ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 5, 2011

$S_{0,3}$ IS NON-RUINOUS

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ABSTRACT. R. Thompson's group F is nonamenable if and only if there exists a k such that $S_{0,k}$ is ruinous. We show that $S_{0,3}$ is not ruinous.

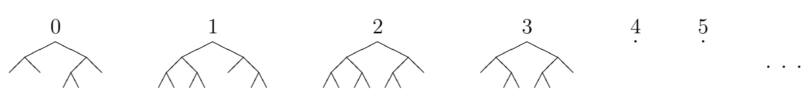
1. Introduction. A theorem of Ore states that if M is a cancellative monoid which has common right multiples, then M embeds as a submonoid into a group G such that the following two conditions are satisfied:

(i) For each $g \in G$, there exist $x, y \in M$ such that $g = xy^{-1}$.

(ii) If M has (monoid) presentation $\langle A \mid R \rangle$, then G is isomorphic to the group defined by the presentation $\langle A \mid R \rangle$.

The group G is called the *group of right fractions* of M, and M is called the *positive monoid* of G. We define Richard Thompson's group F to be the group of right fractions of the monoid P which is given by the presentation $\langle x_0, x_1, x_2, \ldots \mid x_n x_m = x_m x_{n+1} \text{ for } n > m \rangle$.

A (k, m)-*binary forest* $\mathcal{F}$ is a sequence (starting the count at zero) of rooted binary trees such that $\mathcal{F}$ has a total of exactly m carets, and such that for each $i \geq k+1$, the ith tree of $\mathcal{F}$ is the trivial tree, which consists of just a single point. The following is an example of a (3,21)-binary forest.

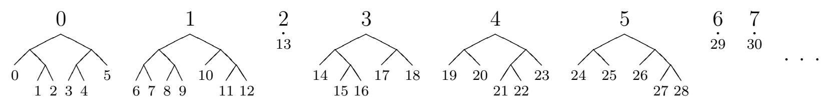


The set of all (k, m)-binary forests is denoted by $S_{k,m}$. If $\mathcal{F}$ is a binary forest, then we denote the i^{th} tree of $\mathcal{F}$ by $\tau_{\mathcal{F}}(i)$.

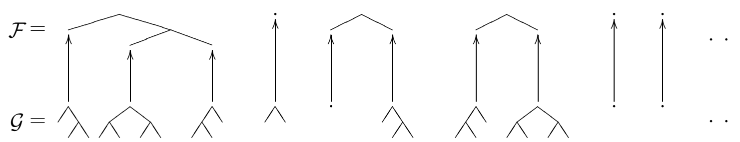
Received by the editors on December 3, 2007, and in revised form on June 25, 2008.

DOI:10.1216/RMJ-2011-41-5-1449

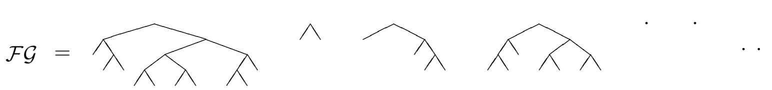
We enumerate the leaves of a binary forest $\mathcal{F}$ from left to right. The following example shows the enumeration on the leaves of a (5,23)-binary forest.



Using this enumeration on the leaves of a binary forest, we define a multiplication on the set of all binary forests. Given two binary forests $\mathcal{F}$ and $\mathcal{G}$, construct the binary forest $\mathcal{F}\mathcal{G}$ by attaching the ith tree of $\mathcal{G}$ to the ith leaf of $\mathcal{F}$. For example, we multiply a (3,4)-binary forest $\mathcal{F}$ with a (7,15)-binary forest $\mathcal{G}$:



to get the following binary forest $\mathcal{F}\mathcal{G}$:



With this multiplication, the set of all binary forests forms a monoid which is isomorphic to the monoid P. More specifically, for $n \geq 0$, the generator x_n of P is the binary forest all of whose trees are trivial except for the nth tree, which consists of a single caret. We say that a subset $A \subseteq S_{0,k}$ is *ruinous* if, for each $m \geq 0$, and for each subset $U \subseteq S_{k,m}$, $|AU| \geq 2|U|$. We say that a subset $U \subseteq S_{k,m}$ is *thin* with respect to A if $|AU| < 2|U|$.

The theory of amenable groups began in the early 20th century with the work of Banach, when he showed that there exists an invariant mean on the set of all bounded real valued functions on $\mathbf{R}$ [**1**]. Von Neumann then showed that, for high enough dimension, the rotation group of the n-sphere has free subgroups on two generators [**9**]. This elaborated on a previous result of Hausdorff, which shows that no mean, which is invariant under all rotations of the 2-sphere, exists on the set of bounded functions on the 2-sphere [**8**]. Von Neumann proved that if a group G contains a free subgroup on two generators, then G is nonamenable [**9**]. The converse of this result has been shown to be false in general. In particular, Olshanskii has constructed an example of a nonamenable group which contains no free subgroup on two generators [**10**]. Furthermore, in [**11**], Ol'shanskii and Sapir construct an example of a finitely presented, nonamenable group which has no free subgroup on two generators. Geoghegan has conjectured that the group F is also an example of a finitely presented, nonamenable group which has no free subroup on two generators [**7**]. In [**2**], Brin and Squier show that F has no free subgroup on two generators. However, the question of whether or not the group F is amenable is still open, and has resisted the efforts of many mathematicians for more than 20 years [**2, 3, 7**].

In [**4**] we show that F is nonamenable if and only if there exists $k \geq 0$ such that $S_{0,k}$ is ruinous. It follows immediately by cancellativity in P that $S_{0,0}$ and $S_{0,1}$ are not ruinous. Since $S_{0,2}$ consists of the two elements x_0x_0 and x_0x_1, then we see that $S_{0,2}\{x_0, x_2\} = \{x_0x_0x_0, x_0x_1x_2, x_0x_0x_2\}$, which implies that $|S_{0,2}\{x_0, x_2\}| < 2|\{x_0, x_2\}|$. Thus, $\{x_0, x_2\}$ is thin with respect to $S_{0,2}$, which implies that $S_{0,2}$ is not ruinous. In Section 3 of this paper, we show that $S_{0,3}$ is not ruinous. We do this by showing that, for m sufficiently large, there exists a set $T \subset S_{3,m}$ such that T is thin with respect to $S_{0,3}$. This set T "prevents" $S_{0,3}$ from being ruinous.

Since for each $k \geq 0$, $S_{0,k}$ is finite, then we can ask what is the minimal number of elements that a set A must contain for A to be ruinous. In [**5**], we show that no subset $A \subseteq S_{0,k}$, such that $|A| \leq 3$, is ruinous. We then expand upon this result in [**6**] by showing that, for sufficiently large m, $S_{3,m}$ is thin with respect to any three element subset of $S_{0,3}$. Thus, the results of this paper improve upon the results in [**6**]. Since the set T that we construct here is thin with respect to all of $S_{0,3}$, then it is clearly thin with respect to any subset of $S_{0,3}$, in particular, any three element subset of $S_{0,3}$.

2. Properties of $S_{k,m}$. The set $S_{0,m}$ is essentially the set of all rooted binary trees, each of which has exactly m carets. It is well known that the number of elements in this set is the mth Catalan number $\frac{(2m)!}{m!(m+1)!}$ [**12**]. The following lemma is proven in [**6**].

Lemma 1. *Let* $m \geq 0$.

(i) $|S_{1,m}| = |S_{0,m+1}|$.

(ii) *For each* $k \geq 1$, $|S_{k+1,m}| = |S_{k,m+1}| - |S_{k-1,m+1}|$.

Applying Lemma 1, we see that

$$\begin{aligned}|S_{3,m}| &= |S_{2,m+1}| - |S_{1,m+1}| = (|S_{1,m+2}| - |S_{0,m+2}|) - |S_{0,m+2}| \\ &= |S_{0,m+3}| - 2|S_{0,m+2}|.\end{aligned}$$

The proof of the following lemma is left to the reader.

Lemma 2. *Let* $m \geq 0$.

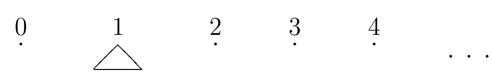
(i) $S_{0,3}S_{3,m} = S_{0,m+3}$.

(ii) *For each* $k \geq 0$, $\lim_{m\to\infty} \frac{|S_{0,m+k}|}{|S_{0,m}|} = 4^k$.

3. A proof that $S_{0,3}$ is not ruinous.

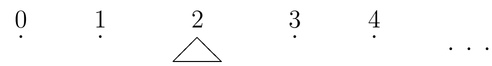
Theorem 1. $S_{0,3}$ *is not ruinous.*

Proof. Let $m \geq 1$. Let $H_{1,m} = \{\mathcal{F} \in S_{3,m} \mid \tau_{\mathcal{F}}(0), \tau_{\mathcal{F}}(2)$, and $\tau_{\mathcal{F}}(3)$ are trivial$\}$. Note that $H_{1,m}$ is the set of all forests of the form



where $\triangle$ is used to denote the single nontrivial tree in $\mathcal{F}$, namely $\tau_{\mathcal{F}}(1)$. Note that there are no restrictions on $\tau_{\mathcal{F}}(1)$, other than that it have a total of m carets. Clearly, $|H_{1,m}| = |S_{0,m}|$.

Let $H_{2,m} = \{\mathcal{F} \in S_{3,m} \mid \tau_{\mathcal{F}}(0), \tau_{\mathcal{F}}(1)$, and $\tau_{\mathcal{F}}(3)$ are trivial$\}$. Note that $H_{2,m}$ is the set of all forests of the form



where $\triangle$ is used to denote the single nontrivial tree in $\mathcal{F}$, namely $\tau_{\mathcal{F}}(2)$. Note that there are no restrictions on $\tau_{\mathcal{F}}(2)$, other than that it have a total of m carets. Again, it is easy to see that $|H_{2,m}| = |S_{0,m}|$.

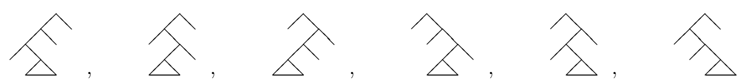
For each $m \geq 1$, let $U_m = S_{3,m} \backslash (H_{1,m} \coprod H_{2,m})$. Thus, if $\mathcal{F} \in U_m$, then exactly one of the following is true:

(1) $\tau_{\mathcal{F}}(0)$ is nontrivial, and all the other trees of $\mathcal{F}$ are trivial.

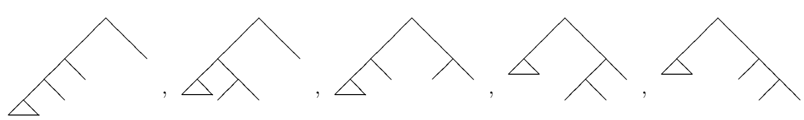
(2) $\tau_{\mathcal{F}}(3)$ is nontrivial, and all the other trees of $\mathcal{F}$ are trivial.

(3) At least two of the trees $\tau_{\mathcal{F}}(0)$, $\tau_{\mathcal{F}}(1)$, $\tau_{\mathcal{F}}(2)$, or $\tau_{\mathcal{F}}(3)$ are nontrivial.

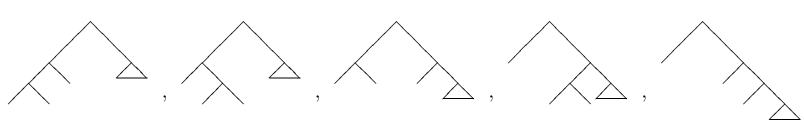
We will show that, for sufficiently large m, U_m is thin with respect to $S_{0,3}$. Let E be the set of all $(0, m+3)$-binary forests $\mathcal{G}$ such that $\tau_{\mathcal{G}}(0)$ is in one of the following forms:



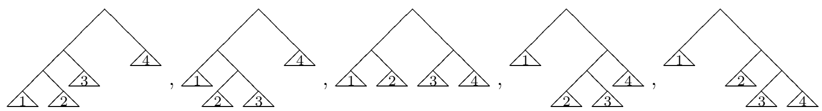
where in each of the six cases above $\triangle$ is used to denote a nontrivial subtree of $\tau_{\mathcal{G}}(0)$. Note that there are no restrictions on this subtree other that it have a total of m carets. Thus, there are $|S_{0,m}|$ possibilities for the subtree, and it is easy to see that $|E| = 6|S_{0,m}|$. Given $\mathcal{H} \in S_{0,3}U_m$, then $\tau_{\mathcal{H}}(0)$ is in one of the following forms:



where in each of the above cases $\triangle$ is used to denote a nontrivial subtree of $\tau_{\mathcal{H}}(0)$,



again, where in each of the above cases $\triangle$ is used to denote a nontrivial subtree of $\tau_{\mathcal{H}}(0)$, or



where in each of the above cases $\triangle_1$, $\triangle_2$, $\triangle_3$, and $\triangle_4$ denote subtrees of $\tau_{\mathcal{H}}(0)$, at least two of which are nontrivial. Because none of the above possibilities for $\tau_{\mathcal{H}}(0)$ are in any of the forms necessary for $\mathcal{H}$ to be an element of E, then it follows that $E \cap (S_{0,3}U_m) = \varnothing$. Therefore, $S_{0,3}U_m \subseteq (S_{0,3}S_{3,m})\backslash E$, which implies that $|S_{0,3}U_m| \leq |S_{0,3}S_{3,m}| - |E|$. Therefore, we see that

$$\begin{aligned}
\limsup_{m\to\infty} \frac{|S_{0,3}U_m|}{|U_m|} &\leq \lim_{m\to\infty} \frac{|S_{0,3}S_{3,m}| - |E|}{|S_{3,m}| - |H_{1,m} \coprod H_{2,m}|} \\
&= \lim_{m\to\infty} \frac{|S_{0,m+3}| - 6|S_{0,m}|}{|S_{3,m}| - 2|S_{0,m}|} \\
&= \lim_{m\to\infty} \frac{|S_{0,m+3}| - 6|S_{0,m}|}{|S_{0,m+3}| - 2|S_{0,m+2}| - 2|S_{0,m}|} \\
&= \lim_{m\to\infty} \frac{\left(\frac{|S_{0,m+3}|}{|S_{0,m}|} - \frac{6|S_{0,m}|}{|S_{0,m}|}\right)}{\left(\frac{|S_{0,m+3}|}{|S_{0,m}|} - \frac{2|S_{0,m+2}|}{|S_{0,m}|} - \frac{2|S_{0,m}|}{|S_{0,m}|}\right)} \\
&= \frac{58}{30} < 2.
\end{aligned}$$

Thus, there exists a $t \in \mathbf{Z}$ such that $\frac{|S_{0,3}U_t|}{|U_t|} < 2$. This implies that U_t is thin with respect to $S_{0,3}$ and, consequently, that $S_{0,3}$ is not ruinous. $\square$

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