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THERE ARE ONLY FINITELY MANY $D(4)$-QUINTUPLES

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ABSTRACT. A $D(4)$-m-tuple is a set of m positive integers with the property that the product of any two of them increased by 4 is a perfect square. It is known that there does not exist a $D(4)$-sextuple. In this paper we show that the number of $D(4)$-quintuples is less than 10^{323}. Moreover, we prove that if $\{a, b, c, d, e\}$ is a $D(4)$-quintuple, then $\max\{a, b, c, d, e\} < 10^{10^{28}}$.

1. Introduction. Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The first one who studied the problem of finding such sets was Diophantus in the case $n = 1$. He found a set of four positive rational numbers with the above property: $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. However, Fermat was the first who found a $D(1)$-quadruple, which was the set $\{1, 3, 8, 120\}$. Euler was later able to add the fifth positive rational, $\frac{777480}{8288641}$, to Fermat's set (see [**3**], [**4**, pages 103–104, 232]. Recently, Gibbs [**13**] found several examples of $D(n)$-sextuples. It is conjectured that there does not exist a $D(1)$-quintuple. The first result supporting this conjecture was given by Baker and Davenport [**1**], who proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella [**6**] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples. This implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples (see [**7**]). The author [**8, 9, 10**] improved that result by proving that there does not exist a $D(4)$-sextuple and that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element.

For $n = 4$ it is conjectured that there does not exist a $D(4)$-quintuple. Moreover, there is an even stronger version of that conjecture.

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Conjecture 1 (cf. [**7**, Conjecture 1]). *There does not exist a $D(4)$-quintuple. Moreover, if $\{a,b,c,d\}$ is a $D(4)$-quadruple such that $a<b<c<d$, then*

$$d = a+b+c+\frac{1}{2}(abc+rst),$$

where r,s,t are positive integers defined by

$$ab+4=r^2,\ ac+4=s^2,\ bc+4=t^2.$$

If $d=a+b+c+(abc+rst)/2$, then $\{a,b,c,d\}$ is a $D(4)$-quadruple. We denote this number by d_+. We also define the number $d_- = a+b+c-(abc+rst)/2$. If $d_- \neq 0$, then $\{a,b,c,d_-\}$ is also a $D(4)$-quadruple, but $d_- < c$.

Definition 1. A $D(4)$-quadruple $\{a,b,c,d\}$ such that $d>\max\{a,b,c\}$, is called regular if $d=d_+$.

Mohanty and Ramasamy [**15**] were the first who have studied the problem of the nonextendability of $D(4)$-m-tuples. They proved that $D(4)$-quadruple $\{1,5,12,96\}$ cannot be extended to a $D(4)$-quintuple. Kedlaya [**14**] later proved that if $\{1,5,12,d\}$ is a $D(4)$-quadruple, then $d=96$.

There are some generalizations of this result that support Conjecture 1. One was given by Dujella and Ramasamy in [**7**], where they proved Conjecture 1 for a parametric family of $D(4)$-quadruples. They proved that if k and d are positive integers and

$$\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$$

is a $D(4)$-quadruple, then $d=4L_{2k}F_{4k+2}$, where F_k and L_k are Fibonacci and Lucas numbers, respectively. The second generalization was given by Fujita in [**12**]. He proved that if $k\geq 3$ is an integer and $\{k-2,k+2,4k,d\}$ is a $D(4)$-quadruple, then $d=4k^3-4k$.

Our main result is the following theorem.

Theorem 1. *The number of $D(4)$-quintuples is less than 10^{323}.*

The proof of the theorem will need the results in [**8**–**10**] and go along the same lines as [**11**], where Fujita improved the bound for the number of $D(1)$-quintuples due to Dujella [**5**].

We are interested in bounding the number of $D(4)$-quintuples. So, we first consider the number of $D(4)$-quintuples $\{a,b,c,d,e\}$ such that $a<b<c<d<e$ for a fixed triple $\{a,b,c\}$. In [**8**] we have proved that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element. It implies the following lemma, which will be used in several places in this paper.

Lemma 1. *If $\{a,b,c,d,e\}$ is a $D(4)$-quintuple such that $a<b<c<d<e$, then $d=d_+$, i.e., the $D(4)$-quadruple $\{a,b,c,d\}$ is regular.*

Since that implies d is unique, it suffices to bound the number of e's. Moreover, we are able to get an upper bound for the maximal element in $D(4)$-quintuple, i.e., we prove the following theorem.

Theorem 2. *If $\{a,b,c,d,e\}$ is a $D(4)$-quintuple, then* $\max\{a,b,c,d,e\} < 10^{10^{28}}$.

2. System of Pellian equations. In this section we will transform the problem of extending a fixed $D(4)$-triple to a quintuple into the problem of solving three simultaneous Pellian equations. And we will see it leads to finding intersections of binary recurrence sequences.

Let $\{a,b,c\}$ be a $D(4)$-triple such that $a<b<c$. Furthermore, let r,s,t be positive integers defined by

$$ab+4=r^2,\ ac+4=s^2,\ bc+4=t^2. \tag{1}$$

Moreover, assume that $\{a,b,c,d,e\}$ is a $D(4)$-quintuple with $c<d<e$. and put

$$ad+4=x^2,\ bd+4=y^2,\ cd+4=z^2,$$

where x,y,z are positive integers. Then there exist integers $\alpha,\beta,\gamma,\delta$ such that

$$ae+4=\alpha^2,\ be+4=\beta^2,\ ce+4=\gamma^2,\ de+4=\delta^2.$$

If we eliminate e, we get the system of simultaneous Pellian equations

$$a\delta^2 - d\alpha^2 = 4(a-d), \tag{2}$$

$$b\delta^2 - d\beta^2 = 4(b-d), \tag{3}$$

$$c\delta^2 - d\gamma^2 = 4(c-d). \tag{4}$$

We can describe the sets of solutions of equations (2), (3) and (4) in the following lemma. From Lemma 1 we know $d = d_+$.

Lemma 2. *Let* (δ, α), (δ, β), (δ, γ) *be positive solutions of* (2), (3) *and* (4), *respectively. Then there exist solutions* (δ_0, α_0), (δ_1, β_1), (δ_2, γ_2) *of* (2), (3) *and* (4), *respectively, in the following ranges*

$$1 \le \alpha_0 < \sqrt{x+2} < 1.236\sqrt[4]{ad}, \ |\delta_0| < \sqrt{\frac{d\sqrt{d}}{\sqrt{a}}} < 0.468d,$$

$$1 \le \beta_1 < \sqrt{y+2} < 1.122\sqrt[4]{bd}, \ |\delta_1| < \sqrt{\frac{d\sqrt{d}}{\sqrt{b}}} < 0.360d,$$

$$1 \le \gamma_2 < \sqrt{z+2} < 1.122\sqrt[4]{cd}, \ |\delta_2| < \sqrt{\frac{d\sqrt{d}}{\sqrt{c}}} < 0.360d,$$

such that

$$\delta\sqrt{a} + \alpha\sqrt{d} = (\delta_0\sqrt{a} + \alpha_0\sqrt{d})\left(\frac{x+\sqrt{ad}}{2}\right)^l, \tag{5}$$

$$\delta\sqrt{b} + \beta\sqrt{d} = (\delta_1\sqrt{b} + \beta_1\sqrt{d})\left(\frac{y+\sqrt{bd}}{2}\right)^m, \tag{6}$$

$$\delta\sqrt{c} + \gamma\sqrt{d} = (\delta_2\sqrt{c} + \gamma_2\sqrt{d})\left(\frac{z+\sqrt{cd}}{2}\right)^n, \tag{7}$$

for some integers $l, m, n \ge 0$.

Proof. The statement of the lemma follows immediately from [**7**, Lemma 2] and [**9**, Lemma 1]. □

Let $(\alpha, \beta, \gamma, \delta)$ be a solution of the system of the equations (2), (3) and (4). Then from (5) we get $\delta = u_l$ for some integer $l \ge 0$, where

$$u_0 = \delta_0, \quad u_1 = (x\delta_0 + d\alpha_0)/2, \quad u_{l+2} = xu_{l+1} - u_l. \tag{8}$$

From (6) we conclude $\delta = v_m$ for some integer $m \geq 0$, where

$$v_0 = \delta_1, \quad v_1 = (y\delta_1 + d\beta_1)/2, \quad v_{m+2} = yv_{m+1} - v_m. \tag{9}$$

In the same way from (7) we get $\delta = w_n$ for some integer $n \geq 0$, where

$$w_0 = \delta_2, \quad w_1 = (z\delta_2 + d\gamma_2)/2, \quad w_{n+2} = zw_{n+1} - w_n. \tag{10}$$

Lemma 3. *If $\delta = u_l = v_m = w_n$, then l, m and n are all even and $\delta_0 = \delta_1 = \delta_2 = \pm 2$.*

Proof. Let us assume l is odd. Then from [**9**, Lemma 9] we conclude $|\delta_0| = y$ and $|\delta_0| = z$, which is a contradiction. So l is even. Analogously we get n and m are even. But then the same lemma implies $\delta_0 = \delta_1 = \delta_2$. Define $e_0 = (\delta_0^2 - 4)/d$. Then

$$ae_0 + 4 = \alpha_0^2, \quad be_0 + 4 = \beta_1^2, \quad ce_0 + 4 = \gamma_2^2, \quad de_0 + 4 = \delta_0^2.$$

It implies that $\{a, b, c, d, e_0\}$ is a $D(4)$-quintuple with $e_0 < d$. Now from Lemma 1 we conclude that the $D(4)$-quadruple $\{a, b, c, e_0\}$ should be regular. But if $d = d_+$, then the $D(4)$-quadruple $\{a, b, c, e_0\}$ is not regular unless $e_0 = d_-$. In that case we would have $d_+d_- + 4$ is a perfect square, which is impossible. Therefore, we obtain $e_0 = 0$, i.e., $|\delta_0| = 2$. □

Lemma 4. *If $\delta = u_l = v_m = w_n$, then $8 \leq n \leq m \leq l \leq 2n$.*

Proof. From [**9**, Lemma 5] we conclude

$$n - 1 \leq m \leq 2n + 1, \qquad m - 1 \leq l \leq 2m + 1.$$

Now, because l, m and n are all even we conclude $n \leq m \leq l \leq 2n$. The inequality $n \geq 8$ follows from [**8**, Lemma 5]. □

3. Lower bounds for solutions.

Lemma 5. *If $\{a,b,c,d\}$ is a regular $D(4)$-quadruple, then $d > b^5$ or $d > c^{5/4}$.*

Proof. Let us first assume $c \geq b^4$. Then $d > abc \geq b^5$. If $c < b^4$, we conclude $d > abc \geq bc > c^{5/4}$. □

Lemma 6. *Suppose that $\delta = u_l = v_m = w_n$ and $d > 10^{292}$.*

(i) *If $d > b^5$, then $m > d^{0.025}$.*

(ii) *If $d > c^{5/4}$, then $m > d^{0.08}$.*

Proof. (i) The case when $d > b^5$ can be proven in exactly the same way as the first case in [**9**, Lemma 13].

(ii) Now let $d > c^{5/4}$, and assume $m \leq d^{0.08}$. Then if we consider equation $u_l = w_n$, [**9**, Lemma 12] implies

$$\pm al^2 + xl \equiv \pm cn^2 + zn \pmod{d}. \tag{11}$$

We also have the estimate

$$\begin{aligned} u_l &> (x-1)^{l-1}(d-x) > (\sqrt{ad}-1)^{l-1}(d-\sqrt{ad+4}) \\ &> (d^{0.5}-1)^{l-1}(d-1.0007d^{0.9}) > d^{0.49(l-1)}d^{0.9} = d^{0.49l+0.41}. \end{aligned}$$

Moreover,

$$\begin{aligned} w_n &< (d+z)z^{n-1} = (d+\sqrt{cd+4})(\sqrt{cd+4})^{n-1} \\ &< (d+1.0001d^{0.9})d^{0.91(n-1)} < d^{1.01}d^{0.91(n-1)} \\ &= d^{0.91n+0.1}. \end{aligned}$$

So $u_l = w_n$ implies $l < 1.858n$. It is easy to check that both sides of congruence (11) are less than d, so it actually becomes an equation. We also have

$$\begin{aligned} \frac{al}{x} &< \frac{l\sqrt{a}}{\sqrt{d}} < \frac{1.858d^{0.08}\cdot d^{0.17}}{d^{0.5}} = 1.858d^{-0.25} < 0.001, \\ \frac{cn}{z} &< \frac{n\sqrt{c}}{\sqrt{d}} < \frac{d^{0.08}\cdot d^{0.4}}{d^{0.5}} = d^{-0.02} < 0.001. \end{aligned}$$

That gives us

$$1.001xl > 0.999zn,$$

i.e.,

$$\frac{l}{n} > \frac{0.999z}{1.001x} > 0.998\sqrt{\frac{c}{a}} > 1.996,$$

which contradicts $l < 1.858n$. □

4. Upper bounds. In this section we will find upper bounds for b and d. For that, we first need the lemma we have already proven in [**10**].

Lemma 7. *If $u_l = v_m = w_n$, then*

$$\frac{m}{\log(m+1)} < 6.543 \cdot 10^{15} \log^2 d,$$
$$\frac{l}{\log(l+1)} < 6.543 \cdot 10^{15} \log^2 d.$$

We can now prove the following proposition.

Proposition 1. *Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple such that $a < b < c < d < e$. Then $d < 1.1 \cdot 10^{972}$ and $b < 2.57 \cdot 10^{194}$.*

Proof. Let us first assume $d > b^5$. Then from Lemmas 6 and 7 we have

$$\frac{m}{\log(m+1)} < 6.543 \cdot 10^{15} \log^2 d,$$

which together with $m > d^{0.025}$ implies $m < 2 \cdot 10^{24}$ and $d < 1.1 \cdot 10^{972}$. Inequality $b < 2.57 \cdot 10^{194}$ follows easily from $b^5 < d$.

If $d \leq b^5$, then $c^{5/4} < d$. If we take $d > 10^{292}$, then again from the Lemmas 6 and 7 we have $m < 2 \cdot 10^{23}$ and $d < 1.84 \cdot 10^{291}$, which is a contradiction. If $d < 10^{292}$, we would get an even better upper bound for b because $b^2 < d$. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple such that $a < b < c < d < e$. Then we have just proved $d < 1.1 \cdot 10^{972}$. It furthermore implies $n \leq m < 2 \cdot 10^{24}$, where $de + 4 = w_n^2$. So, we can conclude

$$e < \frac{w_n^2}{d} < \frac{((z+d)z^{n-1})^2}{d} < \frac{2d \cdot d^{n-1}}{d} = 2d^{n-1} < 10^{10^{28}}. \qquad \square$$

We will now use the methods from [**2**] and in the same way as [**11**] to prove that, for a fixed $D(4)$-triple $\{a, b, c\}$ such that $a < b < c$, there are at most four $D(4)$-quintuples $\{a, b, c, d, e\}$ with $c < d < e$.

Let $\{A, B, C\}$ be a $D(4)$-triple such that $A < B < C$, and let $S = \sqrt{AC + 4}$, $T = \sqrt{BC + 4}$. If we have the system of simultaneous Pellian equations

$$AZ^2 - CX^2 = 4(A - C), \tag{12}$$

$$BZ^2 - CY^2 = 4(B - C), \tag{13}$$

we may express its solutions as $Z = V_j = W_k$, $j, k \geq 0$, with some binary recurrent sequences $(V_j)_{j \geq 0}$ and $(W_k)_{k \geq 0}$.

Lemma 8. *Suppose that $V_0 = W_0 = \pm 2$ and that there exist three positive solutions (X_i, Y_i, Z_i), $i = 1, 2, 3$, of equations* (12) *and* (13) *with $Z_1 < Z_2 < Z_3$ that come from the same fundamental solution $Z = V_0 = W_0$. Put $Z_i = V_{j_i} = W_{k_i}$ for $i = 1, 2, 3$, with $j_1 < j_2 < j_3$. Then we have*

$$j_3 - j_3 > \frac{\Delta}{2AC} \xi^{2j_1} \log \eta,$$

where

$$\xi = \frac{S + \sqrt{AC}}{2},$$

$$\eta = \frac{T + \sqrt{BC}}{2},$$

$$\Delta = \begin{vmatrix} k_2 - k_1 & k_3 - k_2 \\ j_2 - j_1 & j_3 - j_2 \end{vmatrix} > 0.$$

Proof. Let $\varepsilon = V_0/2 = W_0/2$. From $Z = V_j = W_k$ we conclude

$$\begin{aligned} Z &= \frac{1}{\sqrt{A}}\left(\left(\sqrt{C}+\varepsilon\sqrt{A}\right)\xi^j - \left(\sqrt{C}-\varepsilon\sqrt{A}\right)\xi^{-j}\right) \\ &= \frac{1}{\sqrt{B}}\left(\left(\sqrt{C}+\varepsilon\sqrt{B}\right)\eta^k - \left(\sqrt{C}-\varepsilon\sqrt{B}\right)\eta^{-k}\right). \end{aligned}$$

So we can find three points

$$(p_i, q_i) = (j_i \log\xi, k_i \log\eta), \quad i = 1,2,3,$$

on the curve

$$\begin{aligned} F(p,q) = &\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^q - \left(\sqrt{C}-\varepsilon\sqrt{B}\right)e^{-q} \\ &- \sqrt{\frac{B}{A}}\left(\left(\sqrt{C}+\varepsilon\sqrt{A}\right)e^p - \left(\sqrt{C}-\varepsilon\sqrt{A}\right)e^{-p}\right) = 0. \end{aligned}$$

Since

$$\frac{\partial F(p,q)}{\partial q} = \left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^q + \left(\sqrt{C}-\varepsilon\sqrt{B}\right)e^{-q} > 0,$$

for all p and q, we may implicitly differentiate $F(p,q)$ to obtain

$$\begin{aligned} &\left(\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^q + \left(\sqrt{C}-\varepsilon\sqrt{B}\right)e^{-q}\right)\frac{dq}{dp} \\ &\qquad = \sqrt{\frac{B}{A}}\left(\left(\sqrt{C}+\varepsilon\sqrt{A}\right)e^p + \left(\sqrt{C}-\varepsilon\sqrt{A}\right)e^{-p}\right), \end{aligned}$$

which yields

(14)

$$\begin{aligned} \frac{dq}{dp} &= \sqrt{\frac{((\sqrt{C}+\varepsilon\sqrt{A})e^p - (\sqrt{C}-\varepsilon\sqrt{A})e^{-p})^2 + 4(C-A)}{A\left(((\sqrt{C}+\varepsilon\sqrt{B})e^q - (\sqrt{C}-\varepsilon\sqrt{B})e^{-q})^2 + 4(C-B)\right)/B}} \\ &= \sqrt{\frac{((\sqrt{C}+\varepsilon\sqrt{A})e^p - (\sqrt{C}-\varepsilon\sqrt{A})e^{-p})^2 + 4(C-A)}{((\sqrt{C}+\varepsilon\sqrt{A})e^p - (\sqrt{C}-\varepsilon\sqrt{A})e^{-p})^2 + 4((AC/B)-A)}} > 1. \end{aligned}$$

Similarly we obtain

$$\left(\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^{q}+\left(\sqrt{C}-\varepsilon\sqrt{B}\right)e^{-q}\right)\frac{d^2q}{dp^2}$$
$$+\left(\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^{q}-\left(\sqrt{C}-\varepsilon\sqrt{B}\right)e^{-q}\right)\left(\frac{dq}{dp}\right)^2$$
$$=\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^{q}+\left(\sqrt{C}+\varepsilon\sqrt{B}\right)e^{-q},$$

which yields

$$\frac{d^2q}{dp^2}=\left(1-\left(\frac{dq}{dp}\right)^2\right)\frac{(\sqrt{C}+\varepsilon\sqrt{B})e^{q}-(\sqrt{C}-\varepsilon\sqrt{B})e^{-q}}{(\sqrt{C}+\varepsilon\sqrt{B})e^{q}+(\sqrt{C}+\varepsilon\sqrt{B})e^{-q}}<0. \tag{15}$$

From (14) and (15) together with the mean value theorem, we conclude

$$0<\frac{q_2-q_1}{p_2-p_2}-\frac{q_3-q_2}{p_3-p_2}<\frac{q_2-q_1}{p_2-p_1}-1. \tag{16}$$

On the other hand, we may transform $V_{j_i}=W_{k_i}$, for $i=1,2$, into inequalities for linear forms in logarithms of algebraic numbers (see [**9**, Lemma 10]) to obtain

$$0<p_i-q_i+\log\mu<2AC\xi^{-2j_i},$$

where

$$\mu=\frac{\sqrt{B}(\sqrt{C}+\varepsilon\sqrt{A})}{\sqrt{A}(\sqrt{C}+\varepsilon\sqrt{B})}.$$

Hence, using (16), we get

$$0<p_2-q_2+\log\mu<p_1-q_1+\log\mu<2AC\xi^{2j_1},$$

which implies

$$0<(q_2-q_1)-(p_2-p_1)<2AC\xi^{2j_1}. \tag{17}$$

It follows from (16) and (17) that

$$0<\frac{q_2-q_1}{p_2-p_1}-\frac{q_3-q_2}{p_3-p_2}<\frac{(q_2-q_1)-(p_2-p_1)}{p_2-p_1}<\frac{2AC}{(p_2-p_1)\xi^{2j_1}}.$$

If we substitute $p_i = j_i \log \xi$ and $q_i = k_i \log \eta$, we obtain

$$0 < \frac{k_2 - k_1}{j_2 - j_1} - \frac{k_3 - k_2}{j_3 - j_2} < \frac{2AC}{(j_2 - j_1)\xi^{2j_1} \log \eta}.$$

Therefore,

$$j_3 - j_2 > \frac{\Delta}{2AC} \xi^{2j_1} \log \eta. \qquad \square$$

Let us mention that, due to the results we have proven in [**8**, Lemma 5] and [**10**], where we have checked that all $D(4)$-triples $\{a, b, c\}$ such that $a < b < c$ and $ab^2c < 10^7$ can be extended to a quadruple in a unique way, we may assume $ad > d > 29240$ and $cd > bd > 5 \cdot 10^6$.

Proposition 2. *Let $\{a, b, c\}$ be a $D(4)$-triple with $a < b < c$. Then the number of $D(4)$-quintuples $\{a, b, c, d, e\}$ such that $c < d < e$ is at most four.*

Proof. Let us assume that there exist five $D(4)$-quintuples $\{a, b, c, d, e\}$ such that $c < d < e$, containing the fixed triple $\{a, b, c\}$. Since $d = d_+$, we have five such e's, which gives us five positive solutions

$$\delta = u_{l_i} = v_{m_i} = w_{n_i}, \quad 1 \leq i \leq 5,$$

with $l_1 < l_2 < l_3 < l_4 < l_5$, of equations (2), (3) and (4). We know that three of those solutions belong to the same fundamental solution, so we can use the last lemma, together with lower bounds for ad and bd. We also know $\Delta \geq 4$, because all indices are even. We obtain

$$l_5 > \frac{4(\sqrt{ad})^{2l_1}}{2ad} \log \sqrt{bd} > 2(ad)^7 \log \sqrt{bd} > 2.81 \cdot 10^{32}.$$

But this is a contradiction with (see Lemma 7)

$$\frac{l_5}{\log(l_5 + 1)} < 6.543 \cdot 10^{15} \log^2 d,$$

which yields $d > 10^{10^7}$. $\square$

5. Proof of Theorem 1. We are now ready to prove the main theorem. In the proof we will use the same methods as Dujella used in [**5**] and Fujita in [**11**].

Proof of Theorem 1. Suppose $\{a, b, c, d, e\}$ is a $D(4)$-quintuple such that $a < b < c < d < e$. By Proposition 1 we know that $d < 1.1 \cdot 10^{972}$ and $b < 2.57 \cdot 10^{194}$. We can first bound the number of the pairs $\{a, b\}$. If $b \le 10^{102}$, then this number is at most 10^{204}. If $10^{102} < b < 2.57 \cdot 10^{194}$, by (8) in [**5**] we know

$$\log b > \frac{1}{2}\omega(b) \log \omega(b),$$

where $\omega(b)$ is the number of distinct prime factors of b. If $2^{\omega(b)} \ge b^{0.3}0$, we get $\omega(b) < 101.6$, which gives us $b < 10^{102}$, a contradiction. So we can take $2^{\omega(b)} < b^{0.3}$, and the number of corresponding pairs $\{a, b\}$ is less than (see [**5**, Theorem 4])

$$\sum_{b=10^{102}+1}^{2.57\cdot 10^{194}-1} 2^{\omega(b)+1} < 2 \sum_{b=10^{102}+1}^{2.57\cdot 10^{194}-1} b^{0.3} < 2 \int_{10^{102}}^{2.57\cdot 10^{194}} b^{0.3} db$$
$$< 8.32 \cdot 10^{252}.$$

Therefore, the number of pairs $\{a, b\}$ is less than $8.32 \cdot 10^{252}$. Now, for the fixed pair $\{a, b\}$ we want to find the number of possible c's. But the number c belongs to the union of finitely many binary recurrence sequences, and the number of the sequences is less than or equal to the number of solutions of congruence relation $t_0^2 \equiv 4 \pmod{b}$, such that $-b^{0.75} < t_0 < b^{0.75}$. We get the last inequalities considering the Pellian equation $at^2 - bs^2 = 4(a - b)$, which gives us the extension of $D(4)$-pair $\{a, b\}$. We see that this equation is of the same type as (2), (3) and (4), so from Lemma 2 we have estimate for the fundamental solution $|t_0| < \sqrt{(b\sqrt{b})/a} < b^{0.75}$.

If $b \le 10^{58}$. then the number of sequences is less than or equal to $2 \cdot 10^{43.5} < 10^{44}$. If $10^{58} < b < 2.57 \cdot 10^{194}$, then we see $2^{\omega(b)} < b^{1/3}$. Hence, the number of sequences is less than or $2 \cdot 2^{\omega(b)+1} < 4b^{1/3} < 2.55 \cdot 10^{65}$. Furthermore, the elements contained in each of the sequence grows exponentially with base $> ab \ge 5$, so the number of elements contained in each of the sequence is less than $\log_5(1.1 \cdot 10^{972}) < 1391$. Therefore, the number of c's is less than or $2.54 \cdot 10^{65} \cdot 1391$. So at the end, because the number d is unique, and we can have at most four e's, we see that the number of $D(4)$-quintuples is less than

$$8.32 \cdot 10^{252} \cdot 2.55 \cdot 10^{65} \cdot 1391 \cdot 4 < 10^{323},$$

which finishes the proof of Theorem 1. □

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