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ON ANNIHILATOR IDEALS IN MATRIX NEAR-RINGS

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ABSTRACT. This paper focuses on how the structure of a faithful R-group of a near-ring R determines the ideal structure of the matrix near-ring, $\mathbf{M}_n(R)$, associated with R. Intersections of annihilating ideals of *monogenic* R-groups or $\mathbf{M}_n(R)$-groups are referred to as *Annihilator ideals.* However, it is known that there exist some *non-monogenic* R-groups, say Δ, for which Δ^n is monogenic as an $\mathbf{M}_n(R)$-group. Taking cognizance of these non-monogenic R-groups helps us draw conclusions on relationships between some Jacobson ν-radicals of R and those of $\mathbf{M}_n(R)$, $\nu = 0, s, 2$. In particular, and contrary to Meldrum-Meyer's conjecture in [**9**], it is herein shown that $(J_0(R))^+ \not\subset J_0(\mathbf{M}_n(R))$.

1. Introduction. Relationships between ideals of a near-ring R and ideals of its associated matrix near-ring $\mathbf{M}_n(R)$ has been the subject of a number of research papers on near-rings. For instance, the Jacobson ν-radicals are shown to be related as $(J_\nu(R))^* \supseteq J_\nu(\mathbf{M}_n(R))$ where $\nu = 0, s, 2$, see [**3, 13**]. A similar relationship was also proved in [**5**] for the socle ideals. That is, $(Soi(R))^* \supseteq Soi(\mathbf{M}_n(R))$, where R satisfies the $DCCL$. In order to draw any conclusion on the relationship between $(J_0(R))^+$ and $J_0(\mathbf{M}_n(R))$ one needs to pay attention to non-monogenic R-groups, say Δ, such that Δ^n is a monogenic $\mathbf{M}_n(R)$-group. These non-monogenic R-groups are identified and referred to as R-groups of *ν_n-form*, according to the type-ν of Δ^n as an $\mathbf{M}_n(R)$-group, $\nu = 0, s, \mathcal{K}$. R-groups of ν_n-form are used to construct an example of a near-ring R such that $(J_0(R))^+ \not\subset J_0(\mathbf{M}_n(R))$. This is despite the fact that $(J_s(R))^+ \subseteq J_s(\mathbf{M}_n(R))$ for a near-ring R such that $\mathbf{M}_n(R)$ satisfies the DCCL, see [**4**]. It is because of the R-groups of ν_n-form that we could construct a near-ring, R, such that $J_0(\mathbf{M}_n(R)) \neq J_s(\mathbf{M}_n(R)) \neq J_2(\mathbf{M}_n(R))$ while $J_0(R) = J_s(R) = J_2(R)$.

Throughout this paper R denotes a right-distributive near-ring with multiplicative identity. If the near-ring R satisfies the descending

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chain condition for left ideals, we say that R satisfies the $DCCL$. Similarly, if the near-ring R satisfies the descending chain condition for R-subgroups, we say that R satisfies the $DCCS$. We emphasize that a *near-ring direct sum* means a direct sum, $\Gamma = \bigoplus_{\lambda \in \Lambda} \Delta_\lambda$, where each Δ_λ is an R-kernel of Γ. The set of zero fixing maps from the group Ω to itself is denoted by $M_0(\Omega)$. For basic information and results on near-rings we refer the reader to Pilz [**12**] and Meldrum [**7**].

We use matrix near-rings as defined by Meldrum and van der Walt in [**11**]. For a natural number n, R^n is defined to be the direct sum of n copies of the group $(R,+)$, and an element $\overline{\rho}$ in R^n is denoted by $\overline{\rho} = \langle r_1, r_2, \dots, r_n \rangle$, $r_i \in R$. For $r \in R$ and $1 \leq i,\ j \leq n$, define the function $f_{ij}^r : R^n \to R^n$ by $f_{ij}^r(\overline{\rho}) = \iota_i(r\pi_j(\overline{\rho}))$ for each $\overline{\rho} \in R^n$, where $\iota_i : R \to R^n$ and $\pi_i : R^n \to R$ are the i-th injection and projection functions, respectively. The subnear-ring of $M_0(R^n)$ generated by the set $\{f_{ij}^r \mid r \in R, 1 \leq i,j \leq n\}$ is called the $n \times n$ *matrix near-ring* over R, and it is denoted by $\mathbf{M}_n(R)$. For an ideal A of R there are two ways to construct an ideal in $\mathbf{M}_n(R)$ which relate naturally to A, namely $A^+ := \mathrm{Id}\langle f_{ij}^a \mid a \in A, 1 \leq i,j \leq n\rangle$ and $A^* := \{U \in \mathbf{M}_n(R) | U\overline{\rho} \in A^n, \text{ for all } \overline{\rho} \in R^n\}$, see [**14**]. It is immediate from the above definitions that $A^+ \subseteq A^*$ where A is an ideal in R.

In Section 2 we collect basic results needed in this sequel. In Section 3, properties of monogenic $\mathbf{M}_n(R)$-groups of the form Δ^n are studied, and three forms of non-monogenic R-groups are defined. In Section 4, ideals which are intersections of annihilating ideals of monogenic R-groups or $\mathbf{M}_n(R)$-groups (called *annihilator ideals*) are studied. Sufficient conditions for A^+ to be contained in $\mathcal{A}$ are investigated, where $\mathcal{A}$ is the ideal in $\mathbf{M}_n(R)$ corresponding to A. In section 5, a counter example to Meldrum-Meyer's conjecture in [**9**] is presented.

2. Preliminaries. An R-group Δ is *monogenic* if there exists a δ in Δ such that $\Delta = R\delta$.

Proposition 2.1. *Let R be zero symmetric. If Γ is a monogenic R-group with a near-ring direct sum decomposition, $\Gamma = \Delta \oplus H$, then the R-subgroups, Δ and H, are R-homomorphic images of Γ, hence are monogenic.*

Proof. Let $\Gamma = R\gamma$. Since $\Gamma = \Delta \oplus H$ is a near-ring direct sum, each of the summands, Δ and H, is an R-kernel of Γ. Let the unique representation of γ be $\gamma = \delta + h$, where $\delta \in \Delta$ and $h \in H$. Now, let $d \in \Delta$ and $h' \in H$. Then $d + h' \in \Gamma$. Since Γ is monogenic, we have $d + h' = r\gamma$ for some r in R. By left distributivity over R-kernels, $d+h' = r\gamma = r(\delta+h) = r\delta+rh$; hence, $-r\delta+d = rh-h'$. Since Δ and H are R-subgroups of Γ, $-r\delta \in \Delta$ and $-h', rh \in H$. Thus, we have $-r\delta+d = rh-h' \in \Delta\cap H = (0)$, which gives $d = r\delta$ and $h' = rh$. Since d and h' are arbitrary elements in Δ and H, respectively, we conclude that $R\delta = \Delta$ and $Rh = H$. □

Definition 2.2. A monogenic R-group, $\Omega = R\omega$, for some $\omega \in \Omega$, is of

(i) **type-**0 if it has no non-trivial R-kernels,

(ii) **type-**s if it is of type-0 and for all $\omega' \in \Omega$ with $R\omega' \neq (0)$ we have that there exists a near-ring direct sum decomposition, $R\omega' = \bigoplus_{i=1}^{k} \Delta_i$, where each Δ_i is an R-kernel of $R\omega'$ and an R-group of type-0,

(iii) **type-**2 if it has no non-trivial R-subgroups.

(iv) **type-**$\mathcal{K}$ if it is not of type-0, and it has no type-0 R-kernels as its near-ring direct summands.

Definition 2.3. Let Ω be any R-group.

(i) An ideal A of R is a *ν-primitive* ideal if $A = (0 : \Omega)$ and Ω is of type-ν, $\nu = 0, s, 2$.

(ii) For $\nu = 0, s, 2$, the *Jacobson-type radical*, $J_\nu(R)$, is the intersection of all ν-primitive ideals.

It is known that the socle ideal is characterizable as an intersection of annihilators of R-groups of type-$\mathcal{K}$. We note this in the next theorem.

Theorem 2.4. *Let R be zero symmetric and satisfying the DCCL, and let Ω be a faithful R-group. Then $Soi(R) = \bigcap_{K\in\mathcal{K}(\Omega)} (0\colon K)$ where $\mathcal{K}(\Omega)$ is the set of all type-$\mathcal{K}$ R-subgroups of Ω.*

Now the Jacobson ν-radicals and the socle ideal are intersections of annihilators of monogenic R-groups (or $\mathbf{M}_n(R)$-groups). We refer to such ideals as *Annihilator ideals*.

3. Monogenic $\mathbf{M}_n(R)$-groups. In order to relate annihilator ideals in R to their corresponding ideals in $\mathbf{M}_n(R)$, we include a few results on the relationships between R-groups and $\mathbf{M}_n(R)$-groups. Throughout this section R is assumed to be zero symmetric.

Definition 3.1 [**8**]. Let Ω be any R-group. Then Ω is said to be *locally monogenic* if for any finite subset H of Ω there is an $\omega \in \Omega$ such that $H \subseteq R\omega$.

Note that a monogenic R-group is locally monogenic. Van der Walt defined a natural action of $\mathbf{M}_n(R)$ on Ω^n as follows.

Lemma 3.2 [**13**]. *For Ω a locally monogenic R-group, the group Ω^n is a locally monogenic $\mathbf{M}_n(R)$-group under the action*

$$U\langle \omega_1, \omega_2, \dots, \omega_n \rangle := (U\langle r_1, r_2, \dots, r_n \rangle)\omega$$

for $U \in \mathbf{M}_n(R)$, $\omega_i, \omega \in \Omega$, $r_i \in R$ and $r_i\omega = \omega_i$.

Theorem 3.3 [**13**]. *A locally monogenic R-group, Δ, has no non-trivial R-subgroups (respectively R-kernels) if, and only if, Δ^n has no non-trivial $\mathbf{M}_n(R)$-subgroups (respectively $\mathbf{M}_n(R)$-kernels.)*

Theorem 3.4 [**13**]. *Let Δ be a locally monogenic R-group. Then any $\mathbf{M}_n(R)$-kernel (respectively $\mathbf{M}_n(R)$-subgroup) of Δ^n is of the form H^n, where H is an R-kernel (respectively R-subgroup) of Δ.*

It is a consequence of Theorem 3.3 that, any R-group, $R\delta$, is of type-ν, $\nu = 0, s, 2$, as an R-group if, and only if, $(R\delta)^n$ is of type-ν as an $\mathbf{M}_n(R)$-group. We now extend this result to R-groups of type-$\mathcal{K}$.

Corollary 3.5. *Let Δ be any monogenic R-group. The R-group Δ has a non-trivial R-kernel if, and only if, Δ^n has a non-trivial $\mathbf{M}_n(R)$-kernel. That is, Δ is not of type-*0 *if, and only if, Δ^n is not of type-*0.

Lemma 3.6. *Let Δ be any monogenic R-group. The R-group Δ has no near-ring direct summands of type-*0 *if, and only if, Δ^n has no near-ring direct summands of type-*0 *as $\mathbf{M}_n(R)$-subgroups.*

Proof. $\underline{\Rightarrow}$. Suppose Δ^n has a near-ring direct summand of type-0, say $\Delta^n = D \oplus B$, where B is an $\mathbf{M}_n(R)$-kernel of type-0 and D is some $\mathbf{M}_n(R)$-kernel. By Theorem 3.4, $D = H^n$ and $B = S^n$, where both H and S are R-kernels of Δ. Since $H^n \oplus S^n \cong_{\mathbf{M}_n(R)} (H \oplus S)^n$, it follows that $\Delta = H \oplus S$. Since Δ is monogenic, by Proposition 2.1 both H and S are monogenic, and consequently S is of type-0, by Theorem 3.3. Thus Δ has a near-ring direct summand of type-0.

$\underline{\Leftarrow}$. Conversely, suppose Δ has a near-ring direct summand of type-0, say $\Delta = H \oplus P$, where P is an R-kernel of type-0 and H is an R-kernel. Since Δ is monogenic, it follows from Proposition 2.1 that H is monogenic. Hence Δ^n, H^n and P^n are monogenic $\mathbf{M}_n(R)$-groups, by Lemma 3.2. It is easy to show that both H^n and P^n are $\mathbf{M}_n(R)$-kernels of Δ^n. By Theorem 3.3, P^n is a type-0 $\mathbf{M}_n(R)$-kernel of Δ^n. Since $\Delta^n = (H \oplus P)^n \cong_{\mathbf{M}_n(R)} H^n \oplus P^n$, it follows that Δ^n has a type-0 $\mathbf{M}_n(R)$-kernel as a direct summand. □

Theorem 3.7. *Let Δ be any monogenic R-group. The R-group Δ is of type-$\mathcal{K}$ if, and only if, Δ^n is of type-$\mathcal{K}$ as an $\mathbf{M}_n(R)$-group.*

Lemma 3.8 [**11**, Lemma 1.6]. *If L is a left ideal of R, then the $\mathbf{M}_n(R)$-groups R^n/L^n and $(R/L)^n$ are $\mathbf{M}_n(R)$-isomorphic.*

The next result is a generalization of Lemma 3.8, the proof of which extends easily by applying Lemma 3.2 to any group, $(\Gamma/\Delta)^n$, where $\Gamma = R\gamma$ and Δ an R-kernel of Γ. The action of $\mathbf{M}_n(R)$ on $(\Gamma/\Delta)^n$ is defined by

$$U\langle r_1\gamma + \Delta, r_2\gamma + \Delta, \dots, r_n\gamma + \Delta\rangle := \langle s_1\gamma + \Delta, s_2\gamma + \Delta, \dots, s_n\gamma + \Delta\rangle,$$

where $U\langle r_1, r_2, \dots, r_n\rangle = \langle s_1, s_2, \dots, s_n\rangle$, for some $s_i \in R$, $1 \le i \le n$ and $U \in \mathbf{M}_n(R)$. It is easy to show that the above action is well-defined.

Theorem 3.9. *Let Γ be any monogenic R-group, and let Δ be an R-kernel of Γ. Then $(\Gamma/\Delta)^n \cong_{\mathbf{M}_n(R)} \Gamma^n/\Delta^n$.*

The action of the matrix near-ring $\mathbf{M}_n(R)$ is not limited to $\mathbf{M}_n(R)$-groups of the form Ω^n, where Ω is locally monogenic as in Lemma 3.2. There is another action of $\mathbf{M}_n(R)$ on Ω^n called *Action* 2, for which Ω need not be locally monogenic. Meldrum and Meyer used Action 2 to show that type-0 $\mathbf{M}_n(R)$-groups exist in several non-isomorphic ways, see [**8**].

We now explain the action of $\mathbf{M}_n(R)$ called *Action* 2. Let $\Omega = \bigoplus_{\lambda=1}^{m} \Omega_\lambda$ be a group theoretic direct sum of monogenic R-subgroups $\Omega_\lambda = R\omega_\lambda$, $\lambda = 1, 2, \dots, m$. Then, as a group theoretic direct sum,

$$\Omega^n := \bigoplus_{i=1}^{n} \left(\bigoplus_{\lambda=1}^{m} \Omega_\lambda \right).$$

There is a group isomorphism Ψ from Ω^n to

$$\bigoplus_{\lambda=1}^{m} \left(\bigoplus_{i=1}^{n} \Omega_\lambda \right) = \bigoplus_{\lambda=1}^{m} \Omega_\lambda^n$$

defined by

$$\begin{aligned}\Psi(\langle\langle r_{11}\omega_1, &r_{12}\omega_2, \dots, r_{1m}\omega_m\rangle, \dots, \langle r_{n1}\omega_1, r_{n2}\omega_2, \dots, r_{nm}\omega_m\rangle\rangle) \\ &= \langle\langle r_{11}\omega_1, \dots, r_{n1}\omega_1\rangle, \langle r_{12}\omega_2, \dots, \\ &\qquad r_{n2}\omega_2\rangle, \dots, \langle r_{1m}\omega_m, \dots, r_{nm}\omega_m\rangle\rangle.\end{aligned}$$

This can be rewritten as

$$\langle\langle r_{11}, \dots, r_{n1}\rangle\omega_1, \langle r_{12}, \dots, r_{n2}\rangle\omega_2, \dots, \langle r_{1m}, \dots, r_{nm}\rangle\omega_m\rangle.$$

Each $\langle r_{1j}, \dots, r_{nj}\rangle$, $1 \leq j \leq m$, is an element in R^n on which a matrix acts naturally. Action 2 is now defined as follows.

Definition 3.10 (**Action 2**) [**8**]. Let $U \in \mathbf{M}_n(R)$ and $\overline{\rho} = \langle\overline{\gamma_1}, \overline{\gamma_2}, \dots, \overline{\gamma_n}\rangle$ be any element in Ω^n with each $\overline{\gamma_i} = \langle r_{i1}\omega_1, r_{i2}\omega_2, \dots, r_{im}\omega_m\rangle \in \Omega = \bigoplus_{\lambda=1}^{m} \Omega_\lambda$. Then $U\overline{\rho} := \Psi^{-1}U\Psi(\langle\overline{\gamma_1}, \overline{\gamma_2}, \dots, \overline{\gamma_n}\rangle)$, is a well-defined $\mathbf{M}_n(R)$ action on Ω^n.

Proposition 3.11. *Let* $\Gamma = \bigoplus_{i=1}^{m} R\delta_i$ *be an* R*-group which is a group theoretic direct sum of monogenic* R*-groups. If* $n \geq m$*, then* Γ^n *is a monogenic* $\mathbf{M}_n(R)$*-group under Action* 2.

Proof. Denote the matrix f_{ij}^r by $[r; i, j]$ for any $r \in R$ and $1 \leq i, j \leq n$. Consider the following matrices, $V := \sum_{j=1}^{m} [r_j; 1, j]$ and $U := \sum_{i=1}^{n} [a_i; i, 1]$. Let

$$\underline{\varepsilon} := \langle\langle \delta_1, 0, \dots, 0\rangle, \langle 0, \delta_2, 0, \dots, 0\rangle, \dots, \langle 0, \dots, 0, \delta_m\rangle, \langle 0, \dots, 0\rangle, \dots, \langle 0, \dots, 0\rangle\rangle.$$

We need only prove that $\underline{\varepsilon}$ is an $\mathbf{M}_n(R)$-generator of Γ^n. Let us note that

$$\begin{aligned}&\big(\Psi^{-1} U V \Psi\big)(\underline{\varepsilon})\\ &= \langle\langle a_1 r_1 \delta_1, a_1 r_2 \delta_2, \dots, a_1 r_m \delta_m\rangle, \dots, \langle a_n r_1 \delta_1, a_n r_2 \delta_2, \dots, a_n r_m \delta_m\rangle\rangle.\end{aligned}$$

For any $\overline{\sigma} = \langle\langle r_{11}\delta_1, r_{12}\delta_2, \dots, r_{1m}\delta_m\rangle, \dots, \langle r_{n1}\delta_1, r_{n2}\delta_2, \dots, r_{nm}\delta_m\rangle\rangle$ in Γ^n, it involves simple calculations to show that

$$\Psi^{-1}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m}[r_{ij}; i, j]\right)\right)\Psi(\underline{\varepsilon}) = \overline{\sigma}.$$

Since $\overline{\sigma}$ is arbitrary, we conclude that $\Gamma^n = \big(\mathbf{M}_n(R)\big)\underline{\varepsilon}$. □

We can now classify non-monogenic R-groups such as Γ in Proposition 3.11, according to the type-ν ($\nu = 0, s, \mathcal{K}$) of their corresponding monogenic $\mathbf{M}_n(R)$-group Γ^n.

Definition 3.12. Let Γ be a non-monogenic R-group which is a group-theoretic direct sum of monogenic R-subgroups of Γ. Let $n \in \mathbf{N}$, $n \geq 1$.

(a) If Γ^n is an $\mathbf{M}_n(R)$-group of type-0 but not of type-s, then Γ is said to be an R-group of $\mathbf{0}_n$*-form*.

(b) If Γ^n is an $\mathbf{M}_n(R)$-group of type-s but not of type-2, then Γ is said to be an R-group of $\mathbf{s}_n$*-form*.

(c) If Γ^n is an $\mathbf{M}_n(R)$-group of type-$\mathcal{K}$, then Γ is said to be an R-group of *$\mathcal{K}_n$-form*.

4. Annihilator ideals.

Lemma 4.1. *Let Δ be any monogenic R-group and r any element in R. Then $r \in (0 : \Delta)$ if, and only if, $f_{ij}^r \in (\overline{0} : \Delta^n)$.*

Proof. Let r be any non-zero element in R, and let $\overline{\rho} = \langle \delta_1, \delta_2, \dots, \delta_n \rangle$ be any element in Δ^n where $\Delta = R\delta$. For each $\delta_i = r_i\delta$, with r_i some element of R, we have $f_{ij}^r \overline{\rho} = f_{ij}^r \langle r_1\delta, r_2\delta, \dots, r_n\delta \rangle = f_{ij}^r \langle r_1, r_2, \dots, r_n \rangle \delta = \langle 0, \dots, 0, rr_j, 0, \dots, 0 \rangle \delta = \langle 0, \dots, 0, r\delta_j, 0, \dots, 0 \rangle$ where $r\delta_j$ appears in the i-th place. Clearly, $r \in (0 : \Delta)$ if, and only if, $f_{ij}^r \in (\overline{0} : \Delta^n)$. □

Theorem 4.2. *Let Ω be a faithful R-group and $\{R\delta_\lambda \mid \lambda \in \Lambda\}$ a collection of monogenic R-subgroups of Ω. Then*

$$\left(\bigcap_{\lambda \in \Lambda} (0 : R\delta_\lambda) \right)^+ \subseteq \bigcap_{\lambda \in \Lambda} (\overline{0} : (R\delta_\lambda)^n).$$

Proof. Let $Q := \bigcap_{\lambda \in \Lambda} (0 : R\delta_\lambda)$, and let U be any matrix in Q^+. We prove the result by induction on the weight of U. Firstly, let $U = f_{ij}^a$, where $a \in Q$. Then $f_{ij}^a \in \bigcap_{\lambda \in \Lambda} (\bar{0} : (R\delta_\lambda)^n)$, by Lemma 4.1.

Secondly, let $\overline{\rho} = \langle \omega_1, \omega_2, \dots, \omega_n \rangle$ be any element in $(R\delta_\lambda)^n$, and let $U = f_{ij}^{a_1}(f_{jk}^{a_2} + f_{jl}^{a_3})$ where $a_1, a_2, a_3 \in Q$. We now have $U\overline{\rho} = \langle 0, \dots, 0, a_1(a_2\omega_k + a_3\omega_l), 0, \dots, 0 \rangle$. Clearly, $U\overline{\rho} = \overline{0}$ because $a_2, a_3 \in Q$ and $\overline{\rho} \in (R\delta_\lambda)^n$.

The above two cases provide a basic step to an inductive proof based on the way in which the ideal Q^+ is generated by

$$\left\{ f_{ij}^a \middle| a \in Q = \bigcap_{\lambda \in \Lambda} (0 : R\delta_\lambda),\ 1 \le i, j \le n \right\}.$$

Assume that the theorem holds for any matrix in Q^+ of weight less than n, where n is a positive integer.

Suppose $U = V + W$ where V and W are matrices in Q^+ each of a weight less than n. Then, for any $R\delta_\lambda$, $\lambda \in \Lambda$, and $\overline{\delta} = \langle \delta_1, \delta_2, \dots, \delta_n \rangle \in (R\delta_\lambda)^n$, we have $U\overline{\delta} = V\overline{\delta} + W\overline{\delta} = \overline{0} + \overline{0}$. Thus, $U \in Q^+$. Now, suppose $U = VW$ where V and W are matrices in Q^+ each of a weight less than n. Then, for any $R\delta_\lambda$, $\lambda \in \Lambda$, and $\overline{\delta} = \langle \delta_1, \delta_2, \dots, \delta_n \rangle \in (R\delta_\lambda)^n$, we have $U\overline{\delta} = V(W(\overline{\delta})) = V(\overline{0}) = \overline{0}$. Hence the result follows. □

Let Ω be a faithful R-group, and let $E_\nu(R)$ denote the class of all type-ν R-subgroups of Ω, and let $\mathbf{E}_\nu(\mathbf{M}_n(R))$ denote the class of all type-ν $\mathbf{M}_n(R)$-subgroups of Ω^n, $\nu = 0, s, 2, \mathcal{K}$.

Theorem 4.3. *Let Ω be a faithful locally monogenic R-group. Then*

$$\bigcap_{\Gamma \in \mathbf{E}_\nu(\mathbf{M}_n(R))} (\overline{0} : \Gamma) \subseteq \bigcap_{\Delta \in E_\nu(R)} (\overline{0} : \Delta^n).$$

Moreover, if Ω has no R-subgroups of ν_n-form, then the two ideals are equal.

Proof. By Theorems 3.3 and 3.7, $\Delta \in E_\nu(R)$ implies $\Delta^n \in \mathbf{E}_\nu(\mathbf{M}_n(R))$. This gives $\mathbf{E}_\nu(\mathbf{M}_n(R)) \supseteq \{\Delta^n \mid \Delta \in E_\nu(R)\}$. Hence,

$$\bigcap_{\Gamma \in \beta E_\nu(\mathbf{M}_n(R))} (\overline{0} : \Gamma) \subseteq \bigcap_{\Delta \in E_\nu(R)} (\overline{0} : \Delta^n),$$

which proves the result. □

Corollary 4.4. *Let Ω be a faithful locally monogenic R-group. If Ω has no R-subgroups of ν_n-form, $\nu = 0, s, \mathcal{K}$, then*

$$\left(\bigcap_{\Delta \in E_\nu(R)} (0 : \Delta) \right)^+ \subseteq \bigcap_{\Gamma \in \mathbf{E}_\nu(\mathbf{M}_n(R))} (\overline{0} : \Gamma).$$

The next proposition follows readily from Theorem4.3 and Corollary 4.4.

Proposition 4.5. *Let R satisfy the DCCS, and let Ω be a faithful locally monogenic R-group. If Ω has no R-subgroups of 0_n-form, then*

$$J_0(\mathbf{M}_n(R)) = \bigcap_{\Delta \in E_0(R)} (0 : \Delta^n).$$

Moreover, $(J_0(R))^+ \subseteq J_0(\mathbf{M}_n(R))$.

5. An example of a near-ring. We give this example to illustrate how R-groups of ν_n-form affect ν-primitive ideals of $\mathbf{M}_n(R)$, $\nu = 0, s, \mathcal{K}$. It is a near-ring R with the following properties.

1. $J_0(\mathbf{M}_n(R)) \neq J_s(\mathbf{M}_n(R)) \neq J_2(\mathbf{M}_n(R))$ while $J_0(R)) = J_s(R) = J_2(R))$.

2. $(J_0(R))^+ \not\subset J_0(\mathbf{M}_n(R))$ and $J_s(\mathbf{M}_n(R)) \subsetneq (J_s(R))^*$.

Example 5.1. The symbol $\oplus_G$ denotes a group theoretic direct sum. Consider the group $\Omega := \mathbf{Z}_2 \oplus_G \mathbf{Z}_4 \oplus_G \mathbf{Z}_2 \oplus_G \mathbf{Z}_2$, and the following subgroups: $S := \mathbf{Z}_2 \oplus_G \mathbf{Z}_4 \oplus_G \{0\} \oplus_G \{0\}$;

$\Gamma := \{0\} \oplus_G \{0, 2\} \oplus_G \mathbf{Z}_2 \oplus_G \{0\}$;

$H_1 := \{\overline{0}, (1,0,0,0), (0,2,0,0), (1,2,0,0)\}$;

$H_2 := \{\overline{0}, (0,1,0,0), (0,2,0,0), (0,3,0,0)\}$; $T_1 := \{\overline{0}, (1,0,0,0)\}$;

$H_3 := \{\overline{0}, (0,2,0,0), (1,1,0,0), (1,3,0,0)\}$; $T_2 := \{\overline{0}, (0,2,0,0)\}$;

$T_3 := \{\overline{0}, (1,2,0,0)\}$; $T_4 := \{\overline{0}, (0,0,1,0)\}$; $T_5 := \{\overline{0}, (0,2,1,0)\}$.

Define a subnear-ring R of $M_0(\Omega)$ as

$$R := \{f \in M_0(\Omega) \mid f(S) \subseteq S;\ f(\Gamma) \subseteq \Gamma;\ f(H_i) \subseteq H_i,\ \forall i = 1,2,3;$$

$$f(T_j) \subseteq T_j, \forall j = 1,2,\dots,5;\ \omega - \omega' \in \Gamma \Rightarrow f(\omega) - f(\omega') \in \Gamma,$$

for all $\omega, \omega' \in \Omega$;

$$h - h' \in T_2 \Longrightarrow f(h) - f(h') \in T_2, \text{ for all } h, h' \in H_l,\ l = 1,2,3\}.$$

Then R is a near-ring with a multiplicative identity under point-wise addition and map composition. Observe that:

1. The group, $\Omega = R(1,1,1,1)$, is a faithful R-group which is not of type-0 because Γ is its non-trivial R-kernel. Since Ω is not a near-ring direct sum of R-groups of type-0, $J_0(R) \neq (0)$.

2. The subgroups T_j, $j = 1, 2, \dots, 5$, of Ω are all of order 2 and hence R-subgroups of type-2. The subgroup H_1 of Ω is non-monogenic, and it is a group theoretic direct sum $H_1 = T_1 \oplus_G T_2$.

3. The group, H_2, is monogenic but not of type-0 because T_2 is its R-kernel. Since H_2 has T_2 as the only R-subgroup of type-0, it has no R-group of type-0 as a near-ring direct summand, hence H_2 is of type-$\mathcal{K}$.

4. Similarly, H_3, is monogenic and it is not of type-0 because T_2 is its R-kernel. Since T_2 is the only R-subgroup of H_3, it follows that H_3 has no R-subgroup of type-0 as a near-ring direct summand. Thus H_3 is an R-group of type-$\mathcal{K}$.

5. The R-group, $S = T_1 \oplus_G H_2$, is non-monogenic and it has no non-trivial R-kernels. The $\mathbf{M}_2(R)$-group, $S^2 = \mathbf{M}_2(R)\langle s_2, s_3\rangle$ where $s_2 = (1,0,0,0)$ and $s_3 = (0,1,0,0)$, is monogenic and has no $\mathbf{M}_2(R)$-kernels. Thus S^2 is of type-0 and thus S is an R-group of 0_2-form.

6. The R-group, $\Gamma = T_2 \oplus_G T_4$, is non-monogenic and none of its R-subgroups, T_2, T_4 and T_5, is an R-kernel. In addition, each R-subgroup of Γ is an R-group of type-2. Since $\Gamma^2 = \mathbf{M}_2(R)\langle \gamma_1, \gamma_2\rangle$ is monogenic, where $\gamma_1 = (0,2,0,0)$ and $\gamma_2 = (0,0,1,0)$, and Γ^2 has no non-trivial R-kernels, then Γ^2 is of type-s. Hence Γ is an R-group of s_2-form.

7. The quotient R-groups, H_l/T_2, $l = 1, 2, 3$, are of type-0, but they are each of order 2 as groups, thus each is of type-2. Since every type-0 R-group has an isomorphic copy in the faithful R-group Ω, see [**3**, Theorem 2.1], these quotient R-groups, H_l/T_2, $l = 1, 2, 3$, and the R-groups, T_j, $j = 1, 2, \dots, 5$, are all the type-0 R-subgroups of Ω, up to isomorphism. Therefore $J_0(R) = J_s(R) = J_2(R)$.

(a) We show that $(J_0(R))^+ \not\subset J_0(\mathbf{M}_2(R))$. Define a map z in R by

$$z(x) = \begin{cases} \overline{0} & \text{if } x \in (H_1 \cup \Gamma) \\ (0,2,0,0) & \text{if } x \in (H_l \setminus T_2),\ l = 2,3 \\ x & \text{if } x \in \Omega \setminus (S \cup \Gamma). \end{cases}$$

Note that z annihilates all type-0 R-groups, thus $z \in J_0(R)$. Hence, each $f_{ij}^z \in (J_0(R))^+$, $1 \le i, j \le 2$. For $s_1 = (1,3,0,0) \in S$,

$$f_{12}^z \langle s_1, s_1 \rangle = \langle z(s_1), \overline{0} \rangle = \langle (0,2,0,0), \overline{0} \rangle \neq \langle \overline{0}, \overline{0} \rangle.$$

Thus $f_{12}^z \notin (0 : S^2)$. Since S^2 is a type-0 $\mathbf{M}_2(R)$-group, it follows that $f_{12}^z \notin J_0(\mathbf{M}_2(R))$.

(b) Here we show that $J_0(\mathbf{M}_2(R)) \neq J_s(\mathbf{M}_2(R))$. Consider the element z of R as defined above, and let $V := f_{11}^z + f_{12}^z$. Then $V\langle r_1, r_2 \rangle = \langle z(r_1) + z(r_2), \overline{0} \rangle$, for any $\langle r_1, r_2 \rangle \in R^2$. Now, for $\langle s_1, \overline{0} \rangle \in S^2$, where $s_1 = (1,3,0,0)$, we have

$$V\langle s_1, \overline{0} \rangle = \langle z(s_1), \overline{0} \rangle = \langle (0,2,0,0), \overline{0} \rangle \neq \langle \overline{0}, \overline{0} \rangle.$$

So $V \notin J_0(\mathbf{M}_2(R))$. The type-$s$ $\mathbf{M}_2(R)$-groups are Γ^2, T_l^2 and $(H_l/T_2)^2$, $l = 1,2,3$. By Theorem 3.9, $(H_l/T_2)^2 \cong_{m_2(R)} H_l^2/T_2^2$, $l = 1,2,3$.

By definition $V \in ((0 : \Gamma) \cap (0 : H_1))^+$, and hence

$$V \in (0 : \Gamma)^+ \cap (0 : H_1)^+ \subseteq (0 : \Gamma^2) \cap (0 : H_1^2)$$

by Lemma 4.1. A simple calculation shows that $V(H_l^2) \subseteq T_2^2$ and hence $V(H_l^2/T_2^2) \subseteq \overline{0} + T_2^2$, for each $l = 1,2,3$. Therefore $V \in J_s(\mathbf{M}_2(R))$.

(c) We now prove that $J_s(\mathbf{M}_2(R)) \neq J_2(\mathbf{M}_2(R))$. Define an R-subgroup of $(R,+)$ as $K := \{0, k_1, k_2, k_3\}$ where

$$k_i(x) = \begin{cases} b_i & \text{if } x = (0,0,0,1) \\ \overline{0} & \text{if } x \neq (0,0,0,1) \end{cases}$$

and $b_1 = (0,2,0,0)$, $b_2 = (0,0,1,0)$ and $b_3 = (0,2,1,0)$.

Note that K is of s_2-form as it is R-isomorphic to $\Gamma = T_2 \oplus T_4$. That is, K^2 is an $\mathbf{M}_2(R)$-group of type-s.

Define elements t and s of R by $t(b_3) = b_3$ and $t(x) = 0$ otherwise, and $s(b_1) = b_1$, $s(b_2) = b_2$ and $s(x) = 0$, otherwise. The matrix $B := f_{11}^t(f_{11}^s + f_{12}^s)$ on R^2 yields $B\langle r_1, r_2 \rangle = \langle t(sr_1 + sr_2), 0 \rangle$. For $\langle r_1, r_2 \rangle = \langle k_1, k_2 \rangle$, observe that $B\langle k_1, k_2 \rangle = \langle t(sk_1 + sk_2), 0 \rangle$ and

$$t(sk_1 + sk_2)(0,0,0,1) = t(s(b_1) + s(b_2)) = t(b_1 + b_2) = t(b_3) = b_3 \neq 0.$$

Thus, $B \notin (0 : K^2)$. That is, there exists an $\mathbf{M}_2(R)$-group of type-s which B does not annihilate. Therefore $B \notin J_s(\mathbf{M}_2(R))$.

Simple calculations show that: if $\langle b, b' \rangle \in (\bigcup_{i=1}^{5} T_i^2)$, then $B\langle b, b' \rangle = \langle 0, 0 \rangle$, and if $\langle b, b' \rangle \in (\bigcup_{l=1}^{3} H_l^2 / T_2^2)$, then $B\langle b, b' \rangle \in T_2^2$. That is, B annihilates every $\mathbf{M}_2(R)$-group of type-2, therefore $B \in J_2(\mathbf{M}_2(R))$.

(d) In addition, since $(J_s(R))^* = (J_2(R))^* = J_2(\mathbf{M}_2(R))$, and $J_s(\mathbf{M}_2(R)) \neq J_2(\mathbf{M}_2(R))$, we conclude that $J_s(\mathbf{M}_2(R)) \subsetneq (J_s(R))^*$.

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