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POSITIVE EXISTENTIAL DEFINABILITY OF PARALLELISM IN TERMS OF BETWEENNESS IN ARCHIMEDEAN ORDERED AFFINE GEOMETRY

FRANZ KALHOFF AND VICTOR PAMBUCCIAN

ABSTRACT. We prove that one can define the relation $\parallel$, with $ab \parallel cd$ to be read as '$a = b$ or $c = d$ or ab and cd are parallel lines (or coincide)' positively existentially in $L_{\omega_1\omega}$ in terms of $\neq$ and the ternary relation B of betweenness, with $B(abc)$ to be read as 'b lies between a and c' in Archimedean ordered affine geometry. We also show that a self-map of an Archimedean ordered translation plane or of a flat affine plane which preserves both B and $\neg B$ must be a surjective affine mapping.

1. Introduction. There is a large literature on what came to be called *characterizations of geometric transformations under mild hypotheses*, in which classical geometric transformations are characterized as mappings (which may be required to be one-to-one or onto or both) required to preserve only a certain geometric notion, which was thought to be too weak to characterize the geometric transformation in question. The better known among these surprising characterizations are:

• Carathéodory's characterization of Möbius or conjugate Möbius transformation as one-to-one self-maps of the closed complex plane that map circles (real circles or lines) onto circles (real circles or lines);

• The Mazur-Ulam theorem, stating that surjective isometries of real normed spaces are affine (i.e., map lines onto lines);

• The Beckman-Quarles theorem, stating that self-maps of finite-dimensional real Euclidean spaces which map points at unit distance into points at unit distance must be isometries;

• The Alexandrov-Zeeman theorem, characterizing elements of the orthochronous inhomogeneous Lorentz group with dilations as self-transformations of the real Minkowski space that preserve causality.

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All such theorems can be rephrased as purely logical statements inside a formalized geometric theory, asserting the definability of a geometric notion in terms of another geometric notion. Rephrasings of a large class of characterizations of geometric transformations under mild hypotheses as definability statements, in which explicit definitions are provided, can be found in [**4, 13**–**21, 25, 26**].

The reason why one expects that such rephrasings as definability statements ought to exist for such theorems is encapsulated in the following combination of Beth's definability, Lyndon's preservation theorem, and Keisler's extension thereof (see [**8**], [**5**, Theorem 6.6.4, Exercise 6.6.2]), which can be transferred from first-order logic to $\mathcal{L}_{\omega_1\omega}$, a logic in which one can form countably infinite conjunctions (and disjunctions) of first-order formulas (cf. [**9, 12**], [**1**, Chapter 8]):

Preservation and definability theorem. *Let* $\mathsf{L} \subseteq \mathsf{L}^+$ *be two first order or* $\mathcal{L}_{\omega_1\omega}$ *languages containing a sign for an identically false formula,* $\mathcal{T}$ *a theory in* L^+*, and* $\varphi(\mathbf{X})$ *an* L^+*-formula in the free variables* $\mathbf{X} = (X_1, \dots, X_n)$*. Then the following assertions are equivalent*:

(i) *there is a positive* (*positive existential*; *positive existential with negated equality allowed*) L*-formula* $\psi(\mathbf{X})$ *such that* $\mathcal{T} \vdash \varphi(\mathbf{X}) \leftrightarrow \psi(\mathbf{X})$;

(ii) *for any* $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}\,(\mathcal{T})$*, and each* L*-epimorphism* (L*-homomorphism*; L*-monomorphism*) $f : \mathfrak{A} \to \mathfrak{B}$*, the following condition is satisfied*:

if $\mathbf{c} \in \mathfrak{A}^n$ *and* $\mathfrak{A} \models \varphi(\mathbf{c})$*, then* $\mathfrak{B} \models \varphi(f(\mathbf{c}))$*.*

This theorem tells us that, for every characterization of a geometric transformation under mild hypotheses, i.e., for all statements of type (ii), there must be a definition of the notions preserved by the transformation, i.e., a statement of type (i). One can ask whether there is any legitimate reason for preferring one of the two equivalent variants over the other. Given that the proof of the above theorem is not constructive, in the sense that it does not provide a method for finding $\psi(\mathbf{X})$ in case we know that (ii) holds, finding the actual definition is preferable to proving theorems regarding mappings. One may know that (ii) is true but still not know of any definition $\psi(\mathbf{X})$, whereas whenever we have a definition satisfying the required syntactic conditions, the inference to (ii) is immediate.

An interesting theorem of type (ii) was proved recently in [**6**]. It states that a self-map of an Archimedean ordered affine Pappian plane which preserves both the betweenness and the non-betweenness relation (i.e., both B and $\neg B$) must be an affine mapping and thus onto.[1]

The aim of this paper is to provide a purely geometric proof of this result, to slightly generalize this result to flat affine planes, as well as to isolate the main reason why the result and the algebraic characterization of such maps given in [**6**] holds, which is better than one would expect (for, by our Preservation and Definability theorem, the equivalent reformulation, as in (i), would ensure only an existential definition in terms of B), the positive existential definability of the parallelism relation in terms of betweenness and $\neq$.

Main references for the theory of ordered affine planes (in terms of betweenness relations) and the theory of ordered projective planes (in terms of separation relations) are [**24**, Chapter 9], [**28**, Chapter 5] and [**33**]. For logical matters relevant to this paper one should consult [**1**, Chapter 8] and [**5**].

2. Positive existential definition of parallelism in terms of betweenness and $\neq$. Ordered affine geometry can be expressed, with only one sort of variables, standing for *points*, in two different languages:

- one in which the only predicate is the ternary predicate B, with $B(abc)$ standing for 'point b is between a and c (or coincides with a or with c', and

- one with two predicates, B and $\|$, with $ab \parallel cd$ to be read as 'the lines determined by (a, b) and (c, d) are parallel or coincide, or $a = b$, or $c = d$.'

Axiom systems of the first type can be obtained in the two-dimensional case by adding the Euclidean parallel axiom, stating the existence and uniqueness of a parallel from a point to a line, to an axiom system for ordered planes, i.e., one with linear order axioms stating that the order is dense and unending, and the Pasch axiom (in the form that implies two-dimensionality, stating that a line which meets one of the sides of a triangle, must meet another side of that triangle as well). In the higher-dimensional or the dimension-free case (in the latter there is only a lower-dimension axiom, stating that there are three non-collinear points, but no upper-dimension axiom), one obtains an axiom system of the first type by replacing every occurrence of $L(xyz)$ (which stands for '$x, y0$ and z are collinear points (not necessarily different points)') in an L-based axiom system for affine geometry (such as that presented in [**11**]) by $B(xyz) \vee B(yzx) \vee B(zxy)$, as well as adding the axioms for dense, unending linear orders and the outer form of the Pasch axiom (see [**32**, page 12]).

Axiom systems of the second type were presented, for the plane case, in [**33**] and, for the dimension-free case with dimension ≥ 3, in [**10**] (order axioms, including the invariance of betweenness under parallel projections, in the form of axiom **oPasch** in [**33**, page 148] (see A13 in the Appendix), need to be added to the axiom system in terms of $\parallel$ presented in [**10**] to get an axiom system for ordered affine spaces of dimension ≥ 3).

It was shown in [**33**, 4.6] and in [**32**, Satz 6.62] that axiom systems of the first type cannot consist entirely of $\forall\exists$-statements (i.e., of statements all of whose universal quantifiers (if any) precede all existential quantifiers (if any) when written in prenex form), whereas the axiom systems of the second type from both [**10, 33**] (together with the order axioms) consist of $\forall\exists$-statements. We conclude that an existential definition of $\parallel$ in terms of B cannot exist, for, if there were such a definition, we could replace it for $\parallel$ in the $\forall\exists$ axiom systems for ordered affine geometry of the second type to get a $\forall\exists$-axiom system of the first type.

However, there exists a positive definition of $\parallel$ in first-order logic in terms of L (and thus, *a fortiori*, in terms of B, as we can consider all occurrences of $L(uvw)$ in the definiens to be abbreviations of $B(uvw) \vee B(vwu) \vee B(wuv)$), the definition being valid in all ordered affine planes. It can be formulated as the following $\forall\exists$-statement:

$$
\begin{aligned}
(1)\quad ab \parallel cd :\Longleftrightarrow (\forall xyz)(\exists uv)\,[&(L(abc) \wedge L(abd)) \\
&\vee c = d \vee (L(abx) \wedge L(aby)) \\
\vee (L(cdx) \wedge L(cdy)) \vee L(xyz) \vee (((&L(xyu) \wedge L(xyv)) \\
\vee (L(xzu) \wedge L(xzv))) \wedge u \neq v &\wedge L(abu) \wedge L(cdv))].
\end{aligned}
$$

That the definiens holds when $ab \parallel cd$ is seen by noticing that the cases $a = b$, $c = d$, or the coincidence of the lines ab and cd are part of the definiens, and if ab and cd are two different parallel lines, and xy

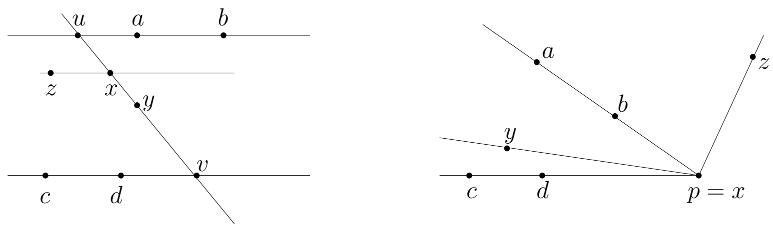

FIGURE 1. The positive definition of $ab \parallel cd$ in terms of L.

and xz are two different lines (i.e., if $\neg L(xyz)$), then at most one of xy and xz can be parallel to ab (and cd), so at least one must intersect the lines ab and cd in two different points u and v.

That the definiens does not hold when $ab \not\parallel cd$, i.e., when ab and cd are two different lines and there is a point p with $L(abp)$ and $L(cdp)$, can be seen by choosing x to be p and y and z to be two points, not on the lines ab or cd, such that $\neg L(xyz)$. In this case, the lines xy and xz intersect both ab and cd in the point x, so there are no different points u and v on one of the lines xy and xz, such that one lies on ab and the other one lies on cd.

That the $u \neq v$ which appears in (1) can be positively defined in terms of L (and thus in terms of B) has been shown, for all affine spaces of finite dimension ≥ 2, in [**13**], and thus the definiens in (1) can be rephrased as a positive statement in L.

The Archimedean nature of an order relation cannot be expressed in first-order logic (if it could, then so would Archimedean ordered fields, which is impossible, as they are isomorphic to subfields of **R**, and, by the Löwenheim-Skolem theorem, any first-order theory with infinite models must have models of any infinite cardinality), but it can be expressed, with $L := L_B$, in $L_{\omega_1\omega}$ (see [**1**, Chapter 8] for a definition and for the main properties of this logic), in weak second order logic $L(II_0)$ (that allows quantification over finite sets, see [**31**]), in logic with the Ramsey quantifier Q^2 (see [**2**]), as well as in deterministic transitive closure logic $L(DTC)$ (see [**3**, 8.6] for a definition).

We will choose to express our positive existential definition of $\parallel$ in terms of B in $L'_{\omega_1\omega}$, where $L' := L_{B\parallel}$. *The definition itself is valid in the theory axiomatized by* $(\alpha \vee \beta)\wedge$ Arch, where by α we have denoted the conjunction of the L'-axioms for ordered affine planes (from [**33**]), by β the conjunction of the L'-axioms for ordered affine spaces of dimensions ≥ 3 (from [**10**], together with the order axioms), and by *Arch* the Archimedean axiom, which in affine ordered geometry, with no configuration theorems (minor or major Desargues, Pappus) among the axioms, will be stated by using the defined notions Z and σ, which are defined by

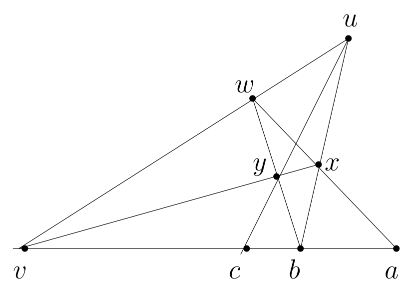

FIGURE 2. The definition of $\sigma(abcuvw)$.

$$Z(abc) :\Longleftrightarrow B(abc) \wedge a \neq b \wedge b \neq c,$$
$$\sigma(abcuvw) :\Longleftrightarrow (\exists xy)\,[B(bxu) \wedge B(axw) \wedge B(vyx) \wedge B(byw) \wedge B(uyc) \wedge B(bcv)].$$

The meaning of Z is plain: it stands for strict betweenness. To have some intuition of what σ stands for, let us note that, if we think of the line vu as the line at infinity, then $\sigma(abcuvw)$ stands for the fact that the point c is the reflection of a in b, and $\sigma(abcuvw)$ can thus be thought of as asserting that c is (a projective geometry view of) the 'reflection' of a into b, constructed with the help of u, v, w, x, y.

We are now ready to state the Archimedean axiom, which states that, given two points a_1 and a_2, a point p on the ray $\overrightarrow{a_1a_2}$, and a line uw (which we may think of as the 'line at infinity') meeting the ray $\overrightarrow{a_1a_2}$ in a point v (which we may think of as 'at infinity'), which is such that p is strictly between a_2 and v, the sequence of points a_i, obtained by iterating the 'reflection' operation, first 'reflecting' a_1 in a_2 to get a_3 (by means of $\sigma(a_1a_2a_3uvw)$), then a_2 in a_3 to get a_4, and so on, will eventually move past p, i.e., for some n, we will find p lying between a_2 and a_{n+2}.

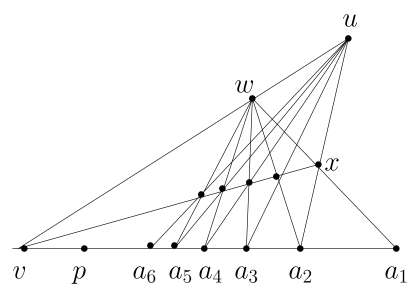

FIGURE 3. The Archimedean axiom.

$$(\forall a_1a_2uvwpx)\,\{\neg L(a_1vw) \wedge Z(a_1xw) \wedge Z(a_2xu)$$
$$\wedge\, Z(vwu) \wedge Z(a_1a_2v) \wedge Z(a_2pv)$$
$$\longrightarrow \left[\bigvee_{n=1}^{\infty}\left((\exists a_3\cdots a_{n+2})\,\bigwedge_{i=1}^{n}\sigma(a_ia_{i+1}a_{i+2}uvw) \wedge B(a_2pa_{n+2})\right)\right]\Bigg\}.$$

To improve the readability of the definition of $\|$ in terms of B, to be stated later, we define further abbreviations:

$$\varphi_n(p,q,r,s,u,v) :\Longleftrightarrow (\exists a_3\cdots a_nu_3\cdots u_n)\,[B(rva_3) \wedge \bigwedge_{i=3}^{n-1}B(ru_ia_{i+1})$$
$$\wedge\, B(uvu_i) \wedge B(su_ia_i) \wedge B(pqa_i)],$$
$$\psi(m,n,p,q) :\Longleftrightarrow (\exists cu_1u_2xy)\,[Z(ncp) \wedge Z(mcq) \wedge Z(mu_1p)$$
$$\wedge\, Z(nu_2q)$$
$$\wedge\, B(u_2u_1x) \wedge B(nmx) \wedge B(yu_2u_1) \wedge B(yqp)$$
$$\wedge \bigwedge_{i=4}^{\infty}\varphi_i(m,n,p,q,u_1,u_2) \wedge \bigwedge_{n=4}^{\infty}\varphi_i(q,p,n,m,u_2,u_1)].$$

We are now ready to state our first result:

Theorem 1. *The relation $\|$ is positively existentially definable in terms of B and $\neq$. The definition, a statement of $L'_{\omega_1\omega}$, is*

(2)

$$a_1a_2 \parallel b_1b_2 \longleftrightarrow (a_1 = a_2 \vee b_1 = b_2 \vee \psi(a_1,a_2,b_1,b_2) \vee \psi(a_1,a_2,b_2,b_1)).$$

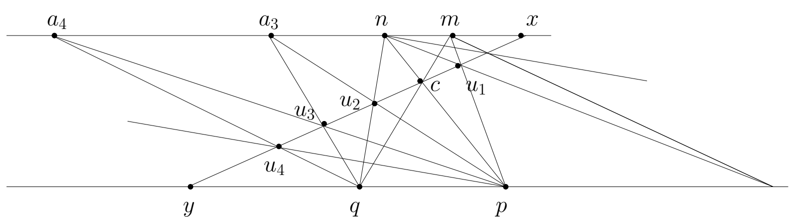

FIGURE 4. $\psi(m, n, p, q)$ states that the process described in the picture can be continued indefinitely.

Proof. To see that this definition is a theorem of $(\alpha \vee \beta) \wedge$ *Arch*, we first notice that the definiens holds whenever $a_1a_2 \parallel b_1b_2$ holds.

The cases $a_1 = a_2$ or $b_1 = b_2$ are taken care of by being part of the definiens. If $a_1 \neq a_2$ and $b_1 \neq b_2$, and the lines determined by (a_1, a_2) and (b_1, b_2) coincide, then all the points that have to exist, as required by the definitions of ψ and φ, can be chosen to be any points lying on the line determined by (a_1, a_2), which satisfy the required betweenness relations. Given that the order is dense, such points always exist, and thus the definiens holds.

If the lines determined by (a_1, a_2) and (b_1, b_2) are parallel (and thus do not coincide), then a_2 and b_2 either lie on the same side of the line determined by (a_1, b_1) in the plane π determined by (a_1, a_2) and (b_1, b_2), or they lie on different sides of (a_1, b_1). If they lie on the same side of (a_1, b_1), then $\psi(a_1, a_2, b_1, b_2)$, which can be seen by choosing c to be the intersection point of segments a_2b_1 and a_1b_2, u_1 to be any point with $Z(a_1u_1b_1)$, x any point with $Z(a_2a_1x)$, the intersections of line xu_1 with lines a_2b_2 and b_1b_2 to be u_2 and y. That $\varphi_i(a_1, a_2, b_1, b_2, u_1, u_2)$ holds for all $i \geq 4$ can be seen by noticing that the u_i are points that lie inside the strip determined by the lines a_1a_2 and b_1b_2 and in the halfplane determined by a_2b_1 in which b_2 lies, and thus that b_1u_i, not being parallel to a_1a_2, must intersect it, and their point of intersection a_{i+1} must be, given the position of u_i, such that $Z(a_1a_2a_{i+1})$; u_{i+1} is defined as the point of intersection of u_1u_2 with b_2a_{i+1} (this point must exist, given that b_2 and a_{i+1} lie on different sides of the line xy (which coincides with the lines u_1u_2)). For similar reasons, $\varphi_i(b_2, b_1, a_2, a_1, u_2, u_1)$ holds for all $i \geq 4$.

To see that the definiens does not hold when $a_1a_2 \nparallel b_1b_2$, notice first that, if a_1, a_2, b_1, b_2 are not all different, but they are not all collinear, i.e., if $a_1 \neq a_2$, $b_1 \neq b_2$, $\neg(L(a_1a_2b_1) \wedge L(a_1a_2b_2))$, $(\exists u)\, L(a_1a_2u) \wedge L(b_1b_2u)$, and $|\{a_1, a_2, b_1, b_2\}| < 4$, then there is no c such that one of $Z(a_2cb_1) \wedge Z(a_1cb_2)$ and $Z(a_1cb_1) \wedge Z(a_2cb_2)$ holds; thus, neither can

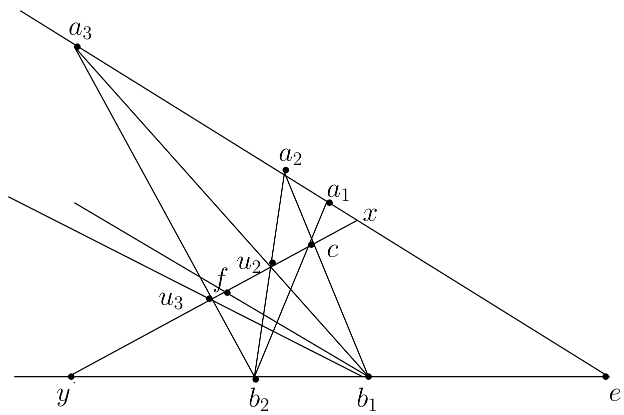

FIGURE 5. If lines a_1a_2 and b_1b_2 intersect in e with $Z(a_2a_1e)$ and $Z(b_2b_1e)$, then the process required by $\psi(a_1, a_2, b_1, b_2)$ breaks at some point (i.e., there are only finitely many a_i's.) In the picture, a_3 exists, but a_4 does not, given that u_3 lies on the same side of b_1f (the parallel from b_1 to ex) as y, so ray $\overrightarrow{bu_3}$ cannot intersect ray $\overrightarrow{ex}$.

$\psi(a_1, a_2, b_1, b_2)$ or $\psi(a_1, a_2, b_2, b_1)$ hold. Suppose now that the points a_1, a_2, b_1, b_2 are all different and that $a_1a_2 \nparallel b_1b_2$. If the segments a_1a_2 and b_1b_2 intersect, then again, there can be no point c such that one of $Z(a_2cb_1) \wedge Z(a_1cb_2)$ and $Z(a_1cb_1) \wedge Z(a_2cb_2)$ holds, and we are done.

Suppose that the segments a_2b_1 and a_1b_2 intersect in c, and the lines a_1a_2 and b_1b_2 intersect in a point e with $Z(a_2a_1e) \wedge Z(b_2b_1e)$. Let f be the point with $Z(xfy)$ and $b_1f \parallel ex$. Given *Arch*, there must be an index n such that $B(u_nfy)$, and thus a_{n+1} required by $\psi(a_1, a_2, b_1, b_2)$ to exist ($\psi(a_1, a_2, b_2, b_1)$ cannot hold, since there is a c such that $Z(a_2cb_1) \wedge Z(a_1cb_2)$) cannot exist. By a similar argument, we find that the sequence of a_n's cannot be infinite, for the case in which the intersection point e of lines a_1a_2 and b_1b_2 is in the position $Z(a_1a_2e) \wedge Z(b_1b_2e)$, as well as for the case in which the segments a_1b_1 and a_2b_2 intersect. □

One may wonder whether $\|$ is existentially definable (without the restriction that the definiens be positive as well) in terms of B by means of a first-order formula, with the definition being valid in, say, affine planes over Archimedean ordered fields. That the answer is negative can be seen from the fact that such an existential definition δ (an existential L_B-formula), with $ab \parallel cd \leftrightarrow \delta(abcd)$ true in all affine planes over Archimedean ordered fields, would have to hold in all affine planes over arbitrary ordered fields as well, given that its field-theoretic counterpart, an $\forall\exists$-statement in the theory of ordered fields, must be true in all ordered fields if it is true in all Archimedean ordered fields (as the $\forall\exists$-theory of Archimedean ordered fields coincides with the $\forall\exists$-theory of ordered fields, given that, when written in prenex disjunctive normal form, an $\forall\exists$-sentence amounts to the statement that one of a finite set of systems of equations and inequalities has a solution, and the existence of solutions for such systems does not depend on the Archimedean nature of the order). However, as noted earlier, we know from [**33**, 4.6] that such a δ cannot exist for ordered affine planes.

Corollary. *Let $\mathfrak{A}$ be an Archimedean ordered affine plane admitting a subplane $\mathfrak{A}'$ which is ordered with respect to the induced ordering. Then $\mathfrak{A}'$ carries the parallelism of $\mathfrak{A}$, i.e., the relation $\|$ of $\mathfrak{A}'$ is a restriction of the relation $\|$ of $\mathfrak{A}$.*

Proof. The relation $\|$ in $\mathfrak{A}'$ can be defined by the positive existential definition (2) in terms of B, and we will refer to it only as the relation defined by the definiens in (2).

If $ab \parallel_{\mathfrak{A}'} cd$, then there are points in the plane $\mathfrak{A}'$ for which the definiens of (2) is satisfied. These points belong to $\mathfrak{A}$ as well, so the definiens holds in $\mathfrak{A}$, and thus, by (2), $ab \parallel_{\mathfrak{A}} cd$.

Suppose now a, b, c, d are points belonging to the universe of $\mathfrak{A}'$, and that $ab \parallel_{\mathfrak{A}} cd$. Were $ab \parallel_{\mathfrak{A}'} cd$ not to hold, then we would have $a \neq b$, $c \neq d$, a, b, c, d are not all on one line, and there would exist a point p in the universe of $\mathfrak{A}'$ with $L(abp)$ and $L(cdp)$. Since the point p belongs to the universe of $\mathfrak{A}$ as well, we couldn't have $ab \parallel_{\mathfrak{A}} cd$. □

To show that this corollary does make a point, we present examples of Pappian ordered affine planes $\mathfrak{A}$ admitting an affine subplane $\mathfrak{A}'$, which is ordered with respect to the induced ordering and which carries a parallelism distinct from that of $\mathfrak{A}$.

Lemma. *Let $\mathfrak{P}$ be an ordered projective plane, let $\mathfrak{A} = \mathfrak{P} \setminus U$ and $\mathfrak{A}'' = \mathfrak{P} \setminus W$ be affine restrictions of $\mathfrak{P}$ with respect to two distinct lines U and W of $\mathfrak{P}$, and let $\mathfrak{A}'$ be an affine subplane of $\mathfrak{A}''$ carrying the parallelism of $\mathfrak{A}''$. Endow $\mathfrak{A}$ and $\mathfrak{A}''$ with the betweenness relations B and B'' respectively, induced by the ordering of $\mathfrak{P}$. If U does not contain any point lying between (with respect to B'') two points of $\mathfrak{A}'$, then $\mathfrak{A}'$ is also a subplane of $\mathfrak{A}$, and thus carries the betweenness relation of $\mathfrak{A}$ (i.e., B and B'' coincide on $\mathfrak{A}'$), but has a distinct parallelism.*

Proof. By the setting above, $\mathfrak{A}$ and $\mathfrak{A}''$ are affine planes, which have U and W as their lines at infinity, have the points of $\mathfrak{P} \setminus (U \cup W)$ in common, and in which three points are collinear, if and only if they are collinear in $\mathfrak{P}$. In particular, we have $\mathfrak{A}' \subset \mathfrak{P} \setminus (U \cup W) \subset \mathfrak{A}$. Since two projective lines are parallel in an affine restriction if and only if they meet on the associated line at infinity, any two lines that are parallel in $\mathfrak{A}$ are not parallel in $\mathfrak{A}''$ and vice versa, unless they meet in $U \cap W$.

Recall that the ordering of $\mathfrak{P}$, defined in terms of a separation relation $xy|uv$ (which stands for 'the points x and y separate the points u and v'), induces betweenness relations on $\mathfrak{A}''$ and $\mathfrak{A}$ by

$$\begin{aligned} B''(bac) &:\Longleftrightarrow aw|bc \text{ where } w \text{ is the point of intersection} \\ &\qquad \text{of the lines } ab \text{ and } W \\ B(bac) &:\Longleftrightarrow au|bc \text{ where } u \text{ is the point of intersection} \\ &\qquad \text{of the lines } ab \text{ and } U \end{aligned}$$

turning $\mathfrak{A}''$ and $\mathfrak{A}$ into ordered affine planes (cf., [**24**] or [**28**]). Of course, generally these two betweenness relations differ (given that the corresponding parallelisms differ), but they coincide on $\mathfrak{A}'$. For, given any three (distinct) collinear points a, b, c in $\mathfrak{A}'$ with w and u the points of intersection of the lines ab with W and U, respectively, we have that $uw|bc$ is false (since U, W do not separate points of $\mathfrak{A}'$), and so that $aw|bc$ is equivalent to $au|bc$. □

Examples. To obtain concrete examples of affine planes as in the above Lemma, we start with an extension $K \subset L$ of ordered fields and an element $t_0 \in L$ such that $t_0 > k$, for all $k \in K$. Let $\mathfrak{A}'$ be the ordered affine plane over K, $\mathfrak{A}''$ the ordered affine plane over L, and $\mathfrak{P}$ the projective closure of $\mathfrak{A}''$ where W denotes the associated line at infinity.

By [**24**, 9.2], [**28**, 5.1] the ordering of $\mathfrak{A}''$ uniquely extends to an ordering of $\mathfrak{P}$ (in terms of a separating relation; in the case of Pappian planes we have $xy|uv$ if and only if the cross ratio of the four points is negative in the underlying field). We thus get the natural embeddings $\mathfrak{A}' \subset \mathfrak{A}'' \subset \mathfrak{P}$, the parallelism of $\mathfrak{A}'$ is a restriction of that of $\mathfrak{A}''$, and the orderings of $\mathfrak{A}'$ and $\mathfrak{A}''$ are induced by that of $\mathfrak{P}$.

If we choose U to be the projective line whose affine part in $\mathfrak{A}''$ is $\{(x, y) \in L^2 \mid x = t_0\}$, the assumptions of the lemma above are obviously fulfilled. Note that there exists a collineation φ of $\mathfrak{P}$ with $\varphi(U) = W$ (i.e., $\varphi \in PGL(3, L)$ and thus order-preserving), which means that $\mathfrak{A}$ and $\mathfrak{A}''$ are isomorphic as ordered affine planes.

For concreteness' sake, we present some examples of this construction:

(a) Take $K := \mathbf{R}$, $L := \mathbf{R}((t))$ to be the field of real Laurent series, fix the unique ordering of L for which t is positive, and let $t_0 := t$. By the above construction, we get a Pappian ordered affine plane $\mathfrak{A}$ admitting an affine subplane $\mathfrak{A}'$, which is even Archimedean ordered with respect to the induced ordering, and which carries a parallelism distinct from that of $\mathfrak{A}$.

(b) Let $\Gamma := \oplus_{i=1}^{\infty} \mathbf{Z}$ be the $\mathbf{N}$-fold direct sum of $(\mathbf{Z}, +)$, ordered lexicographically, i.e., $(z_1, z_2, \dots) > (y_1, y_2, \dots) :\Leftrightarrow z_i = y_i$ for $i = 1, \dots k-1$ and $z_k > y_k$ for some $k \in \mathbf{N}$. Let $L := \mathbf{Q}((\Gamma))$ be the field of formal power series over $\mathbf{Q}$ on Γ (the elements of which are formal sums $f := \sum_{\gamma \in \Gamma} f_\gamma t^\gamma$ with anti-well-ordered support $s(f) := \{\gamma \in \Gamma | f_\gamma \neq 0\}$, with $f_\gamma \in \mathbf{Q}$, the sum and product of which are induced by the rules $f_\gamma t^\gamma + g_\gamma t^\gamma := (f_\gamma + g_\gamma) t^\gamma$ and $t^\gamma \cdot t^\delta := t^{\gamma+\delta}$), and fix that ordering of L in which all t^γ are positive (i.e., $f > 0$ if and only if $f_\gamma > 0$ for $\gamma := \max(s(f))$ and $f \neq 0$), cf. [**28**, II, Section 5].

The order-preserving group monomorphism $\alpha : \Gamma \to \Gamma$, $(z_1, z_2, \dots) \mapsto (0, z_1, z_2, \dots)$ onto $\Delta := \oplus_{i=2}^{\infty} \mathbf{Z}$ induces an order-preserving field monomorphism $\overline{\alpha} : L \to L$, defined by the rule $t^\gamma \mapsto t^{\alpha(\gamma)}$ onto $K := \mathbf{Q}((\Delta))$. Thus $\overline{\alpha}$ is an order-preserving isomorphism from L onto a proper subfield K of L. Further, L admits an element t_0 with $t_0 > k$ for all $k \in K$, namely, $t_0 := t^{(1,0,0,\dots)}$. By the above construction we get a Pappian ordered affine plane $\mathfrak{A}$ admitting an affine subplane $\mathfrak{A}'$, which is ordered with respect to the induced ordering, is even isomorphic to $\mathfrak{A}$ as an ordered affine plane, but which carries a parallelism distinct from that of $\mathfrak{A}$.

(c) Examples where the (proper) affine subplane $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ and carries the same parallelism, may be easier obtained by taking $L := \mathbf{R}((t))$, fixing again its ordering with $t > 0$, and considering the field monomorphism $L \to L$ defined by the rule $t \mapsto t^2$. It yields an order-preserving isomorphism from L onto its subfield $K := \mathbf{R}((t^2))$, and thus an order-preserving isomorphism from the affine plane over L onto its subplane, the affine plane over K.

Also notice that our Corollary does not simply follow from (1), as there are instances of pairs of Pappian affine planes $\mathfrak{A}'$ and $\mathfrak{A}$, with $\mathfrak{A}'$ a subplane of $\mathfrak{A}$, but such that the coordinatizing field K of $\mathfrak{A}'$ is not embeddable in the coordinatizing field L of $\mathfrak{A}$. If we denote by $\mathfrak{A}'$ the minimal affine plane (i.e., the affine plane over $GF(2)$), which embeds into all affine planes, and thus into all (ordered) affine planes $\mathfrak{A}$ over ordered fields L, $\mathfrak{A}'$ is clearly not orderable, the corresponding parallelisms differ, the projective closure of $\mathfrak{A}'$, i.e., the Fano plane (the projective plane over $GF(2)$), does not embed into the projective closure of $\mathfrak{A}$, and $GF(2)$ does not embed into L. Further (non-Desarguesian) examples along this line can be found in [**7**].

The following result will be used for a short proof of our next theorem, and is of interest in its own right.

Proposition. *Let $\mathfrak{P}$ be an Archimedean ordered projective Pappian plane, and let $\mathfrak{P}'$ be a subplane of $\mathfrak{P}$ endowed with the induced ordering. If $\mathfrak{P}$ and $\mathfrak{P}'$ are isomorphic as ordered projective planes, then $\mathfrak{P} = \mathfrak{P}'$.*

Proof. Choosing a coordinatizing frame of $\mathfrak{P}'$ we obtain Archimedean ordered fields $K' \subset K$, where K' coordinatizes $\mathfrak{P}'$ and K coordinatizes $\mathfrak{P}$, and where the order-preserving isomorphism between $\mathfrak{P}$ and $\mathfrak{P}'$ induces an order-preserving isomorphism between K and K', cf. [**24**, 4.1 and 9.3] and [**28**, V.4, Satz 12]. Now the usual proof, showing that any order-preserving automorphism of an Archimedean field is the identity, obviously goes over to order-preserving homomorphisms $\alpha : K \to K' \subset K$: assuming $\alpha(k) \neq k$, say $\alpha(k) < k$, and choosing an element q of the prime field $\mathbf{Q}$ of K' with $\alpha(k) < q < k$, leads to the contradiction $q < k \Rightarrow q = \alpha(q) < \alpha(k)$. So we have $K' = K$, and therefore $\mathfrak{P} = \mathfrak{P}'$. $\square$

We now turn to our main result, which is Theorem 4.5 of [**6**] (and which, incidentally, provides an answer to a question raised by Peano [**23**, page 144]), with an alternate wording, for which we provide a purely geometrical proof.

Theorem 2. *A map* $g : \mathfrak{A} \to \mathfrak{A}$, *which satisfies* $B(abc) \Leftrightarrow B(g(a)g(b)g(c))$ *for all* a, b, c *in* $\mathfrak{A}$, *where* $\mathfrak{A}$ *is an Archimedean ordered Pappian plane, must be surjective.*

Proof. We will present two proofs for this theorem. Our first proof will be purely geometric, while our second proof will use the above Proposition.

1. Let $f : \mathbf{N} \to \{0, 1\}$. We define

$$M_u(amb) :\Longleftrightarrow (\exists u')\,[\neg L(abu) \wedge au' \parallel ub \wedge au \parallel a'b \wedge B(amb) \wedge B(umu')],$$

to be read as 'm is the midpoint of ab (u being an auxiliary point in the construction of m),' as well as the formulas $\delta_n^f(a_1^0, a_2^0, u, x)$, for all $n \in \mathbf{N}$, defined by

$$\delta_n^f(a_1^0, a_2^0, u, x) :\Longleftrightarrow (\exists m_1 \cdots m_{n-1} a_1^1 \cdots a_1^n a_2^1 \cdots a_2^n)\left[\bigwedge_{i=0}^{n-1} M_u(a_1^i m_i a_2^i)\right.$$
$$\left. \wedge\, B(a_1^i x a_2^i) \wedge \bigwedge_{i=1}^{n} a_{2-f(i)}^i = m_{i-1} \wedge a_{f(i)+1}^i = a_{f(i)+1}^{i-1}\right].$$

In Archimedean ordered Pappian planes, the following statement is valid, stating that, given two segments, $a_1^0 a_2^0$ and $b_1^0 b_2^0$, and a point x, lying between a_1^0 and a_2^0, and being found in some sequence of nested intervals, starting with $[a_1^0, a_2^0]$, the nth interval of the sequence being the left or the right half of the $(n-1)$st interval in the sequence (with $n \geq 2$), depending on whether $f(n)$ is 0 or 1, there is a point y between b_1^0 and b_2^0 having the same position with respect to the sequence of nested intervals the function f defines on the interval $[b_1^0, b_2^0]$, i.e., for all $f : \mathbf{N} \to \{0,1\}$, we have:

$$(3) \qquad (\forall a_1^0 a_2^0 u b_1^0 b_2^0 v x)(\exists y) \left[\bigwedge_{n=1}^{\infty} \delta_n^f(a_1^0, a_2^0, u, x) \to \delta_n^f(b_1^0, b_2^0, v, y) \right].$$

The truth of this statement is easily seen by choosing y such that the ratio in which it divides the segment $[b_1^0, b_2^0]$ coincides with the ratio in which x divides the segment $[a_1^0, a_2^0]$.

We now prove that g maps lines onto lines. Let a and b be two different points. Let y be a point between $g(a)$ and $g(b)$. Let $f : \mathbf{N} \to \{0,1\}$ be the map describing y's precise location on the segment $[g(a), g(b)]$ (i.e., $f(1)$ is 0 if y belongs to $[g(a), m]$ and is 1 if y belongs to $[m, g(b)]$, where m denotes the midpoint of $[g(a), g(b)]$, and so on). Let x be the point on the segment $[a, b]$ which lies in all the nested intervals defined by f on $[a, b]$. Given that g preserves, besides B and $\neg B$, midpoints, since it preserves the $\parallel$ relation, $g(x)$ belongs to the same nested sequence of intervals y belongs to. By *Arch*, there can be only one such point, and thus $y = g(x)$. Now let (see Figure 6)

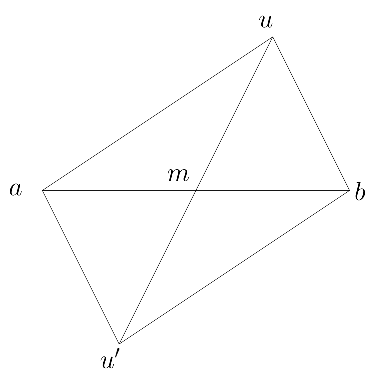

FIGURE 6. The definition of $M_u(amb)$.

$$S(abc) :\Longleftrightarrow (\exists uv)\,[L(abc) \wedge \neg L(abu) \wedge au \parallel bv \wedge bu \parallel vc \wedge uv \parallel ab].$$

Given *Arch*,

$$B(a_1 a_2 x) \longrightarrow \left\{ \bigvee_{n=3}^{\infty} (\exists a_3 \cdots a_n) \left[\bigwedge_{i=1}^{n-3} S(a_i a_{i+1} a_{i+2}) \wedge B(a_1 x a_n) \right] \right\}.$$

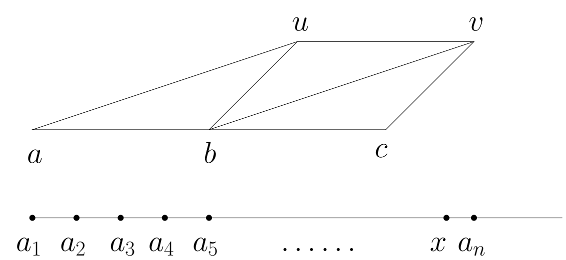

FIGURE 7. Definition of $S(abc)$; getting past any given point x on the ray $\overrightarrow{a_1a_2}$.

Since we also have $S(abc) \Rightarrow S(g(a)g(b)g(c))$ (given that S is definable in terms of L and $\|$), every point y on the line $g(a)g(b)$ lies between two points $g(e)$ and $g(f)$, with e and f on the line ab, and thus, by (3), there is a point x between e and f (i.e., a point x on the line ab) with $g(x) = y$.

Let now a, b, c be three non-collinear points. Then $g(a), g(b), g(c)$ are also non-collinear. For any point x of $\mathfrak{A}$, there exist distinct points u and v on the lines $g(a)g(b)$ and $g(b)g(c)$, respectively, such that x lies on the line uv. Given that g maps lines onto lines, there must be distinct points u' and v' on the lines ab and bc respectively, such that $g(u') = u$ and $g(v') = v$. Given that g maps the line $u'v'$ onto the line uv, there must be an x' on $u'v'$ such that $g(x') = x$.

2. Given an order-preserving (and thus injective) mapping $g : \mathfrak{A} \to \mathfrak{A}$, the image $\mathfrak{A}' = g(\mathfrak{A})$ is an affine subplane of $\mathfrak{A}$, isomorphic to $\mathfrak{A}$ as an ordered affine plane. Furthermore, by the Corollary of Theorem 1 the parallelism of $\mathfrak{A}''$ is indeed that of $\mathfrak{A}$, which means, that the projective closure $\mathfrak{P}'$ of $\mathfrak{A}'$ naturally embeds into the projective closure $\mathfrak{P}$ of $\mathfrak{A}$. Since by [**24**, 9.2] and [**28**, V.1. Satz 8] (Archimedean) orderings of affine planes uniquely extend to (Archimedean) orderings of their projective closure, $\mathfrak{P}$ and $\mathfrak{P}'$ are isomorphic as Archimedean ordered projective planes. By the above Proposition, we obtain $\mathfrak{P} = \mathfrak{P}'$, and so $\mathfrak{A} = \mathfrak{A}'$. □

Notice that, by [**24**, 9.4, Satz 19], Archimedean ordered translation planes (i.e., planes satisfying the minor Desargues axiom, A14 in the Appendix) must be Pappian (see the Appendix for the statement of the Pappus axiom), so we could have stated the above theorem (as well as the Proposition preceding it) under the apparently weaker assumption that $\mathfrak{A}$ is an Archimedean ordered translation plane.

Notice also that Theorem 2 remains valid if $\mathfrak{A}$ is an Archimedean ordered n-dimensional affine space with $n \geq 3$. Such spaces are Desarguesian; thus, by Archimedeanity, Pappian, and so Theorem 2 for this n-dimensional $\mathfrak{A}$ is implied by its plane case. As the following example shows, Theorem 2 is no longer true in infinite-dimensional Archimedean ordered affine spaces. Let the universe of $\mathfrak{A}$ consist of all maps $h : \mathbf{N} \to \mathbf{Q}$, and let $B(abc)$ hold in $\mathfrak{A}$ if and only if there is a $t \in \mathbf{Q}$, $0 \leq t \leq 1$, such that $b = ta + (1-t)b$, where λh is the map defined by $(\lambda h)(n) = \lambda h(n)$, for all $\lambda \in \mathbf{Q}$ and all $h : \mathbf{N} \to \mathbf{Q}$. The map $\varphi : \mathfrak{A} \to \mathfrak{A}$, defined by $\varphi((h))(1) = 0$ and $\varphi(h)(n+1) = h(n)$, for all $n \in \mathbf{N}$ and all $h : \mathbf{N} \to \mathbf{Q}$, preserves both B and $\neg B$, but is not onto.

By a celebrated result of Prieß-Crampe [**27**], the completion construction making any Archimedean ordered field a copy of the reals can be carried over to arbitrary (i.e., not necessarily Desarguesian or Pappian) projective planes: Each Archimedean ordered projective plane can be embedded into a topological projective plane the point space of which is a surface. The latter are called *flat* projective planes, and have been thoroughly studied, see [**29**], [**30**, Chapter 3]. In particular, any flat projective plane carries a unique ordering, is Archimedean and fulfills, by [**29**], the following

Fact. Let $\mathfrak{P}$ be a flat projective plane, and let $\varphi : \mathfrak{P} \to \mathfrak{P}$ be a homomorphism, i.e., a mapping from the point set of $\mathfrak{P}$ into its point set fulfilling:

(i) collinear points are mapped onto collinear points,

(ii) the image of φ contains a frame, i.e., four points no three of which are collinear.

Then φ is an isomorphism (i.e., it is one-to-one and onto, and preserves both L and $\neg L$).

Calling affine planes *flat*, if their projective closures are flat, this result of Salzmann immediately yields the following extension of Theorem 2:

Corollary. *Let $\mathfrak{A}$ be a flat affine plane. Then any map $g : \mathfrak{A} \to \mathfrak{A}$ fulfilling $B(abc) \Leftrightarrow B(g(a)g(b)g(c))$ for all points a, b, c of $\mathfrak{A}$ is an isomorphism, i.e., must be surjective.*

Proof. As above, we have that the image $\mathfrak{A}'$ of g is an affine subplane of $\mathfrak{A}$, isomorphic to $\mathfrak{A}$ as an Archimedean ordered affine plane, and that its projective closure $\mathfrak{P}'$ is a subplane of the projective closure $\mathfrak{P}$ of $\mathfrak{A}$, which is a flat plane. Since any isomorphism of affine planes uniquely extends to an isomorphism of their projective closures, the above Fact gives the assertion. □

APPENDIX

3. The L'-axiom system for ordered affine planes from [**33**] consists of (here $L(abc)$ is an abbreviation for $ab \parallel ac \vee a = b \vee a = c$):

A 1 $ab \parallel cc$,

A 2 $ab \parallel ba$,

A 3 $a \neq b \wedge ab \parallel pq \wedge ab \parallel rs \to pq \parallel rs$,

A 4 $ab \parallel ac \to ba \parallel bc$,

A 5 $(\exists abc)\, \neg(ab \parallel ac)$,

A 6 $(\forall abcp)(\exists q)\, [ab \parallel pq \wedge p \neq q]$,

A 7 $(\forall abcd)(\exists p)\, [\neg(ab \parallel cd) \to (pa \parallel pb \wedge pc \parallel pd)]$,

A 8 $B(abc) \to L(abc)$,

A 9 $L(abc) \to (B(abc) \vee B(bca) \vee B(cab))$,

A 10 $B(abc) \to B(cba)$,

A 11 $B(abc) \wedge B(acd) \to B(bcd)$,

A 12 $B(abc) \wedge B(bcd) \wedge b \neq c \to B(acd)$,

A 13 $\neg L(abb') \wedge L(abc) \wedge L(ab'c') \wedge bb' \parallel cc' \wedge B(abc) \to B(ab'c')$.

The last axiom, A13, is the outer form of the Pasch axiom, **oPasch** in [**33**, page 148], stating the invariance of the betweenness relation under parallel projection. In the presence of the minor Desargues axiom, i.e., of

A 14 $\neg L(abp) \wedge \neg L(abr) \wedge ab \parallel pq \wedge ab \parallel rs \wedge ap \parallel bq \wedge ar \parallel bs \to pr \parallel qs$,

axiom A12 becomes superfluous. If we add *Arch* to A1–A11, A13, A14, then we can prove (see [**23**, 9.4, Satz 19]) the Pappus axiom, i.e.,

A 15 $\left(\bigwedge_{1\leq i<j\leq 3}\bigwedge_{k=1}^{3}\neg L(p_ip_jq_k)\right)\wedge L(p_1p_2p_3)\wedge L(q_1q_2q_3)\wedge p_1q_2 \parallel p_2q_1\wedge p_2q_3\parallel p_3q_2\rightarrow p_1q_3\parallel p_3q_1$.

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ENDNOTES

1. The first who appears to have considered mappings m preserving the strict betweenness relation Z (see Section 2 for its definition) in ordered spaces, in the sense of $Z(abc)\Rightarrow Z(m(a)m(b)m(c))$ was Peano [**23**, pages 143–144], who presciently calls them "affinities."

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Fakultät für Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany
Email address: franz.kalhoff@tu-dortmund.de

News College of Interdisciplinary Arts & Sciences, Arizona State University–West Campus, P.O. Box 37100, Phoenix, AZ 85069-7100
Email address: pamb@math.west.asu.edu