{0\}$, then $s_1 + s_2 + s_3 - m \in S$, because $s_1 - x, s_2 - y, s_3 - z \in S$ for some x, y and z minimal generators of S and $x + y + z - m \in S$. To prove that $\overline{S}$ is an MED-semigroup, we use Lemma 7. It is clear that $\overline{S} \subseteq \mathbf{N}$ and that $0 \in \overline{S}$. Moreover, every positive integer greater than $m + 2\mathrm{F}(S)$ is in $\overline{S}$ and therefore $\mathbf{N}\backslash\overline{S}$ is finite. As $m < 2\mathrm{m}(S)$, we deduce that $m = \min(\overline{S}\backslash\{0\})$. It is obvious that $2m \in \overline{S}$. Let us prove that if $x, y \in \overline{S}\backslash\{0\}$, then $x + y - m \in \overline{S}$. For that we distinguish three cases.

(1) If $x, y \in 2S$, then $x = 2s_1$ and $y = 2s_2$ for some $s_1, s_2 \in S\backslash\{0\}$. If $2s_1 + 2s_2 - m \notin \{m\} + 2S$, then $2(s_1 + s_2 - m) \notin 2S$ and so $s_1 + s_2 - m \notin S$. We already saw that $s_1 + s_2 - m + s_3 \in S$, for every $s_3 \in S\backslash\{0\}$. Consequently, $s_1 + s_2 - m \in \mathrm{PF}\,(S)$, and therefore $2s_1 + 2s_2 - m \in \{m\} + 2\mathrm{PF}\,(S)$.

(2) If $x \in 2S$ and $y \in \{m\} + 2(S \cup \mathrm{PF}\,(S))$, then $x = 2s$ and $y = m + 2t$, for some $s \in S\backslash\{0\}$ and some $t \in S \cup \mathrm{PF}\,(S)$. Therefore, $x + y - m = 2s + m + 2t - m = 2(s + t) \in 2S$.

(3) If $x, y \in \{m\} + 2(S \cup \mathrm{PF}\,(S))$, then $x = m + 2t_1$ and $y = m + 2t_2$ with $t_1, t_2 \in S \cup \mathrm{PF}\,(S)$. Whence $x + y - m = m + 2(t_1 + t_2) \in \{m\} + 2(S \cup \mathrm{PF}\,(S))$, in view of Lemma 8.

Finally, it is clear that $S = \overline{S}/2$. □

We end this section by illustrating the previous theorem with two examples.

Example 11. Let us see that $S = \langle 4, 5, 7\rangle$ is one half of an MED-semigroup. Observe that 5 is an odd integer of S smaller than the double of the multiplicity of S. The reader can easily verify that

$\{x + y + z - 5 \mid x, y, z \in \{4, 5, 7\}\} \subseteq S$. Therefore, by applying the previous theorem we have that S is one half of an MED-semigroup. Besides, $\mathrm{PF}(S) = \{3, 6\}$ and, in view of the proof of Theorem 10, we know that $\overline{S} = (2S) \cup (\{5\} + 2(S \cup \mathrm{PF}(S))) = \langle 5, 8, 11, 14, 17\rangle$ is an MED-semigroup such that $S = \overline{S}/2$.

Example 12. Let us see that $S = \langle 4, 5, 6\rangle$ is not one half of an MED-semigroup. Observe that 5 is the only odd integer of S smaller than $2\mathrm{m}(S) = 8$. Besides, $4+4+4-5 = 7 \notin S$. Therefore, by applying Theorem 10, we can state that S is not one half of an MED-semigroup.

4. Numerical semigroups that are one half of an MED-semigroup with odd multiplicity. From Theorem 6 we know that $\{S \mid S$ is an MED-semigroup$\}$ is equal to $\{S/2 \mid S$ is an MED-semigroup and $\mathrm{m}(S)$ is even$\}$. In this section we are interested in the study of the set $\{S/2 \mid S$ is an MED-semigroup and $\mathrm{m}(S)$ is odd$\}$. This set contains numerical semigroups that are not MED-semigroups (see Example 11) and also there exist MED-semigroups that do not belong to this set. In fact, by applying Lemma 9, we know that $S = \langle 4, 9, 10, 11\rangle$ is an MED-semigroup that is not one half of an MED-semigroup with odd multiplicity.

The next result is deduced from [**3**].

Lemma 13. *Let S be a numerical semigroup. Then S is an* MED-*semigroup if and only if* $(S\backslash\{0\}) + \{-\mathrm{m}(S)\}$ *is also a numerical semigroup.*

Notice that Lemma 13 is a reformulation of Lemma 1 and that $S \subseteq (S\backslash\{0\}) + \{-\mathrm{m}(S)\}$.

The following result is fundamental to the development of this section.

Lemma 14. *Let S be an* MED-*semigroup with odd multiplicity. Then*

$$S = 2\left(\frac{S}{2}\right) \cup \left(\{\mathrm{m}(S)\} + 2\left(\frac{(S\backslash\{0\}) + \{-\mathrm{m}(S)\}}{2}\right)\right).$$

Proof. It is clear that $2(S/2) \subseteq S$. If $x \in [(S\backslash\{0\}) + \{-\mathrm{m}(S)\}]/2$, then $2x + \mathrm{m}(S) \in S$. Thus $\{\mathrm{m}(S)\} + 2(((S\backslash\{0\}) + \{-\mathrm{m}(S)\})/2) \subseteq S$. Therefore, $2(S/2) \cup (\{\mathrm{m}(S)\} + 2(((S\backslash\{0\}) + \{-\mathrm{m}(S)\})/2)) \subseteq S$.

Let us now see that $S \subseteq 2(S/2) \cup (\{\mathrm{m}(S)\} + 2(((S\backslash\{0\}) + \{-\mathrm{m}(S)\})/2))$. If $s \in S$ and s is even, then $s \in 2(S/2)$. If s is odd, then $(s - \mathrm{m}(S))/2 \in [(S\backslash\{0\}) + \{-\mathrm{m}(S)\}]/2$, and therefore $s \in \{\mathrm{m}(S)\} + 2(((S\backslash\{0\}) + \{-\mathrm{m}(S)\})/2)$. □

As an immediate consequence of Lemmas 13 and 14, we have the following.

Lemma 15. *Let* $\overline{S}$ *be an* MED-*semigroup with odd multiplicity, and let* $S = \overline{S}/2$. *Then* $\overline{S} = (2S) \cup (\{\mathrm{m}(\overline{S})\} + 2A)$ *where* A *is a numerical semigroup containing* S.

The next result characterizes the numerical semigroups A of the previous lemma such that $(2S) \cup (\{m\} + 2A)$ is an MED-semigroup with odd multiplicity m.

Lemma 16. *Let* S *and* A *be numerical semigroups such that* $S \subseteq A$, *and let* m *be an odd integer of* S *such that* $m < 2\mathrm{m}(S)$. *Then* $\overline{S} = (2S) \cup (\{m\} + 2A)$ *is an* MED-*semigroup if and only if* $(S\backslash\{0\}) + (S\backslash\{0\}) + \{-m\} \subseteq A \subseteq S \cup \mathrm{PF}\,(S)$.

Proof. To prove the necessary condition, we begin by proving that if $s_1, s_2 \in S\backslash\{0\}$, then $s_1 + s_2 - m \in A$. As $\overline{S}$ is an MED-semigroup with multiplicity m, then by Lemma 1, $2s_1 + 2s_2 - m \in \overline{S}$. Thus, $2s_1 + 2s_2 - m \in \{m\} + 2A$ and so $s_1 + s_2 - m \in A$. We now prove that $A \subseteq S \cup \mathrm{PF}\,(S)$. Let $a \in A\backslash S$. By applying Lemma 1 and that $\overline{S}$ is an MED-semigroup with multiplicity m, we have that if $s \in S\backslash\{0\}$, then $2s + 2a + m - m \in \overline{S}$. Hence, $s + a \in S$. Therefore, $a \in \mathrm{PF}\,(S)$.

For the sufficient condition, we apply Lemma 7 to prove that $\overline{S}$ is an MED-semigroup. It is clear that $\overline{S} \subseteq \mathbf{N}$, $0 \in \overline{S}$, $\mathbf{N}\backslash\overline{S}$ is finite, $m = \min(\overline{S}\backslash\{0\})$ and $2m \in \overline{S}$. Let us prove that if $x, y \in \overline{S}\backslash\{0\}$, then $x + y - m \in \overline{S}$. For that we distinguish three cases.

(1) If $x, y \in 2S$, then $x = 2s_1$ and $y = 2s_2$, for some $s_1, s_2 \in S\backslash\{0\}$. Since, by hypothesis, we have $s_1 + s_2 - m \in A$, we obtain $x + y - m = 2s_1 + 2s_2 - m \in \{m\} + 2A$.

(2) If $x \in 2S$ and $y \in \{m\} + 2A$, then $x = 2s$ and $y = m + 2a$, for some $s \in S\backslash\{0\}$ and some $a \in A$. As $A \subseteq S \cup \mathrm{PF}\,(S)$ by hypothesis, we have that $s + a \in S$. Thus $x + y - m = 2s + m + 2a - m = 2(s + a) \in 2S$.

(3) If $x, y \in \{m\} + 2A$, then $x = m + 2a_1$ and $y = m + 2a_2$, for some $a_1, a_2 \in A$. Note that $a_1 + a_2 \in A$. Therefore $x + y - m = m + 2a_1 + m + 2a_2 - m \in \{m\} + 2A$. □

We are now ready to give the characterization of the set $\{S/2 \mid S \text{ is an MED-semigroup and } \mathrm{m}(S) \text{ is odd}\}$.

Theorem 17. *Let S be a numerical semigroup. Then S is one half of an* MED*-semigroup with odd multiplicity if and only if there exists an odd integer m of S such that $m < 2\mathrm{m}(S)$ and $(S\backslash\{0\}) + (S\backslash\{0\}) + \{-m\} \subseteq S \cup \mathrm{PF}\,(S)$.*

Proof. The necessary condition follows from Lemmas 15 and 16.

If $S = \mathbf{N}$, then $\mathbf{N} = \mathbf{N}/2$. If $S \neq \mathbf{N}$, by Lemma 8 we know that $S \cup \mathrm{PF}\,(S)$ is a numerical semigroup. The proof concludes by applying Lemma 16. □

Given a numerical semigroup S we denote by $\Omega(S)$ the set $\{S' \mid S' \text{ is an MED-semigroup, } \mathrm{m}(S') \text{ is odd and } S = S'/2\}$.

Corollary 18. *Let S be a numerical semigroup. Then $\Omega(S)$ is finite.*

Proof. If $\Omega(S) = \varnothing$, then the result is obvious. If $S = \mathbf{N}$ and $\overline{S} \in \Omega(\mathbf{N})$, then $2 \in \overline{S}$. Moreover, as $\mathrm{m}(\overline{S})$ is an odd integer we deduce that $1 \in \overline{S}$ and so $\overline{S} = \mathbf{N}$. Consequently $\Omega(\mathbf{N}) = \{\mathbf{N}\}$. Suppose now that $S \neq \mathbf{N}$ and $\Omega(S) \neq \varnothing$. From Lemma 15 we know that if $S' \in \Omega(S)$, then S' is fully determined by a pair (m, A) where m is an odd integer of S smaller than $2\mathrm{m}(S)$ and A is a numerical semigroup that contains S. To conclude the proof it suffices to observe that S contains only a finite number of odd integers smaller than $2\mathrm{m}(S)$ and that the set of numerical semigroups that contains S is also finite. □

If S is a numerical semigroup, then the cardinality of the set $\mathbf{N}\backslash S$ is an important invariant of S known either as the singularity degree of S (see [**6**]) or the genus of S (see [**3**]). We denote it by $\mathrm{g}\,(S)$. The elements of $\mathbf{N}\backslash S$ are the so called gaps of S.

Corollary 19. *Let $S \neq \mathbf{N}$ be a numerical semigroup, and let m be an odd integer of S such that $m < 2\mathrm{m}(S)$ and $(S\backslash\{0\}) + (S\backslash\{0\}) + \{-m\} \subseteq S \cup \mathrm{PF}\,(S)$. Then*

(1) $\overline{S} = (2S) \cup (\{m\} + 2(S \cup \mathrm{PF}\,(S)))$ *is the maximum, with respect to the set inclusion order,* MED*-semigroup with multiplicity m such that $S = \overline{S}/2$;*

(2) $\mathrm{F}\,(\overline{S}) = 2\mathrm{F}\,(S)$;

(3) $\mathrm{g}\,(\overline{S}) = 2\mathrm{g}\,(S) + (m-1)/2 - \mathrm{t}\,(S)$.

Proof. (1) It is a consequence of Lemmas 8, 15 and 16.

(2) It is clear that $2\mathrm{F}(S)$ is the greatest even integer that does not belong to $\overline{S}$.

It is also clear that $\mathrm{F}(S) - \mathrm{m}(S) \notin S \cup \mathrm{PF}(S)$. Let us show that $\{\mathrm{F}(S) - \mathrm{m}(S) + 1, \rightarrow\} \subseteq S \cup \mathrm{PF}(S)$. Let $x \in \mathbf{N}\backslash\{0\}$. We have to prove that $\mathrm{F}(S) - \mathrm{m}(S) + x \in S \cup \mathrm{PF}(S)$. If $\mathrm{F}(S) < \mathrm{m}(S)$, then $S = \{0, \mathrm{m}(S), \rightarrow\}$, $\mathrm{F}(S) = \mathrm{m}(S) - 1$, $S \cup \mathrm{PF}(S) = \mathbf{N}$ and so $\mathrm{F}(S) - \mathrm{m}(S) + x \in S \cup \mathrm{PF}(S)$. Otherwise, $\mathrm{F}(S) > \mathrm{m}(S)$ and $\mathrm{F}(S) - \mathrm{m}(S) + x \in \mathbf{N}$. If $\mathrm{F}(S) - \mathrm{m}(S) + x \notin S$, then for every $s \in S\backslash\{0\}$, we have $s + \mathrm{F}(S) - \mathrm{m}(S) + x \in S$, because $s - \mathrm{m}(S) + x > 0$. Thus, $\mathrm{F}(S) - \mathrm{m}(S) + x \in \mathrm{PF}(S)$. Whence we deduce that $m + 2(\mathrm{F}(S) - \mathrm{m}(S)) = 2\mathrm{F}(S) + m - 2\mathrm{m}(S)$ is the greatest odd integer that does not belong to $\overline{S}$.

Notice that $m < 2\mathrm{m}(S)$ by hypothesis, and so $2\mathrm{F}(S) + m - 2\mathrm{m}(S) < 2\mathrm{F}(S)$. Therefore, $\mathrm{F}(\overline{S}) = 2\mathrm{F}(S)$.

(3) It is obvious that $\overline{S}$ has $g(S)$ even gaps and $(m-1)/2 + \mathrm{g}(S) - \mathrm{t}(S)$ odd gaps. Therefore, $\mathrm{g}(\overline{S}) = 2\mathrm{g}(S) + (m-1)/2 - \mathrm{t}(S)$. □

It is well known (see for example [**9**]) that a numerical semigroup $S \neq \mathbf{N}$ is an MED-semigroup if and only if $\mathrm{t}(S) = \mathrm{m}(S) - 1$.

Corollary 20. *Let $S \neq \mathbf{N}$ be an* MED-*semigroup with odd multiplicity. Then $\overline{S} = (2S) \cup (\{\mathrm{m}(S)\} + 2(S \cup \mathrm{PF}(S)))$ is an* MED-*semigroup with $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$, $S = \overline{S}/2$, $\mathrm{F}(\overline{S}) = 2\mathrm{F}(S)$ and $\mathrm{g}(\overline{S}) = 2\mathrm{g}(S) - (\mathrm{m}(S) - 1)/2$.*

Proof. By Lemma 1 we know that $(S\backslash\{0\}) + (S\backslash\{0\}) + \{-\mathrm{m}(S)\} \subseteq S \subseteq S \cup \mathrm{PF}(S)$. The proof follows by applying Corollary 19 and that $\mathrm{t}(S) = \mathrm{m}(S) - 1$. □

As a consequence of Corollary 20 we have that every MED-semigroup with odd multiplicity is one half of an MED-semigroup with odd multiplicity. Recall that, at the beginning of this section, we saw that there exist MED-semigroups with even multiplicity that are not one half of an MED-semigroup with odd multiplicity.

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