ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 1, 2012

CLASSICAL SOLUTIONS OF HYPERBOLIC DIFFERENTIAL SYSTEMS WITH STATE DEPENDENT DELAYS

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ABSTRACT. We consider the generalized Cauchy problem for a system of quasilinear partial functional differential equations of the first order

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,V(z;t,x))\, \partial_{x_j} z_i(t,x) = G_i(t,x,V(z;t,x)), \quad 1 \le i \le m,$$

where V is a nonlinear operator of Volterra type, mapping bounded (with respect to seminorm) subsets of the space of Lipschitz-continuously differentiable functions, into bounded subsets of this space.

Using the method of bicharacteristics and the fixed-point theorem we prove the local existence, uniqueness and continuous dependence on data of classical solutions of the problem.

This approach covers systems of the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi(t,x,z_{(t,x)})})\partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \quad 1 \le i \le m,$$

where $(t,x) \mapsto z_{(t,x)}$ is the Hale operator, and all the components of ψ may depend on $(t,x,z_{(t,x)})$. More specifically, problems with deviating arguments and integro-differential systems are included.

1. Introduction. We formulate the functional differential problem. Let $a > 0$, $h_0 \in \mathbf{R}_+$, $\mathbf{R}_+ = [0,+\infty)$, and $h = (h_1, \dots, h_n) \in \mathbf{R}_+^n$ be given. We define the family of sets

$$E_t = [-h_0, t] \times \mathbf{R}^n, \quad t \in [0,a]$$

2010 AMS *Mathematics subject classification.* Primary 35R10, 35L45.
Keywords and phrases. Partial functional differential systems, classical solutions, local existence, bicharacteristics.
Received by the editors on April 11, 2009, and in revised form on June 29, 2009.
DOI:10.1216/RMJ-2012-42-1-71

and the sets

$$E = E_a \setminus E_0, \quad D = [-h_0, 0] \times [-h, h].$$

For k, l arbitrary positive integers, we denote by $M_{k,l}$ the class of all $k \times l$ matrices with real elements, and we choose the norms in $\mathbf{R}^k$ and $M_{k,l}$ to be ∞-norms: $\|y\| = \|y\|_\infty = \max_{1 \le i \le k} |y_i|$ and $\|A\| = \|A\|_\infty = \max_{1 \le i \le k} \sum_{j=1}^{l} |a_{ij}|$, respectively, where $A = [a_{ij}]_{i=1,\dots,k, j=1,\dots,l}$. The product of two matrices is denoted by "$*$." For $U \subset \mathbf{R}^{1+n}$ and a normed space Y, equipped with the norm $\|\cdot\|_Y$, we define $C(U, Y)$ to be the set of all continuous functions $w : U \to Y$; this space is equipped with the usual supremum norm $\|w\|_{C(U,Y)} = \sup_{P \in U} \|w(P)\|_Y$. We write it simply $C(U)$ when no confusion can arise.

Put $X = C(D, \mathbf{R}^m)$. Let $V : C(E_a, \mathbf{R}^m) \times E \to X$, in variables $(z; t, x)$, be a nonlinear Volterra operator. By the Volterra property we mean that, for z, $\overline{z} \in C(E_a, \mathbf{R}^m)$ and $t \in [0, a]$,

$$z(\tau, x) = \overline{z}(\tau, x) \quad \text{for } \tau \le t$$

implies that

$$V(z; \tau, x) \equiv V(\overline{z}; \tau, x) \quad \text{for } \tau \le t.$$

Let

$$\rho_{ij},\ G_i : \Omega \longrightarrow \mathbf{R} \quad \text{and} \quad \varphi_i : E_{0.i} \longrightarrow \mathbf{R},$$
$$1 \le i \le m,\ 1 \le j \le n,$$

be given, where Ω stand for $E \times X$ and $E_{0.i} = E_{a_i}$, $0 \le a_i < a$, $1 \le i \le m$. We consider the hyperbolic functional differential system

$$\partial_t z_i(t, x) + \sum_{j=1}^{n} \rho_{ij}(t, x, V(z; t, x)) \partial_{x_j} z_i(t, x) = G_i(t, x, V(z; t, x)) \qquad (1)$$
$$1 \le i \le m,$$

augmented with the initial conditions

$$z_i(t, x) = \varphi_i(t, x) \quad \text{on} \quad E_{0.i}, \quad 1 \le i \le m. \qquad (2)$$

A function $\widetilde{z} \in C^1(E_c, \mathbf{R})$, where $\widetilde{a} < c \le a$, $\widetilde{a} = \max\{a_i : 1 \le i \le m\}$, is a classical solution of (1) and (2) if condition (2) holds and, for each $1 \le i \le m$, the ith equation in (1) is satisfied on $[a_i, c] \times \mathbf{R}^n$.

The theory of ordinary differential equations with state dependent delays was prior to investigations of partial equations of this type. Among major works in this field are [**14**] with two consecutive companion papers, on limiting profiles of periodic solutions, and [**19**], in which the general theory of invariant manifolds was developed. Further bibliography may be found in [**5**].

Existence results for abstract partial differential equations with state dependent delay are given in [**6**]. A new class of nonlocal equations with state-selective delay was introduced in [**16**]. Discrete state dependent delay for partial equations was investigated in [**15**].

Note that different models of the functional dependence in partial equations are used in the literature. The first group of results is connected with initial problems for equations

$$\partial_t x(t,x) = G(t,x,z,\partial_x z(t,x)) \tag{3}$$

where the variable z represents the functional argument. This model is suitable for differential functional inequalities generated by initial problems considered on the Haar pyramid. Existence results for (3) can be characterized as follows: theorems have simple assumptions and their proofs are very natural (see [**17, 18**]). Unfortunately, a small class of differential functional problems is covered by this theory. There are a lot of papers concerning initial value problems for equations

$$\partial_t z(t,x) = H(t,x,W[z](t,x),\partial_x z(t,x)) \tag{4}$$

where W is an operator of Volterra type and H is defined on the finite-dimensional Euclidean space. The main assumptions in existence theorems for (4) concern the operator W. They are formulated ([**1, 9**]) in terms of inequalities for norms in some functional spaces.

A new model of a functional dependence is proposed in [**7, 8**]. Partial equations have the form

$$\partial_t z(t,x) = F(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{5}$$

where $z_{(t,x)}$ is a functional variable. This model is well known for ordinary functional differential equations (see, for example, [**4, 12, 13**]). It is also very general since equations with deviating variables, integral differential equations, and equations of forms (3) and (4) can be obtained from (5) by specifying the operator F. In the paper we use the model (5). In existence results [**11**], concerning partial differential equations with state dependent delays, Carathéodory solutions were considered and the functional variable was

$$z_{(\psi_0(t),\psi'(t,x,z_{(t,x)}))}.$$

We deal in this paper with a slightly wider class of deviating functions, admitting functional variables of the form

$$z_{(\psi_0(t,x,z_{(t,x)}),\psi'(t,x,z_{(t,x)}))},$$

and we consider classical solutions of the respective problem.

Cases of more (or less) complicated deviating functions are also covered by our operator formulation.

Delay systems with state dependent delays occur as models for the dynamics of diseases when the mechanism of infection is such that the infectious dosage received by an individual has to reach a threshold before the resistance of the individual is broken down and as a result the individual becomes infectious. A prototype of such model was proposed in [**2**].

The aim of this paper is to prove a theorem on the existence and continuous dependence of classical solutions to (1), (2). Apart from classical solutions, there are two types of generalized solutions to non-linear partial functional differential problems of the first order. Solutions in the Cinquini-Cibrario sense are close to classical solutions: assuming continuity in time, instead of measurability and upper-boundedness by an integrable function, makes Cinquini-Cibrario solutions turn into classical ones. In the case of existence results for Carathéodory solutions, which are investigated in [**11**], there is no refinement of assumptions leading to classical solutions, so a new proof is necessary.

To prove existence of classical solutions, one may replace the spatial derivative of z by a new unknown function u and solve an equivalent, functional integral system for (z, u) by constructing the sequence of successive approximations $(z^{(m)}, u^{(m)})$ with the property $\partial_x z^{(m)} = u^{(m)}$; sending m to infinity, we obtain a solution. Alternatively, one may construct a functional integral problem of the form $z = Fz$, equivalent to the original one; then, the main difficulty lies in finding a closed subset X of a Banach space, such that F maps it into itself and is a contraction. Such an obtained fixpoint of F is a classical solution of our quasilinear problem. Our existence proof goes along these lines.

The paper is organized as follows. In Section 2 we prove a result on the existence and regularity of bicharacteristics, having assumed our conditions on the operator V. In the next section, the method of bicharacteristics is used to transform the Cauchy problem into a system of integral equations. A fixed-point equation is constructed. Section 4 contains the main result. An application of our approach to the systems with state dependent delays is described in the last section.

2. Bicharacteristics. Let $U \subset \mathbf{R}^{1+n}$, and let k be a positive integer. For $z : U \to \mathbf{R}^k$ and $(t,x) \in U$, denote

$$\partial_x z(t,x) = [\partial_{x_j} z_i(t,x)]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}} \in M_{k\times n}$$

and

$$\partial z(t,x) = [\partial_{x_j} z_i(t,x)]_{\substack{i=1,\dots,k,\\ j=0,\dots,n}} \in M_{k\times (n+1)},$$

where $\partial_{x_0} \equiv \partial_t$.

For a fixed $p \in \mathbf{R}_+$, we consider the space

$$C^1[p](U,\mathbf{R}^k) = \{w \in C(U,\mathbf{R}^k) : w \text{ is continuously differentiable and } |\partial w|_{C(U)} \le p\}.$$

Similarly, for $p = (p_1,p_2) \in \mathbf{R}^2_+$, we define

$$C^{1.L}[p](U,\mathbf{R}^k) = \{w \in C^1[p_1](U,\mathbf{R}^k) : |\partial w|_{C^{0.L}(U)} \le p_2\},$$

where $|z|_{C^{0.L}(U)} = \sup_{P\ne\overline{P};P,\overline{P}\in U} \|z(P) - z(\overline{P})\| \cdot \|P-\overline{P}\|^{-1}$. We are now able to define the function space, in which we seek solutions to (1), (2). Given $p \in \mathbf{R}^2_+$, $\varphi_i \in C^{1.L}[p](E_{0.i},\mathbf{R})$, $1 \le i \le m$, and $d \in \mathbf{R}^2_+$ such that $d_j \ge p_j$, $j = 1,2$, we set

$$C^{1.L}_{\varphi.C}[d] = \{z \in C^{1.L}[d](E_c,\mathbf{R}^m) : z_i \equiv \varphi_i \text{ on } E_{0.i},\ 1 \le i \le m\}.$$

We prove that, under suitable assumptions on ρ, G, φ_i, on the parameters p, d, and for sufficiently small $\widetilde{a} \in (0,a)$, $c \in (\widetilde{a}, a]$, there exists a solution $\overline{z}$ of problem (1), (2) such that $\overline{z} \in C^{1.L}_{\varphi.c}[d]$.

Let Y stand for $C(D, M_{m\times(n+1)})$. Write $\rho_i = (\rho_{i1}, \dots, \rho_{in})$, $1 \leq i \leq m$. For the convenience of calculations, we consider m Fréchet derivatives $\partial_w \rho_i(t,x,w) \in L(X, \mathbf{R}^n)$, $1 \leq i \leq m$, rather than mn Fréchet derivatives $\partial_w \rho_{ij}(t,x,w)$, $1 \leq i \leq m$, $1 \leq i \leq n$. We are interested in estimating it in the norm $\|\cdot\|_{L(Y, M_{n\times(n+1)})}$, since we use the notation

(6)
$$\partial_w \rho_i(t,x,w)\delta = \Big(\partial_w \rho_i(t,x,w)\delta_0, \dots, \partial_w \rho_i(t,x,w)\delta_n\Big) \in M_{n\times(n+1)}$$

for $\delta \in Y$, $\delta = (\delta_0, \dots, \delta_n)$, $\delta_j \in X$, $0 \leq j \leq n$. The symbol $\Omega^{1.L}$ is short for $E \times C^{1.L}(D, \mathbf{R}^m)$.

Assumption H $[\rho]$**.** Suppose that $\rho : \Omega \to M_{m\times n}$ in the variables (t,x,w), is continuous and

1) the derivatives: $\partial_x \rho_i(t,x,w)$ and the Fréchet derivative $\partial_w \rho_i(t,x,w)$ exist for $(t,x,w) \in \Omega^{1.L}$, $1 \leq i \leq m$,

2) for $1 \leq i \leq m$, $\partial_x \rho_i$ and $\partial_w \rho_i$ are continuous in t on $\Omega^{1.L}$,

3) there is a non-negative constant A such that, for $1 \leq i \leq m$,

$$\|\rho_i(t,x,w)\|,\ \|\partial_x \rho_i(t,x,w),\ \|\partial_w \rho_i(t,x,w)\|_{L(Y, M_{n\times(n+1)})} \leq A \quad \text{on } \Omega^{1.L}$$

and

$$\begin{aligned}\|\partial_x \rho_i(t,x,w) - \partial_x \rho_i(t,\overline{x},\overline{w})\|,& \\ \|\partial_w \rho_i(t,x,w) - \partial_w \rho_i(t,\overline{x},\overline{w})\|_{L(Y, M_{n\times(n+1)})}& \\ \leq A\big(\|x-\overline{x}\| + \|w-\overline{w}\|_X\big)&\end{aligned}$$

for (t,x,w), $(t,\overline{x},\overline{w}) \in \Omega^{1.L}$.

Assumption H $[V]$**.** The operator $V : C(E_a, \mathbf{R}^m) \times E \to X$ is such that for every $d \in \mathbf{R}^2_+$ there are $\overline{d} \in \mathbf{R}^2_+$, $L \in \mathbf{R}_+$ such that:

1) for $z \in C^1[d_1](E_a, \mathbf{R}^m)$, $(t,x) \in E$,

$$\|\partial V(z;t,x)\|_Y \le \overline{d}_1,$$

2) for $z \in C^{1.L}[d](E_a, \mathbf{R}^m)$, $(t,x) \in E$,

$$\|\partial V(z;t,x)\|_Y \le \overline{d}_1, \quad \|\partial V(z;t,x) - \partial V(z;t,\overline{x})\|_Y \le \overline{d}_2\|x-\overline{x}\|,$$

3) for every z, $\overline{z} \in C^1[d_1](E_a, \mathbf{R}^m)$ and (t,x) the following holds

$$\|V(z;t,x) - V(\overline{z};t,x)\|_X \le L\|z-\overline{z}\|_{C(E_t)}.$$

Suppose that $\varphi_i \in C^{1.L}[p](E_{0.i}, \mathbf{R})$, $1 \le i \le m$, and $z \in C^{1.L}_{\varphi.c}[d]$. For $1 \le i \le m$, and a point $(t,x) \in [a_i, c] \times \mathbf{R}^n$, we consider the Cauchy problem

$$\eta'(\tau) = \rho_i(\tau, \eta(\tau), V(z;\tau,\eta(\tau))), \quad \eta(t) = x, \tag{7}$$

and denote by $g_i[z](\cdot,t,x) = (g_{i1}[z](\cdot,t,x), \dots, g_{in}[z](\cdot,t,x))$ its classical solution. This function is the bicharacteristic of the ith equation of (1), corresponding to z. Write

$$Q_i[z](\tau,t,x) = (\tau, g_i[z](\tau,t,x), V(z;\tau,g_i[z](\tau,t,x))).$$

We prove a lemma on bicharacteristics.

Lemma 2.1. *Suppose that Assumptions $H[\rho]$, $H[V]$ are satisfied, and let φ_i, $\overline{\varphi}_i \in C^{1.L}[p](E_{0.i}, \mathbf{R})$ such that $\|\varphi_i - \overline{\varphi}_i\|_{C(E_{0.i})} < +\infty$, $1 \le i \le m$, and $z \in C^{1.L}_{\varphi.c}[d]$, $\overline{z} \in C^{1.L}_{\overline{\varphi}.c}[d]$, be given. Then, for $1 \le i \le m$, the solutions $g_i[z](\cdot,t,x)$ and $g_i[\overline{z}](\cdot,t,x)$ exist on the interval $[a_i, c]$ and are unique. Moreover, the estimates*

$$\begin{gathered} \|\partial g_i[z](\tau,t,x)\| \le C, \\ \|\partial g_i[z](\tau,t,x) - \partial g_i[z(\tau,\overline{t},\overline{x})]\| \le Q\max\{|t-\overline{t}|, \|x-\overline{x}\|\} \end{gathered} \tag{8}$$

and

$$\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\| \le \overline{A}\left|\int_t^\tau \|z-\overline{z}\|_{C(E_s)}\,ds\right| \tag{9}$$

hold for $\tau \in [a_i, c]$, (t, x), $(\bar{t}, \bar{x}) \in [a_i, c] \times \mathbf{R}^n$, *where*

$$
\begin{aligned}
C &= (A+1)e^{cB}, \\
Q &= [(1+C)B + \widetilde{C}]e^{cB}, \\
\widetilde{C} &= C^2 cA[(1+\overline{d}_1)^2 + \overline{d}_2], \\
B &= A(1+\overline{d}_1)
\end{aligned}
\tag{10}
$$

and

$$\overline{A} = ALe^{cB}$$

and $\overline{d} = (\overline{d}_1, \overline{d}_2) \in \mathbf{R}^2_+$ *is the parameter from Assumption* $H[V]$, *corresponding to* d.

Proof. Let $z \in C^{1.L}_{\varphi.c}[d]$. The existence and uniqueness of solutions of (7) follow from the theorem on classical solutions of initial problems. From another classical theorem on differentiation of solutions with respect to the initial data it follows that the derivative $\partial g_i[z](s, t, x)$ exists and fulfills the integral equations

$$
\begin{aligned}
\partial g_i[z](\tau, t, x) = {} & \big[-\rho_i(t, x, V(z; t, x)) \mid I \big] \\
& + \int_t^\tau \Big[\partial_x \rho_i(Q_i[z](s, t, x)) \\
& + \partial_w \rho_i(Q_i[z](s, t, x)) \partial V(z; s, g_i[z](s, t, x)) \Big] \\
& \qquad * \partial g_i[z](s, t, x)\, ds
\end{aligned}
\tag{11}
$$

where $[-\rho_i(t, x, V(z; t, x)) \mid I]$ denotes concatenation of the matrix $-\rho_i(t, x, V(z; t, x))$ with the identity matrix. It follows from (11), from Assumptions $H[\rho]$, $H[V]$ that $\partial g_i[z](\cdot, t, x)$ satisfy integral inequalities

$$\|\partial g_i[z](\tau, t, x)\| \le A + 1 + B \left| \int_t^\tau \|\partial g_i[z](s, t, x)\|\, ds \right|,$$

and from the Gronwall lemma we get the first estimate in (8). Hence we derive the inequality

$$
\begin{aligned}
&\|\partial g_i[z](\tau, t, x) - \partial g_i[z](\tau, \bar{t}, \bar{x})\| \\
&\qquad \le (B + \widetilde{C}) \max\{|t - \bar{t}|, \|x - \bar{x}\|\} + CB|t - \bar{t}| \\
&\qquad\quad + B \left| \int_t^\tau \|\partial g_i[z](s, t, x) - \partial g_i[z](s, \bar{t}, \bar{x})\|\, ds \right|
\end{aligned}
$$

which, by the Gronwall lemma, implies that

$$\begin{aligned}\|\partial g_i[z](\tau,t,x) - \partial g_i[z](\tau,\bar{t},\bar{x})\| \\ &\le Q_1 \max\{|t-\bar{t}|, \|x-\bar{x}\|\} + Q_0|t-\bar{t}| \\ &\le (Q_0+Q_1)\max\{|t-\bar{t}|, \|x-\bar{x}\|\},\end{aligned}$$

with $Q_0 = CB\exp(cB)$ and $Q_1 = (\widetilde{C}+B)\exp(cB)$, yielding the second estimate in (8).

We now prove (9). The function $g_i[z](\tau,t,x)$ satisfies the following relation:

$$g_i[z](\tau,t,x) = x + \int_t^\tau \rho_i(s, g_i[z](s,t,x), V(z; s, g_i[z](s,t,x)))\, ds.$$

This leads to

$$\begin{aligned}\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \\ &\le B\left|\int_t^\tau \|g_i[z](s,t,x) - g_i[\bar{z}](s,t,x)\|\, ds\right| \\ &\quad + AL\left|\int_t^\tau \|z-\bar{z}\|_{C(E_s)} ds\right|.\end{aligned}$$

Again, from the Gronwall inequality we obtain

$$\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \le AL\exp(cB)\left|\int_t^\tau \|z-\bar{z}\|_{C(E_s)}\, ds\right|.$$

This completes the proof. □

3. Functional integral system. Let W stand for $L(C(D, M_{m\times n}), M_{1\times n})$. The expression $\partial_w G_i(t,x,w)\delta$ for $\delta \in C(D, M_{m\times n})$ is to be interpreted in a way analogous to (6); for the sake of simplicity of calculations, we use $\|\cdot\|_W$ (rather than $\|\cdot\|_{L(X,\mathbf{R})}$) for measuring the values of $\partial_w G_i$.

Assumption $H[\rho, G]$**.** Assumption $H[\rho]$ is fulfilled, $G : \Omega \to \mathbf{R}^m$, in the variables (t,x,w), is continuous and, for $1 \le i \le m$,

1) the derivative $\partial_x G(t,x,w)$ and the Fréchet derivative $\partial_w G(t,x,w)$ exist for $(t,x,w) \in \Omega^{1.L}$,

2) for (t,x,w), $(\bar{t},\overline{x},\overline{w}) \in \Omega^{1.L}$,

$$\begin{gathered} \|G(t,x,w)\|, \quad \|\partial_x G(t,x,w)\|, \quad \|\partial_w G_i(t,x,w)\|_W \leq A, \\ \|G_i(t,x,w) - G_i(\bar{t},x,w)\| \leq A|t-\bar{t}|, \\ \|\partial_x G_i(t,x,w) - \partial_x G_i(t,\overline{x},\overline{w})\|, \\ \|\partial_w G_i(t,x,w) - \partial_w G_i(t,\overline{x},\overline{w})\|_W \leq A\big(\|x-\overline{x}\| + \|w-\overline{w}\|_X\big). \end{gathered}$$

with the same constant A as in the Assumption $H[\rho]$.

We define the operator $F = (F_1, \dots, F_m)$ on $C^{1.L}_{\varphi.c}[d]$ by the formula

$$\begin{gathered} F_i z(t,x) = \varphi_i(a_i, g_i[z](a_i,t,x)) + \int_{a_i}^{t} G_i(Q_i[z](s,t,x))\, ds \\ \text{for } (t,x) \in [a_i,c] \times \mathbf{R}^n, \quad 1 \leq i \leq m, \\ F_i z \equiv \varphi_i \quad \text{on} \quad E_{0.i}, \quad 1 \leq i \leq m. \end{gathered} \tag{12}$$

Remark 3.1. The right-hand side of (1) is obtained in the following way. We consider each equation of (1) along its bicharacteristic:

$$\begin{aligned} \partial_t z_i(\tau, g_i[z](\tau,t,x)) + \partial_x z_i(\tau, g_i[z](\tau,t,x)) \\ * \rho_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x))) \\ = G_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x))) \end{aligned}$$

from which, using (7), we get

$$\frac{d}{d\tau} z_i(\tau, g_i[z](\tau,t,x)) = G_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x))).$$

By integrating the latter equation with respect to τ, we get a right-hand side of (12).

Assumption $H[\varphi,c,d,V]$**.** Assumption $H[V]$ is fulfilled, and the constants $a_i > 0$, $1 \leq i \leq m$, and $c > \max\{a_i : 1 \leq i \leq m\}$ are small enough so to satisfy, together with d and p,

$$d_1 \geq p_1 C + A + cCB, \tag{13}$$

$$d_2 \geq B + p_1 Q + p_2 C^2 + B(C + cQ) + \widetilde{C}, \tag{14}$$

with constants B, C, $\widetilde{C}$, Q defined in (10). Moreover, the compatibility condition

$$\partial_t\varphi_i(a_i,x)+\partial_x\varphi_i(a_i,x)*\rho_i(a_i,x,V(z;a_i,x))=G_i(a_i,x,V(z;a_i,x)),\quad 1\le i\le m, \tag{15}$$

holds for any $x\in\mathbf{R}^n$, $z\in C^{1.L}_{\varphi.c}[d]$.

Lemma 3.2. *Suppose that Assumptions $H[\rho,G]$ and $H[\varphi,c,d,V]$ are satisfied. Then the operator F maps $C^{1.L}_{\varphi.c}[d]$ into itself.*

Proof. Let $z\in C^{1.L}_{\varphi.c}[d]$. From (12) it follows that

$$\begin{aligned}\partial F_i z(t,x)&=\partial_x\varphi_i(a_i,g_i[z](a_i,t,x))*\partial g_i[z](a_i,t,x)\\&\quad+\big[G_i(t,x,V(z;t,x))\mid 0\big]\\&\quad+\int_{a_i}^t\Big[\partial_x G_i(Q_i[z](s,t,x))\\&\quad+\partial_w G_i(Q_i[z](s,t,x))\partial_x V(z;s,g_i[z](s,t,x))\Big]\\&\qquad\qquad*\partial g_i[z](s,t,x)\,ds,\end{aligned}\tag{16}$$

where $\big[G_i(t,x,V(z;t,x))\mid 0\big]=\big(G_i(t,x,V(z;t,x)),0,\dots,0\big)\in\mathbf{R}^{1+n}$, and $\partial_w G_i(Q_i)\partial_x V(z;\tau,y)$ is to be interpreted column-wise. It follows that $\|\partial Fz(t,x)\|\le p_1C+CBc+A$ on E_c, which, by Assumption $H[\varphi,c,d,V]$, implies that $\|\partial Fz\|_{C(E_c)}\le d_1$. Furthermore, for $1\le i\le m$ and for (t,x), $(\bar t,\bar x)\in[a_i,c]\times\mathbf{R}^n$,

$$\begin{aligned}&\|\partial F_i z(t,x)-\partial F_i z(\bar t,\bar x)\|\\&\quad\le\Big\|\partial_x\varphi_i(a_i,g_i[z](a_i,t,x))*\partial g_i[z](a_i,t,x)\\&\qquad-\partial_x\varphi_i(a_i,g_i[z](a_i,\bar t,\bar x))*\partial g_i[z](a_i,\bar t,\bar x)\Big\|\\&\quad+\Big|G_i(t,x,V(z;t,x))-G_i(\bar t,\bar x,V(z;\bar t,\bar x))\Big|\\&\quad+\int_{a_i}^t\Big\|\partial_x G_i(Q_i[z](s,t,x))*\partial g_i[z](s,t,x)\\&\qquad\qquad-\partial_x G_i(Q_i[z](s,\bar t,\bar x))*\partial g_i[z](s,\bar t,\bar x)\Big\|\,ds\end{aligned}$$

$$+ \int_{a_i}^{t} \Big\| \partial_w G_i(Q_i[z](s,t,x)) \partial_x V(z;s,g_i[z](s,t,x)) * \partial g_i[z](s,t,x)$$
$$- \partial_w G_i(Q_i[z](s,\bar{t},\overline{x})) \partial_x V(z;s,g_i[z](s,\bar{t},\overline{x})) * \partial g_i[z](s,\bar{t},\overline{x}) \Big\| \, ds$$
$$+ \Big| \int_{t}^{\bar{t}} \Big\| \Big[\partial_x G_i(Q_i[z](s,\bar{t},\overline{x}))$$
$$+ \partial_w G_i(Q_i[z](s,\bar{t},\overline{x}) \partial_x V(z;s,g_i[z](s,\bar{t},\overline{x})) \Big] * \partial g_i[z](s,\bar{t},\overline{x}) \Big\| \, ds \Big| .$$

From the above inequalities, Assumption $H[\rho, G]$ and Lemma 2.1 it follows that

$$\|\partial F_i z(t,x) - \partial F_i z(\bar{t},\overline{x})\| \le (p_1 Q + p_2 C^2 + B + cBQ + \widetilde{C}) \cdot \max\{|t-\bar{t}|, \|x-\overline{x}\|\} + BC|t-\bar{t}|,$$

for (t,x), $(\bar{t},\overline{x}) \in [a_i,c] \times \mathbf{R}^n$, which, in view of the second inequality from the Assumption $H[\varphi,c,d,V]$, gives $|\partial F z|_{C^{0.L}(E_c)} \le d_2$.

The fact that $F_i z$ are continuous extensions of φ_i is a simple consequence of the definition (12); it remains to prove that this extension is of class C^1. From (11), (16), and from the compatibility condition (15) we obtain

$$\begin{aligned}
\partial_t F_i z(a_i,x) &= \partial_x \varphi_i(a_i,x) \\
&\quad * \partial_t g_i[z](a_i,a_i,x) + G_i(a_i,x,V(z;a_i,x)) \\
&= -\partial_x \varphi_i(a_i,x) * \rho_i(a_i,x,V(\varphi_i;a_i,x)) \\
&\quad + G_i(a_i,x,V(\varphi_i;a_i,x)) \\
&= \partial_t \varphi_i(a_i,x), \quad 1 \le i \le m.
\end{aligned}$$

By the same token we get

$$\begin{aligned}
\partial_x F_i z(a_i,x) &= \partial_x \varphi_i(a_i,x) * \partial_x g_i[z](a_i,a_i,x) \\
&= \partial_x \varphi_i(a_i,x) * I = \partial_x \varphi_i(a_i,x), \quad 1 \le i \le m.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

4. Existence of solutions.

Theorem 4.1. *Suppose that* $\varphi_i \in C^{1.L}[p](E_{0.i}, \mathbf{R})$, $1 \le i \le m$, *and Assumptions* $H[\rho, G]$, $H[\varphi, c, d, V]$ *are satisfied. Then there exists exactly one solution* $\overline{z} \in C^{1.L}_{\varphi.c}[d]$ *of problem* (1), (2). *Moreover, there is a* $\Lambda_c \in \mathbf{R}_+$ *such that*

$$(17)\qquad \|v - \overline{v}\|_{C(E_t)} \le \Lambda_c \max_{1 \le i \le m} \|\varphi_i - \overline{\varphi}_i\|_{C(E_{0.i})}, \quad 0 \le t \le c,$$

for $\overline{\varphi_i} \in C^{1.L}[p](E_{0.i}, \mathbf{R})$ *satisfying* $\|\varphi_i - \overline{\varphi_i}\|_{C(E_{0.i})} < +\infty$, $1 \le i \le m$, *and* $\overline{v} \in C^{1.L}_{\overline{\varphi}_i.c}[d]$ *being a solution of* (1) *with the initial condition* (2) *with* φ_i *replaced by* $\overline{\varphi_i}$, $1 \le i \le m$.

Proof. We prove that there exists exactly one $\overline{z} \in C^{1.L}_{\varphi.c}[d]$ satisfying the equation $z = F[z]$. Lemma 3.2 shows that $F : C^{1.L}_{\varphi.c}[d] \to C^{1.L}_{\varphi.c}[d]$. Note that the uniform boundedness of temporal derivatives implies the finiteness of $\|z - \widetilde{z}\|_{C(E_c)}$ for $z, \widetilde{z} \in C^{1.L}_{\varphi.c}[d]$. From the definition (12) of F, and from the Lipschitz continuity of g_i with respect to z (see (9)), the existence of an $L^* > 0$ follows easily such that

$$(18)\qquad \|F_i z(t,x) - F_i \widetilde{z}(t,x)\| \le L^* \int_{a_i}^{t} \|z - \widetilde{z}\|_{C(E_s)}\, ds$$

for z, $\widetilde{z} \in C^{1.L}_{\varphi.c}[d]$, $(t,x) \in [a_i, c] \times \mathbf{R}^n$, $1 \le i \le m$. Let $\lambda > L^*$. We define a metric in $C^{1.L}_{\varphi.c}[d]$ by

$$d_\lambda(z, \widetilde{z}) = \sup\{\|(z - \widetilde{z})(t,x)\| e^{-\lambda t} : (t,x) \in [0,c] \times \mathbf{R}^n\}.$$

We now prove that a $q \in [0,1)$ exists such that

$$(19)\qquad d_\lambda(Fz, F\widetilde{z}) \le q d_\lambda(z, \widetilde{z}).$$

According to (18),

$$\begin{aligned} \|Fz(t,x) - F\widetilde{z}(t,x) &\le L^* \int_0^t \|z - \widetilde{z}\|_{C(E_s)}\, ds \\ &= L^* \int_0^t \|z - \widetilde{z}\|_{C(E_s)} e^{-\lambda s} e^{\lambda s} ds \\ &\le L^* d_\lambda(z, \widetilde{z}) \int_0^t e^{\lambda s} ds = \frac{L^*}{\lambda} d_\lambda(z, \widetilde{z})(e^{\lambda t} - 1) \\ &\le \frac{L^*}{\lambda} d_\lambda(z, \widetilde{z}) e^{\lambda t} \end{aligned}$$

for $(t,x) \in [0,c] \times \mathbf{R}^n$. Then

$$\|Fz(t,x) - F\widetilde{z}(t,x)\| e^{-\lambda t} \le \frac{L^*}{\lambda} d_\lambda(z,\widetilde{z}) \quad \text{for all } (t,x) \in [0,c] \times \mathbf{R}^n,$$

which gives (19) with $q = L^* \lambda^{-1}$. By the Banach fixed point theorem, there exists the unique fix-point of F. Denoting this fix-point by $\overline{z}$ we prove that it is a solution of equation (1). For $(t,x) \in [a_i, c] \times \mathbf{R}^n$, there is

$$\overline{z}_i(t,x) = \varphi_i(a_i, g_i[\overline{z}](a_i,t,x)) + \int_{a_i}^t G_i(s, g_i[\overline{z}](s,t,x), V(\overline{z}; s, g_i[\overline{z}](s,t,x)))\, ds.$$

For a given $x \in \mathbf{R}^n$, we put $y = g_i[\overline{z}](a_i,t,x)$. It follows from Lemma 2.1 that $g_i[\overline{z}](s,t,x) = g_i[\overline{z}(s,a_i,y)$ for $s,t \in [a_i,c]$ and $x = g_i[\overline{z}](t,a_i,y)$. Then we get

$$\overline{z}_i(t, g[\overline{z}](t,a_i,y)) = \varphi_i(a_i,y) + \int_{a_i}^t G_i(s, g_i[\overline{z}](s,a_i,y), V(\overline{z}; s, g_i[\overline{z}](s,a_i,y)))\, ds \quad 1 \le i \le m. \tag{20}$$

Relations $y = g_i[\overline{z}](a_i,t,x)$ and $x = g_i[\overline{z}](t,a_i,y)$ are equivalent for $x,y \in \mathbf{R}^n$. By differentiating (20) with respect to t and again putting $x = g_i[\overline{z}](t,a_i,y)$ we conclude that $\overline{z}$ satisfies (1). Since $\overline{z}$ satisfies initial condition (2), it is a solution of our problem.

We now prove relation (17). The function $\overline{v}$ satisfies integral functional system

$$z(t,x) = Fz(t,x)$$

and initial condition (2) with $\overline{\varphi}_i$ instead of φ_i. It follows easily that there is a $\Lambda \in \mathbf{R}_+$ such that the integral inequality

$$\|v - \overline{v}\|_{C(E_t)} \le \max_{1 \le i \le m} \|\varphi_i - \overline{\varphi}_i\|_{C(E_{0.i})} + \Lambda \int_0^t \|v - \overline{v}\|_{C(E_s)}\, ds, \quad 0 \le t \le c,$$

is satisfied. Using the Gronwall inequality, we obtain (17) with $\Lambda_c = \exp(\Lambda c)$. This proves the theorem. □

Remark 4.2. Inequalities (13), (14), given in Assumption $H[\varphi, c, d, V]$, have the following impact on the conditions on operator V.

We indicate how to solve those inequalities. Put, for example, $d_1 = A + 2(1+A)(1+p_1)$; condition 1) of Assumption $H[V]$ then produces a corresponding constant $\overline{d}_1 \geq 0$. By performing easy calculations, one can see that the condition $c \leq A^{-1}(1+\overline{d}_1)^{-1}\log 2$, on c, assures the fulfilment of (13).

After the construction of $\overline{d}_1$, an example of a suitable value of d_2, *in terms of* $\overline{d}_1$ and of given constants, may be found using (14) (we shall assume that $c\overline{d}_2$ is appropriately bounded). Since at this stage d_1 and d_2 are fixed, condition 2) of Assumption $H[V]$ gives $\overline{d}_2$. This leads to one more constraint on c, of which we assume the stronger one.

The above explained dependence of choice of d_2 on $\overline{d}_1$ shows that condition 2) of the Assumption being considered does not suffice for solvability of inequalities from Assumption $H[\varphi, c, d, V]$, but that condition 1) has to be added.

5. Systems with state dependent delays. Suppose that $z : E_a \to \mathbf{R}$ and $(t,x) \in E$ are fixed. We define the function $z_{(t,x)} : D \to \mathbf{R}$ as follows

$$z_{(t,x)}(\tau,\xi) = z(t+\tau, x+\xi), \quad (\tau,\xi) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-h_0,t]\times[x-h,x+h]$, and this restriction is shifted to the set D. For $z\colon E_a \to \mathbf{R}^m$, $z = (z_1,\dots,z_m)$, write $z_{(t,x)} = ((z_1)_{(t,x)},\dots,(z_m)_{(t,x)})$.

Let $\psi_{ij} : \Omega^{1.L} \to \mathbf{R}$, $1 \leq i \leq m$, $0 \leq j \leq n$, be given. Consider the function

$$\Big((z_1)_{\psi_1(t,x,w)},\dots,(z_m)_{\psi_m(t,x,w)}\Big) \in X,$$

where $\psi_i = (\psi_{i0},\dots,\psi_{in})$, $1 \leq i \leq m$, and $z : E_a \to \mathbf{R}^m$. We write it $z_{\psi(t,x,w)}$ for brevity. We show that the operator V, defined by

$$V(z;t,x) = z_{\psi(t,x,z_{(t,x)})} \tag{21}$$

satisfies Assumption $H[V]$, provided that certain regularity conditions on ψ are met.

Assumption $H[\psi]$**.** Deviating function $\psi : \Omega \to M_{m\times(n+1)}$ is continuous and, for $1 \leq i \leq m$,

1) the inequality $\psi_{i\,0}(t,x,w) \le t$ holds on Ω,

2) derivatives: $\partial\psi_i$ and the Fréchet derivative $\partial_w\psi_i$ exist on $\Omega^{1.L}$,

3) there is a non-negative constant A_1 independent of i and such that, for (t,x,w), $(t,\overline{x},\overline{w}) \in \Omega^{1.L}$,

$$\|\partial\psi_i(t,x,w)\|, \qquad \|\partial_w\psi_i(t,x,w)_{L(Y,M_{(n+1)\times(n+1)})} \le A_1$$

and

$$\|\partial\psi_i(t,x,w) - \partial\psi_i(t,\overline{x},\overline{w})\|,$$
$$\|\partial\psi_i(t,x,w) - \partial_w\psi_i(t,\overline{x},\overline{w})\|_{L(Y,M_{(n+1)\times(n+1)})}$$

are bounded from above by $A_1(\|x-\overline{x}\| + \|w-\overline{w}\|_X)$.

In view of Assumption $H[\psi]$, differentiation of (21) gives

$$\partial V_i(z;t,x) \equiv (\partial z_i)_{\psi_i(t,x,z_{(t,x)})} * \left[\partial\psi_i(t,x,z_{(t,x)}) + \partial_w\psi_i(t,x,z_{(t,x)})(\partial z)_{(t,x)}\right] \text{ on } D,$$

and, consequently, for $z \in C^{1.L}_{\varphi.c}[d]$ and $(t,x) \in [a_i,c] \times \mathbf{R}^n$,$\|\partial V_i(z;t,x)\|_{C(D,M_{1\times(n+1)})} \le d_1A_1(1+d_1)$ and

$$\|\partial V_i(z;t,x) - \partial V_i(z;t,\overline{x})\|_{C(D,M_{1\times(n+1)})} \le A_1[d_1d_2 + (1+d_1)^2(d_1+d_2A_1)] \cdot \|x-\overline{x}\|.$$

Taking maximum (with respect to i) on the left-hand sides of these estimates, we obtain conditions 1), 2) of Assumption $H[V]$ with $\overline{d}_1 = d_1A_1(1+d_1)$ and $\overline{d}_2 = A_1[d_1d_2 + (1+d_1)^2(d_1+d_2A_1)]$. Fulfilment of condition 3) of that assumption follows from the estimates

$$\begin{aligned}\|V_i(z;t,x) - V_i(\overline{z};t,x)\|_{C(D)} &\le \|(z_i)_{\psi_i(t,x,z_{(t,x)})} - (z_i)_{\psi_i(t,x,\overline{z}_{(t,x)})}\|_{C(D)} \\ &\quad + \|(z_i - \overline{z}_i)_{\psi_i(t,x,\overline{z}_{(t,x)})}{}_{C(D)} \\ &\le d_1A_1\|z-\overline{z}\|_{C(E_t)} + \|z_i - \overline{z}_i\|_{C(E_t)} \\ &\le (d_1A_1+1)\|z-\overline{z}\|_{C(E_t)}, \quad 1 \le i \le m.\end{aligned}$$

Thus we have proved the following

Theorem 5.1. *Suppose that $\varphi_i \in C^{1.L}[p](E_{0.i},\mathbf{R})$ and Assumptions $H[\rho,G]$, $H[\psi]$ are satisfied. Furthermore, assume that the inequalities*

(13), (14) *hold, as well as the compatibility condition* (15). *Then there exists exactly one solution* $\overline{z} \in C^{1.L}_{\varphi.c}[d]$ *of the system*

(22)
$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi(t,x,z_{(t,x)})})\partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \quad 1 \le i \le m,$$

augmented with the generalized Cauchy condition (2). *Moreover, there is a* $\Lambda_c \in \mathbf{R}_+$ *such that the Lipschitz condition* (17), *with respect to initial data, holds for*

$$\overline{\varphi_i} \in C^{1.L}[p](E_{0.i}, \mathbf{R}), \qquad \|\varphi_i - \overline{\varphi}_i\|_{C(E_{0.i})} < +\infty, \quad 1 \le i \le m,$$

and for $\overline{v} \in C^{1.L}_{\overline{\varphi}.c}[d]$ *being a solution of* (22) *with the initial condition* $z \equiv \varphi_i$ *on* $E_{0.i}$, $1 \le i \le m$.

Assumption $H[\overline{\rho}, \overline{G}]$**.** Functions $\overline{\rho} : E \times \mathbf{R}^m \to M_{m\times n}$, $\overline{G} : E \times \mathbf{R}^m \to \mathbf{R}^m$, in variables (t,x,y), are continuous, uniformly bounded, and

1) $\overline{G}$ is Lipschitz continuous in t,

2) the derivatives $\partial_x\overline{\rho}$, $\partial_y\overline{\rho}$, $\partial_x\overline{G}$, $\partial_y\overline{G}$ exist on $E\times\mathbf{R}^m$, are continuous in t, and uniformly bounded,

3) these derivatives are Lipschitz continuous in x and y.

Example 5.2. Suppose that Assumption $H[\overline{\rho}, \overline{G}]$ is satisfied, and set

$$\rho(t,x,w) = \overline{\rho}(t,x,w(0,0)), \qquad G(t,x,w) = \overline{G}(t,x,w(0,0)).$$

Then Assumption $H[\rho, G]$ is fulfilled and the system (22) takes the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z(\psi(t,x,z_{(t,x)})))\partial_{x_j} z_i(t,x) = G_i(t,x,z(\psi(t,x,z_{(t,x)}))), \quad 1 \le i \le m,$$

that is, it becomes a system of equations with deviating argument where the deviation is state dependent.

Example 5.3. Suppose that Assumption $H[\overline{\rho},\overline{G}]$ is satisfied, and set

$$\rho(t,x,w) = \overline{\rho}\Big(t,x,\int_D w(\tau,\xi)\,d\tau\,d\xi\Big),$$
$$G(t,x,w) = \overline{G}\Big(t,x,\int_D w(\tau,\xi)\,d\tau\,d\xi\Big).$$

Then Assumption $H[\rho,G]$ is fulfilled, and the system (22) takes the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}\Big(t,x,\int_D z_{\psi(t,x,z_{(t,x)})}(\tau,\xi)\,d\tau\,d\xi\Big)\partial_{x_j} z_i(t,x)$$
$$= G_i\Big(t,x,\int_D z_{\psi(t,x,z_{(t,x)})}(\tau,\xi)\,d\tau\,d\xi\Big), \quad 1 \le i \le m,$$

that is, it becomes a system of integro-differential equations, where the domain of integration is state dependent.

REFERENCES

1. P. Brandi and R. Ceppitelli, *Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equation of the first order*, Ann. Polon. Math. **47** (1986), 121–136.

2. K.L. Cooke, *Functional differential systems*: *Some models and perturbation problems*, in Inter. Symp. Differential Equations Dynam. Syst., J. Hale and J. LaSalle, eds., Academic Press, New York, 1967.

3. A. Gołaszewska and J. Turo, *Carathéodory solutions to quasi-linear hyperbolic systems of partial differential equations with state dependent delays*, Funct. Differ. Equations **14** (2007), 257–278.

4. J.K. Hale, L. Verduyn and M. Sjoerd, *Introduction to functional-differential equations*, Appl. Math. Sci. **99**, Springer-Verlag, New York, 1993.

5. F. Hartung, T. Krisztin, H.-O. Walther and J. Wu, *Functional differential equations with state dependent delays*: *Theory and applications*, in *Handbook of differential equations: ordinary differential equations*, Vol. 3, Elsevier, Amsterdam, 2006.

6. E. Hernández, A. Prokopczyk and L. Ladeira, *A note on partial functional differential equations with state-dependent delay*, Nonlinear Anal. Real World Appl. **7** (2006), 510–519.

7. D. Jaruszewska-Walczak, *Existence of solutions of first order partial differential-functional equations*, Boll. Un. Mat. Ital. **4** (1990), 57–82.

8. Z. Kamont, *Hyperbolic functional differential inequalities and applications*, Math Appl. **486**, Kluwer Academic Publishers, Dordrecht, 1999.

9. Z. Kamont, *On the local Cauchy problem for Hamilton-Jacobi equations with a functional dependence*, Rocky Mountain J. Math. **30** (2000), 587–608.

10. ———, *First order partial functional differential equations with state dependent delays*, Nonlinear Stud. **12** (2005), 135–157.

11. Z. Kamont and J. Turo, *Carathédory solutions to hyperbolic functional differential systems with state dependent delays*, Rocky Mountain J. Math. **35** (2005), 1935–1952.

12. V Kolmanovskii and A. Myshkis, *Introduction to the theory and applications of functional-differential equations*, Math. Appl. **463**, Kluwer Academic Publishers, Dordrecht, 1999.

13. V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications*, Math. Sci. Engng. **55**, Academic Press, New York, 1969.

14. J. Mallet-Paret and R.D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time lags*, Arch. Rat. Mech. Anal. **120** (1992), 99–146.

15. A.V. Rezounenko, *Partial differential equations with discrete and distributed state-dependent delays*, J. Math. Anal. Appl. **326** (2007), 1031–1045.

16. A.V. Rezounenko and J. Wu, *A non-local PDE model for population dynamics with state-selective delay: Local theory and global attractors*, J. Comput. Appl. Math. **190** (2006), 99–113.

17. J. Szarski, *Generalized Cauchy problem for differential-functional equations with first order partial derivatives*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 575–580.

18. ———, *Cauchy problem for an infinite system of differential-functional equations with first order partial derivatives*, Comment. Math. Special Issue **1** (1978), 293–300.

19. H.-O. Walther, *The solution manifold and C^1-smoothness for differential equations with state-dependent delay*, J. Differential Equations **195** (2003), 46–65.

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