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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF A FIRST-ORDER NEUTRAL DIFFERENTIAL EQUATION OF EULER FORM

KAIZHONG GUAN AND JIANHUA SHEN

ABSTRACT. In this paper we give a necessary and sufficient condition for the oscillation of the first-order neutral differential equations of Euler form with variable unbounded delays

$$\frac{d}{dt}\left(x(t) - cx(\alpha t)\right) + \frac{1}{t}\sum_{i=1}^{n} p_i x(\beta_i t) = 0, \quad t \geq t_0 > 0,$$

where $0 \leq c < 1$, $0 < \alpha < 1$, $0 < \beta_i < 1$, $p_i > 0$, $i = 1, 2, \dots, n$. Some relevant results in the literature are also extended and improved.

1. Introduction. The oscillation theory of delay differential equations and neutral differential equations has drawn much attention in recent years. This is evidenced by extensive references in books of Györi and Ladas [**7**], Erbe et al. [**5**] and Ladde et al. [**10**].

The oscillation of all solutions of neutral differential equation with constant delays and constant coefficients of the form

$$(x(t) - cx(t-\tau))' + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \quad t \geq t_0, \tag{1.1}$$

where $0 \leq c < 1$, $\tau, \tau_i, p_i \in (0, \infty)$, $i = 1, 2, \dots, n$, has been investigated by many authors. See, for example, [**4, 8, 9, 12**] and the references cited therein. In particular, the following well-known oscillation results are established.

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Theorem 1.1 ([**7, 9, 11**]). *All solutions of equation* (1.1) *oscillate if and only if*

$$-\lambda + c\lambda e^{\lambda\tau} + \sum_{i=1}^{n} p_i e^{\lambda\tau_i} > 0, \quad \text{for all } \lambda > 0. \tag{1.2}$$

Recently, there have been some publications on neutral differential equations with variable delays and variable coefficients. For example, see [**2, 3, 6, 11**] and the references cited therein. In particular, Guan and Shen [**6**] investigated the oscillatory behavior of all solutions of the following first-order neutral differential equation of Euler form with variable delays

$$\frac{d}{dt}\left(x(t) - cx(\alpha t)\right) + \frac{p}{t}x(\beta t) = 0, \quad t \geq t_0 > 0, \tag{1.3}$$

where $0 \leq c < 1$, $0 < \alpha$, $\beta < 1$, $p > 0$, and the more general one

$$\frac{d}{dt}\left(x(t) - cx(\alpha t)\right) + \frac{1}{t}\sum_{i=1}^{n} p_i x(\beta_i t) = 0, \quad t \geq t_0 > 0, \tag{1.4}$$

where $0 \leq c < 1$, $0 < \alpha < 1$, $0 < \beta_i < 1$, $p_i > 0$, $i = 1, 2, \dots, n$.

For equation (1.3), the authors established the following necessary and sufficient condition for all solutions of (1.3) to oscillate, which is similar to Theorem 1.1.

Theorem 1.2. *Assume that* $p > 0$, $0 \leq c < 1$, *and* $0 < \beta < \alpha < 1$. *Then every solution of equation* (1.3) *oscillates if and only if*

$$F(\lambda) = -\lambda + c\lambda\alpha^{-\lambda} + p\beta^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{1.5}$$

However, for equation (1.4) as $n > 1$, they only established a sufficient condition for the oscillation of all solutions. The question is posed whether one can establish the conclusion which is similar to Theorem 1.2. We also note that the condition "$\beta < \alpha$" which is essential in the proof of the corresponding result in [**6**] actually restricted the applications of the result. These stimulate the strong interest in investigating further equation (1.4).

The main purpose of our paper is to introduce a new technique which is different from [**6**] to obtain the necessary and sufficient conditions as well as some explicit sufficient conditions for every solution of equation (1.4) to oscillate. Our results show that the condition "$\beta < \alpha$" in Theorem 1.2 is superfluous, and hence the results presented extend and improve those of [**6**].

By a solution of equaton (1.4) we understand a function $x(t) \in C([\rho\bar{t}, \infty), R)$ for some $\bar{t} \geq t_0$, such that $x(t) - cx(\alpha t)$ is continuously differentiable, and $x(t)$ satisfies equation (1.4) for all $t \geq \bar{t}$, where $\rho = \min\{\alpha, \min_{1\leq i\leq n}\{\beta_i\}\}$.

As is customary, a nontrivial solution of equaton (1.4) is said to be oscillatory if it has arbitrarily large zeroes and nonoscillatory if it is either eventually positive or eventually negative.

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large t.

2. Lemmas. We need the following lemmas for the proofs of our main results.

Lema 2.1 ([**1, 6**]). *Suppose that $p > 0$, $0 < \alpha < 1$, and that $x(t)$ is an eventually positive solution of the delay differential inequality*

$$x'(t) + \frac{p}{t}x(\alpha t) \leq 0. \tag{2.1}$$

Then

$$x(\alpha t) \leq \frac{1}{(p \ln \alpha/2)^2} x(t). \tag{2.2}$$

Lemma 2.2. *Assume that $p > 0$, $0 < \alpha < 1$, and that $\alpha(t) \in C([\alpha T, +\infty), (0, +\infty))$ satisfies the inequality*

$$\alpha(t) - p \exp\left(\int_{\alpha t}^{t} \frac{\alpha(s)}{s}\, ds\right) \geq 0, \quad t \geq T > 0. \tag{2.3}$$

Then

$$\liminf_{t\to+\infty}\int_{\alpha t}^{t}\frac{\alpha(s)}{s}\,ds<+\infty. \tag{2.4}$$

Proof. Set

$$x(t)=\exp\bigg(-\int_{T}^{t}\frac{\alpha(s)}{s}\,ds\bigg),\quad \text{for } t\geq T/\alpha.$$

Then

$$\frac{x(\alpha t)}{x(t)}=\exp\bigg(\int_{\alpha t}^{t}\frac{\alpha(s)}{s}\,ds\bigg) \tag{2.5}$$

and so (2.3) implies that

$$x'(t)+\frac{p}{t}x(\alpha t)\leq 0,\quad t\geq T/\alpha. \tag{2.6}$$

It follows from Lemma 2.1 that

$$\frac{x(\alpha t)}{x(t)}\leq\frac{1}{(\frac{p\ln\alpha}{2})^2},$$

or

$$\exp\bigg(\int_{\alpha t}^{t}\frac{\alpha(s)}{s}\,ds\bigg)\leq\frac{1}{(p\ln\alpha/2)^2}.$$

This implies that $\liminf_{t\to+\infty}\int_{\alpha t}^{t}[(\alpha(s))/s]\,ds<+\infty$, and so the proof is complete. □

Lemma 2.3 ([**6**]). *Let* $x(t)$ *be an eventually positive solution of equation* (1.4), *and set*

$$y(t)=x(t)-cx(\alpha t).$$

Then $y(t)>0$ *and* $y'(t)<0$ *for sufficiently large* t.

3. Main results.

Theorem 3.1. *Assume that* $0 \le c < 1$, $0 < \alpha < 1$, $0 < \beta_i < 1$, *and* $p_i > 0$, $i = 1, 2, \dots, n$. *Then every solution of equation* (1.4) *oscillates if and only if*

$$F(\lambda) = -\lambda + c\lambda\alpha^{-\lambda} + \sum_{i=1}^{n} p_i \beta_i^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{3.1}$$

Proof. When $c = 0$, Theorem 1 of [**1**] can be applied. Below we consider the case $0 < c < 1$.

Assume firstly that (3.1) does not hold, which means that there exists at least a positive number λ_1 such that $F(\lambda_1) \le 0$. Since $F(0) = \sum_{i=1}^{n} p_i > 0$, there exists a $\lambda_0 \in (0, \lambda_1]$ such that $F(\lambda_0) = 0$. It is easy to check that $x(t) = t^{-\lambda_0}$ is a nonoscillatory solution of equation (1.4), a contradiction.

Assume, conversely, that not all solutions of equation (1.4) oscillate. This will imply that there exists at least one nonoscillatory solution of equation (1.4). Without loss of generality, we assume that $x(t)$ is an eventually positive solution. Let

$$y(t) = x(t) - cx(\alpha t).$$

By Lemma 2.3, then $y(t) > 0$, $y'(t) < 0$ for $t \ge T \ge t_0$.

From (1.4), we have

$$\begin{aligned} \frac{d}{dt} y(t) &= -\frac{1}{t} \sum_{i=1}^{n} p_i x(\beta_i t) \\ &= -\frac{1}{t} \sum_{i=1}^{n} p_i \left[y(\beta_i t) + cx(\alpha\beta_i t) \right] \\ &= -\frac{1}{t} \sum_{i=1}^{n} p_i y(\beta_i t) + c \frac{d}{dt} y(\alpha t). \end{aligned} \tag{3.2}$$

Set

$$\frac{\lambda(t)}{t} = -\frac{y'(t)}{y(t)}, \quad t \ge T.$$

Then $\lambda(t) > 0$, and it follows from (3.2) that

$$\lambda(t) = c\lambda(\alpha t)\exp\left(\int_{\alpha t}^{t} \frac{\lambda(s)}{s}\,ds\right) + \sum_{i=1}^{n} p_i \exp\left(\int_{\beta_i t}^{t} \frac{\lambda(s)}{s}\,ds\right) \tag{3.3}$$

$$\geq \sum_{i=1}^{n} p_i \exp\left(\int_{\beta t}^{t} \frac{\lambda(s)}{s}\,ds\right), \tag{3.4}$$

where $\beta = \max\limits_{1\leq i\leq n}\{\beta_i\}$.

It follows from (3.1) that

$$\inf_{\lambda>0}\left\{c\alpha^{-\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} p_i\beta_i^{-\lambda}\right\} > 1,$$

or

$$\inf_{t\geq T,\ \lambda>0}\left\{c\exp\left(\lambda\int_{\alpha t}^{t}\frac{1}{s}\,ds\right) + \frac{1}{\lambda}\sum_{i=1}^{n} p_i \exp\left(\lambda\int_{\beta_i t}^{t}\frac{1}{s}\,ds\right)\right\} > 1. \tag{3.5}$$

This implies that there exists a $\delta \in (0,1)$ such that

$$\delta\inf_{\lambda>0}\left\{c\alpha^{-\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} p_i\beta_i^{-\lambda}\right\} > 1, \tag{3.6}$$

or

$$\delta\inf_{t\geq T,\ \lambda>0}\left\{c\exp\left(\lambda\int_{\alpha t}^{t}\frac{1}{s}\,ds\right) + \frac{1}{\lambda}\sum_{i=1}^{n} p_i \exp\left(\lambda\int_{\beta_i t}^{t}\frac{1}{s}\,ds\right)\right\} > 1.$$

On the other hand, using Lemma 2.2 and (3.4), we have

$$\liminf_{t\to+\infty}\int_{\beta t}^{t}\frac{\lambda(s)}{s}\,ds < +\infty.$$

This and the expression $\int_{\beta t}^{t}(1/s)\,ds = \ln(1/\beta) > 0$ imply that $0 \leq \liminf\limits_{t\to+\infty}\lambda(t) < +\infty$.

Now we turn to showing that $\liminf_{t\to+\infty} \lambda(t) > 0$. Otherwise, if $\liminf_{t\to+\infty} \lambda(t) = 0$, then there exists a sequence $\{t_k\}$ such that $\min_{1\le i\le n}\{\alpha t_k, \beta_i t_k\} \ge T$, $\lim_{k\to+\infty} t_k = +\infty$, and $\lambda(t_k) \le \lambda(t)$, for $t \in [T, t_k]$, $k = 1, 2, \dots$.

From (3.3), we have

$$\lambda(t_k) \ge c\lambda(\alpha t_k)\exp\left(\lambda(t_k)\int_{\alpha t_k}^{t_k}\frac{1}{s}\,ds\right) + \sum_{i=1}^{n} p_i \exp\left(\lambda(t_k)\int_{\beta_i t_k}^{t_k}\frac{1}{s}\,ds\right),$$

i.e.,

$$c\exp\left(\lambda(t_k)\int_{\alpha t_k}^{t_k}\frac{1}{s}\,ds\right) + \frac{1}{\lambda(t_k)}\sum_{i=1}^{n} p_i \exp\left(\lambda(t_k)\int_{\beta_i t_k}^{t_k}\frac{1}{s}\,ds\right) \le 1,$$

which contradicts (3.5), and therefore,

$$\liminf_{t\to+\infty} \lambda(t) = \lambda_0 \in (0, +\infty).$$

Then there exists a $T_1 \ge T$ such that $\lambda(t) \ge \delta\lambda_0$, $t \ge T_1$.

Thus, by (3.3), we have

$$\lambda(t) \ge c\delta\lambda_0\alpha^{-\delta\lambda_0} + \sum_{i=1}^{n} p_i\beta_i^{-\delta\lambda_0}, \quad t \ge T_1.$$

Taking the inferior limit as $t \to \infty$, we get

$$\lambda_0 \ge c\delta\lambda_0\alpha^{-\delta\lambda_0} + \sum_{i=1}^{n} p_i\beta_i^{-\delta\lambda_0}. \tag{3.7}$$

Letting $\lambda_1 = \delta\lambda_0$ in (3.7) yields

$$\delta\left(c\alpha^{-\lambda_1} + \frac{1}{\lambda_1}\sum_{i=1}^{n} p_i\beta_i^{-\lambda_1}\right) \le 1. \tag{3.8}$$

Since $\lambda_1 > 0$, (3.8) contradicts (3.6) and so the proof is complete. □

Remark 3.2. When $n = 1$, Theorem 3.1 reduces to Theorem 1.2, namely, Theorem 3.1 of [**6**]. And moreover, it is easy to see that the condition "$\beta < \alpha$" is superfluous and our method employed is different from that of [**6**].

Theorem 3.1 is of theoretical interest. But the assumptions are not easy to verify. Hence, it is necessary to establish some explicit sufficient conditions for the oscillation of every solution of equation (1.4). Here we give the following theorems.

Theorem 3.3. *Assume that* $0 < \alpha$, $\beta_i < 1$, $p_i > 0$, $i = 1, 2, \dots, n$, *and* $0 \le c < 1$. *Then every solution of equation* (1.4) *oscillates if*

$$e\left(\prod_{i=1}^{n} p_i\right)^{1/n} \sum_{i=1}^{n} \ln\left(1/\beta_i\right) \ge 1 - c\alpha^{-\left(\prod_{i=1}^{n} p_i\right)^{1/n}/n(1-c)}. \tag{3.9}$$

Proof. Clearly, (3.1) can be rewritten as

$$g(\lambda) = -1 + c\alpha^{-\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} p_i \beta_i^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{3.10}$$

Using the arithmetic-geometric means inequality, we have

$$g(\lambda) \ge -1 + c\alpha^{-\lambda} + \frac{n\left(\prod_{i=1}^{n} p_i\right)^{1/n}}{\lambda} \exp\left(\frac{\lambda}{n}\sum_{i=1}^{n} \ln(1/\beta_i)\right), \quad \lambda > 0.$$

Set

$$f(\lambda) = -1 + c\alpha^{-\lambda} + \frac{n\left(\prod_{i=1}^{n} p_i\right)^{1/n}}{\lambda} \exp\left(\frac{\lambda}{n}\sum_{i=1}^{n} \ln(1/\beta_i)\right).$$

By Theorem 3.1, the proof will be completed if we can prove $f(\lambda) > 0$ for all $\lambda > 0$.

Let

$$f_1(\lambda) = \frac{n\left(\prod_{i=1}^{n} p_i\right)^{1/n}}{\lambda} \exp\left(\frac{\lambda}{n}\sum_{i=1}^{n} \ln(1/\beta_i)\right),$$

and

$$f_2(\lambda) = 1 - c\alpha^{-\lambda}.$$

It is not difficult to verify that $f_1(\lambda)$ has only a global minimum at $\lambda_0 = n/\left(\sum\limits_{i=1}^{n} \ln(1/\beta_i)\right)$ and the minimum value is $e\left(\prod_{i=1}^n p_i\right)^{1/n} \sum_{i=1}^n \ln(1/\beta_i)$.

Calculating the value of $f(\lambda) = f_1(\lambda) - f_2(\lambda)$ at $\lambda = n/\left(\mu \sum\limits_{i=1}^{n} \ln(1/\beta_i)\right)$ $(\mu > 0)$, we have

$$\begin{aligned} f(\lambda)|_{\lambda = n/(\mu \sum_{i=1}^n \ln(1/\beta_i))} &= \mu \left(\prod_{i=1}^n p_i \right)^{1/n} \sum_{i=1}^n \ln(1/\beta_i) e^{1/\mu} \\ &\quad + c\alpha^{-n/\left(\mu \sum_{i=1}^n \ln(1/\beta_i)\right)} - 1 \\ &> \mu \left(\prod_{i=1}^n p_i \right)^{1/n} \sum_{i=1}^n \ln(1/\beta_i) + c - 1. \end{aligned}$$

Thus, if $\mu > (1-c)/\left(\left(\prod_{i=1}^n p_i\right)^{1/n} \sum_{i=1}^n \ln(1/\beta_i)\right)$, then $f(\lambda) = f_1(\lambda) - f_2(\lambda) > 0$. This shows that $f(\lambda) = f_1(\lambda) - f_2(\lambda) > 0$, $\lambda \in \left(0, \left(\left(\prod_{i=1}^n p_i\right)^{1/n}\right)/(n(1-c))\right)$.

Now we are in a position to consider the case where $\lambda \geq \left(\prod_{i=1}^n p_i\right)^{1/n} / n(1-c)$ and note,

$$\begin{aligned} f(\lambda) &\geq \left(\prod_{i=1}^n p_i \right)^{1/n} e \sum_{i=1}^n \ln(1/\beta_i) - (1 - c\alpha^{-\lambda}) \\ &\geq \left(\prod_{i=1}^n p_i \right)^{1/n} e \sum_{i=1}^n \ln(1/\beta_i) + c\alpha^{-\left(\prod_{i=1}^n p_i\right)^{1/n}/n(1-c)} - 1. \end{aligned}$$

By (3.9), we obtain

$$f(\lambda) = f_1(\lambda) - f_2(\lambda) > 0, \quad \text{for } \lambda \geq \frac{\left(\prod_{i=1}^n p_i\right)^{1/n}}{n(1-c)}.$$

Up to this, we have shown that

$$g(\lambda) \geq f(\lambda) = f_1(\lambda) - f_2(\lambda) > 0, \quad \text{for all } \lambda > 0.$$

Therefore, (3.10) holds and so the proof is complete. □

Theorem 3.4. *Assume that* $0 < \alpha,\ \beta_i < 1$, $p_i > 0$, $i = 1, 2, \dots, n$, *and* $0 \le c < 1$. *Then all solutions of equation (1.4) are oscillatory if*

$$e \sum_{i=1}^{n} p_i \ln(1/\beta_i) > 1 - c. \tag{3.11}$$

Proof. Let g be defined as in (3.10), i.e., $g(\lambda) = -1 + c\alpha^{-\lambda} + (1/\lambda) \sum_{i=1}^{n} p_i \beta_i^{-\lambda}$, $\lambda > 0$. One can easily find that, for every fixed i ($i = 1, 2, \dots, n$), the function $f_i(\lambda) = (1/\lambda) p_i \beta_i^{-\lambda}$ ($\lambda > 0$) has only a global minimum at $\lambda_i = 1/\ln(1/\beta_i)$, and the minimum value is $ep_i \ln(1/\beta_i)$.

Thus, it follows from (3.11) that

$$g(\lambda) \ge -1 + c\alpha^{-\lambda} + e \sum_{i=1}^{n} p_i \ln(1/\beta_i) > -1 \\ + c + e \sum_{i=1}^{n} p_i \ln(1/\beta_i) > 0, \quad \text{for all } \lambda > 0.$$

By Theorem 3.1, all solutions of equation (1.4) are oscillatory and so the proof is complete.

Set $\beta = \max_{1 \le i \le n} \{\beta_i\}$. Using Corollary 4 of [**11**], we can obtain the following

Corollary 3.5. *All solutions of equation* (1.4) *are oscillatory if*

$$e \sum_{i=1}^{n} p_i \ln(1/\beta) > 1. \tag{3.12}$$

Remark 3.6. It is easy to see that $\sum_{i=1}^{n} p_i \ln(1/\beta_i) \ge \sum_{i=1}^{n} p_i \ln(1/\beta)$. This shows that (3.11) improves (3.12).

4. Examples. In this section, we give some examples to show the applications of our results.

Example 4.1. Consider the neutral differential equation

$$(4.1)\quad \frac{d}{dt}\left[x(t)-\frac{1}{2}x(e^{-\pi}t)\right]+\frac{2}{t}x(e^{-5/2\pi}t)+\frac{1}{2t}x(e^{-3/2\pi}t)=0,\quad t\geq 2.$$

Calculation shows that $\sqrt{p_1p_2}\left(\ln\frac{1}{\beta_1}+\ln\frac{1}{\beta_2}\right)e=4\pi e>1-\frac{1}{2}e^{\pi}=1-c\alpha^{-\sqrt{p_1p_2}/2(1-c)}$. Thus, equation (4.1) satisfies the conditions of Theorem 3.3 and so all solutions of (4.1) oscillate. Indeed, $x(t)=\sin(\ln t)$ is such a solution.

Again, it is easy to see that

$$p_1\ln\frac{1}{\beta_1}+p_2\ln\frac{1}{\beta_2}=\frac{23\pi}{4}>\frac{1-c}{e}=\frac{1}{2e}.$$

Therefore, equation (4.1) also satisfies all conditions of Theorem 3.4 and so every solution oscillates.

Example 4.2. Consider the neutral differential equation

$$(4.2)\quad \frac{d}{dt}\left[x(t)-\frac{1}{4}x\left(\frac{t}{2}\right)\right]+\frac{1}{9t}x(e^{-1}t)+\frac{1}{t}x(e^{-1/9}t)=0,\quad t\geq 1,$$

Straightforward computing gives us

$$\sqrt{p_1p_2}\left(\ln\frac{1}{\beta_1}+\ln\frac{1}{\beta_2}\right)e=\frac{10}{27}e$$

and

$$1-c\alpha^{-\sqrt{p_1p_2}/2(1-c)}=1-\left(\frac{1}{2}\right)^{16/9}.$$

Thus, Theorem 3.3 implies that every solution of (4.2) oscillates.

Here we also note that $p_1 \ln \frac{1}{\beta_1} + p_2 \ln \frac{1}{\beta_2} = \frac{2}{9} < \frac{3}{4e} = \frac{1-c}{e}$, which implies that Theorem 3.4 cannot be applied.

Example 4.3. Consider the neutral differential equation

$$\frac{d}{dt}\left[x(t) - \left(1 - \frac{2}{e}\right)x\left(\left(\frac{2}{3}\right)^4 t\right)\right] + \frac{4}{et}x(e^{-1/4}t) + \frac{1}{4et}x(e^{-1/8}t) = 0, \quad t \geq 1. \tag{4.3}$$

Simple calculation shows that

$$p_1 \ln \frac{1}{\beta_1} + p_2 \ln \frac{1}{\beta_2} = \frac{33}{32e} > \frac{1-c}{e} = \frac{2}{e^2}.$$

By Theorem 3.4, every solution of (4.3) is oscillatory.

However, calculating yields

$$\sqrt{p_1 p_2}\left(\ln \frac{1}{\beta_1} + \ln \frac{1}{\beta_2}\right)e = \frac{3}{8} \quad \text{and} \quad 1 - c\alpha^{-\sqrt{p_1 p_2}/2(1-c)} = \frac{3}{e} - \frac{1}{2}.$$

Since $\frac{3}{8} < \left(\frac{3}{e} - \frac{1}{2}\right)$, Theorem 3.3 cannot be applied to equation (4.3).

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School of Mathematics and Computing Science, Wuyi University, Jiangmen, Guangdong 529020, China
Email address: kaizhongguan@yahoo.com.cn

Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China
Email address: jhshen2ca@yahoo.com