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NORMAL DIFFERENTIAL OPERATORS OF FIRST-ORDER WITH SMOOTH COEFFICIENTS

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ABSTRACT. In this work, in terms of boundary values all normal extensions of a class of formally normal minimal operators, generated by a linear differential-operator expression of first-order with smooth operator coefficients are described in Hilbert space of vector-functions in a finite interval. The structure of the spectrum of the normal extensions is investigated.

1. Introduction. A densely defined closed operator N in a Hilbert space H is called *formally normal* if $D(N) \subset D(N^*)$ and $\|Nf\|_H = \|N^*f\|_H$ for all $f \in D(N)$. If a formally normal operator has no formally normal non-trivial extension, then it is called a *maximal formally normal operator*. If a formally normal operator N satisfies the condition $D(N) = D(N^*)$, then it is called a *normal operator* [**4**]. The densely defined closed operator N is normal if and only if $NN^* = N^*N$ [**13**].

The first results in the area of normal extension of unbounded formally normal operators in a Hilbert space are due to Kilpi [**14–16**] and Davis [**5**]; furthermore, Coddington [**4**], Biriuk and Coddington [**2**] and Stochel and Szafraniec [**20–22**] established and developed it as a general theory. However, application of this theory to the theory of differential operators in Hilbert space has not received the attention it deserves [**10**].

Evidently, for the formal normality of the minimal operator generated by a linear differential expression of any natural order with variable operator coefficients in the Hilbert space of vector-functions in finite interval, these operator coefficients must satisfy some additional conditions. The detailed analysis of the relationship between the normality property in the Hilbert space $L^2(H, (a, b))$, H a separable Hilbert space,

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$a, b \in \mathbf{R}$ and operator coefficients of the differential-operator expression which generates the minimal operator was given in [**6, 10, 12, 18**]. These papers also contain some facts about the structure of such normal operators.

In the case of bounded constant normal operator coefficients, description of all the normal extensions in terms of boundary values and spectral analysis of these extensions has been researched in [**11**]. Moreover, the case of unbounded constant normal operator coefficients may be found in [**9**].

The main purpose of this work is to generalize the results obtained in this area to a differential operator of first-order in $L^2(H,(a,b))$ with a variable operator coefficient when a strong derivative of this coefficient in (a, b) is equal to zero almost everywhere. Note that even if $\dim H = 1$ and the derivative of a function $f : [a, b] \to \mathbf{R}$ is zero, it doesn't imply that f = constant almost everywhere in (a, b). For example, strongly monotone and continuous functions exist such that their derivatives are equal to zero almost everywhere (see [**7, 19**]).

Throughout the paper it is supposed that H is a separable Hilbert space, $B(H)$ is the space of linear bounded operators in H and $L^2 := L^2(H,(a,b))$ is the Hilbert space of vector-functions from a finite interval $[a, b]$ to H [**6**].

2. Description of normal extensions. In the space L^2 consider a linear differential-operator expression of first-order in the form

$$l(u) = u'(t) + A(t)u(t), \tag{2.1}$$

where $A(t) : [a, b] \to B(H)$ is a linear bounded, strongly continuous and a self-adjoint operator function.

In this case the family $A(t)$, $t \in [a, b]$, is uniformly bounded on $[a, b]$, and the Cauchy problem for the equation corresponding to (2.1) is solvable and well posed. It is clear that the formally adjoint expression in space L^2 is in the form

$$l^+(v) = -v'(t) + A(t)v(t). \tag{2.2}$$

Now let us define the operator L'_0 in L^2 on the dense manifold of the vector-functions

$$D_0' := \Big\{ u(t) \in L^2 : u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k,\ \varphi_k(t) \in C_0^\infty(a,b),\ f_k \in H, \\ k = 1, 2, \dots, n, \quad n \in \mathbf{N} \Big\}$$

as $L_0' := l(u)$.

The closure of L_0' in L^2 is called the *minimal operator* generated by differential-operator expression (2.1), and it is denoted by L_0.

In a similar way the minimal operator L_0^+ in L^2 generated by the differential-operator expression (2.2) may be defined. The adjoint operator of L_0^+ (L_0) in L^2 is called the *maximal operator* generated by (2.1) ((2.2)) and is denoted by L (L^+) [**1, 8**]. It is clear that $L_0 \subset L$, $L_0^+ \subset L^+$, $D(L_0) = \overset{0}{W}{}_2^1(H,(a,b))$ and $D(L) = W_2^1(H,(a,b))$.

Next we prove the following theorem which connects a formally normal property of the minimal operator L_0 and the operator coefficients of the differential-operator expression (2.1) which generates L_0.

Theorem 2.1. *Suppose that the operator-function $A(t)$ is strongly differentiable in the interval (a,b) and $\|A'(t)\| \in L^2(a,b)$. Then the minimal operator L_0 is formally normal in the space L^2 if and only if $A'(t) = 0$ almost everywhere in (a,b).*

Proof. Suppose that L_0 is a formally normal operator in L^2; then, for each $u(t) \in D(L_0) \subset D(L^+)$, the following equality holds

$$\|L_0 u\|_{L^2}^2 - \|L^+ u\|_{L^2}^2 = 2\left[(u', A(t)u)_{L^2} + (A(t)u, u')_{L^2}\right] = 0.$$

By using this, for special vector-functions $u(t) = \varphi(t) f \in D(L_0)$, $\varphi = \overline{\varphi}$, $\varphi(a) = \varphi(b) = 0$, $f \in H$, we obtain

$$\int_a^b \sigma_f'(t) |\varphi(t)|^2 dt = 0$$

where $\sigma_f(t) := (A(t)f, f)_H$, $a \le t \le b$.

Since the last relation holds for each real-valued function φ from $\overset{0}{W}{}_2^1(a,b)$, then for all $\varphi, \psi \in \overset{0}{W}{}_2^1(a,b)$ we get

$$\int_a^b \sigma'_f(t)|\varphi(t)+\psi(t)|^2 dt = 2\int_a^b [\sigma'_f(t)\varphi(t)]\psi(t)\,dt = 0.$$

Now $\psi \in \overset{0}{W}{}_2^1(a,b)$ is an arbitrary function and it follows from this equality that

$$\sigma'_f(t)\varphi(t) = 0 \text{ almost everywhere in } (a,b).$$

Since the function φ is an arbitrary function from $\overset{0}{W}{}_2^1(a,b)$, then from the last equality it is seen that $\sigma'_f(t) = 0$ almost everywhere in (a,b). On the other hand, the last relation holds for every $f \in H$ which is why for any $f, g \in H$ $(A'(t)f, g)_H = 0$ almost everywhere in (a,b). Hence $A'(t) = 0$ almost everywhere in (a,b). Vice versa, if $A'(t) = 0$ almost everywhere in (a,b), then the formal normality of the minimal operator L_0 in the space L^2 can be easily verified. □

Note that, as mentioned in the Introduction, in general the condition $A'(t) = 0$ almost everywhere in (a,b) does not imply that the operator-function $A(t)$, $t \in [a,b]$, is piecewise constant.

In the next section our main purpose is, in terms of boundary values, to describe all normal extensions of the minimal operator L_0 when $A'(t) = 0$ almost everywhere in (a,b).

Theorem 2.2. *Let, for each $t \in [a,b]$, $A(t)$ be a linear bounded self-adjoint operator in H, $A(a), A(b) \geq E$, where $E : H \to H$ is the identity operator, let the operator-function $A(t) : [a,b] \to B(H)$ be strongly differentiable in H on $[a,b]$ and $A'(t) = 0$ almost everywhere in (a,b).*

Then every normal extension L_n, $L_0 \subset L_n \subset L$, of the minimal operator L_0 in L^2 is generated by the differential-operator expression (2.1) *and the boundary condition*

$$u(b) = Wu(a) \tag{2.3}$$

where W is a unitary operator in H with property $WA(a) = A(b)W$. The unitary operator W is determined uniquely by the extension L_n, i.e., $L_n = L_W$.

On the contrary, the restriction of the maximal operator L to the manifold of vector-functions $u(t) \in W_2^1(H, (a, b))$ that satisfy (2.3) for a unitary operator W in H with the property $WA(a) = A(b)W$, is a normal extension of the minimal operator L_0 in L^2.

Proof. Let L_n be a normal extension of L_0. In this case

$$\begin{aligned} \mathrm{Re}\,(L_n)u &= A(t)u(t), \quad u \in D(L_n), \\ \mathrm{Im}\,(L_n)u &= -i\frac{d}{dt}u(t), \quad u \in D(L_n) \end{aligned}$$

are self-adjoint operators in L^2.

It is easily verified that the triple $(\mathcal{H}, \gamma_1, \gamma_2)$, where

$$\begin{aligned} \mathcal{H} &:= H, \\ \gamma_1(u) &:= \frac{u(a) + u(b)}{\sqrt{2}}, \\ \gamma_2(u) &:= \frac{u(a) - u(b)}{i\sqrt{2}} \end{aligned}$$

is a boundary value space for the minimal operator $\mathrm{Im}\,(L_0)$ in L^2 [**8**].

It is known that a self-adjoint extension $\mathrm{Im}\,(L_n)$ of the minimal operator $\mathrm{Im}\,(L_0)$ in L^2 is described as the following boundary condition

$$(W - E)\gamma_1(u) + i(W + E)\gamma_2(u) = 0,$$

with a uniquely unitary operator W in H, i.e.,

$$u(b) = Wu(a), \quad u \in D(L_n).$$

On the other hand since the extension L_n is a normal operator, then for every $u(t) \in D(L_n)$ the following equality holds

$$(\mathrm{Re}\,(L_n)u, \mathrm{Im}\,(L_n)u)_{L^2} = (\mathrm{Im}\,(L_n)u, \mathrm{Re}\,(L_n)u)_{L^2}\,.$$

In other words, for every $u(t) \in D(L_n)$,

$$(u', A(t)u)_{L^2} + (A(t)u, u')_{L^2} = 0$$

is satisfied. From this relation it is obtained that

$$(u, A(t)u)'_{L^2} + (A'(t)u, u)_{L^2} = 0, \quad u \in D(L_n),$$

i.e.,

$$\begin{aligned}(u, A'(t))'_{L^2} &= (u(b), A(b)u(b))_H - (u(a), A(a)u(a))_H \\ &= \|A^{1/2}(b)u(b)\|_H^2 - \|A^{1/2}(a)u(a)\|_H^2 = 0.\end{aligned}$$

Hence, for every $u \in D(L_n)$, the equality $\|A^{1/2}(b)u(b)\|_H^2 = \|A^{1/2}(a) \times u(a)\|_H^2$ holds. Therefore, an isometric operator $V : H \to H$ exists such that $A^{1/2}(b)u(b) = VA^{1/2}(a)u(a)$, $u \in D(L_n)$.

Let us set $U := A^{-1/2}(b)VA^{1/2}(a)$. In this case we have

$$(U - E)\gamma_1(u) + i(U + E)\gamma_2(u) = 0, \quad u \in D(L_n).$$

Since the unitary operator W is determined uniquely by the expression $\mathrm{Im}\,(L_n)$, then U is a unitary operator and $U = W$, i.e.,

$$A^{-1/2}(b)VA^{1/2}(a) = W, \qquad VA^{1/2}(a) = A^{1/2}(b)W.$$

Also it follows from this result that V is a unitary operator and $V = A^{1/2}(b)WA^{-1/2}(a)$. Hence, $VV^* = V^*V = E$ and consequently $WA(a) = A(b)W$. It is clear that the unitary operator W is determined uniquely by the extension L_n.

Now let L_W be an operator generated by differential-operator expression $l(u)$ with the boundary condition (2.3) in L^2, that is,

$$L_W u = l(u), \qquad D(L_W) = \left\{u \in W_2^1(H, (a,b)) : u(b) = Wu(a)\right\},$$

where W is a unitary operator in H with property $WA(a) = A(b)W$.

In this case the adjoint operator L_W^* is generated by the differential-operator expression $l^*(v) = -v'(t) + A(t)v(t)$ with the boundary condition $v(a) = W^*v(b)$, $v(t) \in D(L_W^*)$. It is easy to see that $D(L_W) = D(L_W^*)$ and the other condition of normal extensions in L^2 can be easily verified. $\square$

Remark 2.1. An analog of Theorem 2.2 can be established in the case where, for each $t \in [a,b]$, $A(t)$ is a linear bounded normal operator in H, $A_R(t)$ is strongly differentiable in $[a,b]$ and $A'_R(t) = 0$ almost everywhere in (a,b).

Remark 2.2. By Theorem 2.2 Dirichlet's extension ($W = E$) is a normal operator in L^2 if and only if an operator $A^{1/2}(b)A^{-1/2}(a)$ is unitary in H.

Remark 2.3. In the case where $A(t) = A = \text{constant}$, $t \in [a,b]$, the result obtained in Theorem 2.2 has been established in [**11**].

Example 2.1. Consider in the space $L^2(-(\pi/4),(\pi/4))$ the differential expression

$$l(u) = u'(t) + \tau(\sin^2 t)u(t), \quad t \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right),$$

where $\tau(\cdot)$ is a Cantor function.

By Theorem 2.2, all normal extensions l_φ of the minimal operator l_0 in $L^2(-(\pi/4),(\pi/4))$ are generated by $l(u)$, and the boundary condition is

$$u\left(\frac{\pi}{4}\right) = e^{i\varphi}u\left(-\frac{\pi}{4}\right), \quad \varphi \in [0,2\pi),$$

because $\tau(\sin^2(-(\pi/4)))e^{i\varphi} = e^{i\varphi}\tau(\sin^2(\pi/4))$ holds for every $\varphi \in [0,2\pi)$.

3. The spectrum of normal extensions. In this section the spectrum of the normal extension L_W of minimal operator L_0 generated by the linear differential-operator expression (2.1) in L^2 and boundary conditions (2.3) with the unitary operators W and $A^{1/2}(b)WA^{-1/2}(a)$ in H will be investigated.

Now let $U(t,s)$, $t,s \in [a,b]$ be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} U'_t(t,s)f + A(t)U(t,s)f = 0 & t,s \in [a,b] \\ U(s,s)f = f & f \in H \end{cases}$$

(for a more detailed analysis, see [**3, 17**]).

Theorem 3.1. *The spectrum of the normal extension L_W has the form*

$$\sigma(L_W) = \Big\{\lambda \in \mathbf{C} : \lambda = \lambda_0 + \frac{2k\pi i}{b-a}, \text{ where } \lambda_0 \text{ takes values from the set of solutions of the equation } e^{-\lambda_0(b-a)} - \mu = 0, \mu \in \sigma\left(W^*U(b,a)\right), \ k \in \mathbf{Z}\Big\}.$$

Proof. Consider a problem for the spectrum for the normal extension L_W, i.e.,

$$L_W u = u'(t) + A(t)u(t) = \lambda u(t) + f(t), \quad u \in D(L_W), \ f \in L^2, \ \lambda \in \mathbf{C}.$$

It is clear that the general solution of this differential equation in L^2 has the form

$$u_\lambda(t) = e^{\lambda(t-a)}U(t,a)f + \int_a^t e^{\lambda(t-s)}U(t,s)f(s)\,ds, \quad f \in H.$$

In this case from the boundary condition $u_\lambda(b) = Wu_\lambda(a)$, where W and $A^{1/2}(b)WA^{-1/2}(a)$ are unitary operators in H, the following relation is found

$$\left(W^*U(b,a) - e^{-\lambda(b-a)}\right) f = -W^* \int_a^b e^{\lambda(a-s)}U(b,s)f(s)\,ds.$$

From this it is easy to see that $\lambda \in \mathbf{C}$ is a point of the spectrum of normal extension L_W if and only if

$$e^{-\lambda(b-a)} = \mu \in \sigma(W^*U(b,a)).$$

Therefore,

$$\lambda = \lambda_0 + \frac{2k\pi i}{b-a},$$

where $e^{-\lambda_0(b-a)} \in \sigma(W^*U(b,a))$, $k \in \mathbf{Z}$. □

From this theorem the validity of the following results is clear.

Corollary 3.1. *The spectrum $\sigma(L_W)$ of the normal extension L_W can be defined as*

$$\sigma(L_W) = \Big\{\lambda \in \mathbf{C} : \lambda = \frac{1}{b-a}\left(\ln|\mu| + i(\arg\mu + 2k\pi)\right), \\ \mu \in \sigma(W^*U(b,a)),\ k \in \mathbf{Z}\Big\}.$$

Corollary 3.2. *The spectrum $\sigma(L_W)$ of the normal extension L_W is nonempty and at infinity.*

Corollary 3.3. *If $\dim H < +\infty$, then each normal extension L_W has a pure point spectrum and their eigenvalues have the same asymptotics*

$$\lambda_k(L_W) \sim \frac{2k\pi}{b-a}, \quad \text{as } k \to \infty.$$

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