ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 4, 2011

WHEN RADICAL OF PRIMARY SUBMODULES ARE PRIME SUBMODULES

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ABSTRACT. If R is a commutative ring with identity, then the radical of a primary ideal of R is a prime ideal of R. We will try to study and generalize this property to modules. It is proved that if one of the following holds, then for any primary submodule Q of an R-module M, we have $\operatorname{rad} Q = M$ or $\operatorname{rad} Q$ is a prime submodule of M.

(1) R is a ZPI-ring, an almost multiplication ring, an arithmetical ring with locally ACC on principal ideals, or a ring with DCC on principal ideals.

(2) M is a special module, a secondary representable module, a module with DCC on cyclic submodules, or a module with DCC on the submodules of the form $\{r^n M \mid n \in \mathbf{N}\}$, for each $r \in R$.

1. Introduction. Throughout this note, all rings are commutative with identity and all modules are unitary. Also we consider R to be a commutative ring with identity and M a unitary R-module.

For a submodule N of M, the set $\{r \in R \mid rM \subseteq N\}$ is denoted by $(N : M)$. If N is a proper submodule of M such that $(N : M) = P$ and $rm \in N$, $r \in R$, $m \in M$ implies either $m \in N$ or $r \in P$, then the ideal P will be a prime ideal of R, and we say N is a P-*prime submodule* of M. Prime submodules are generalizations of prime ideals (see [**2, 3, 6, 8**–**13**]).

Recall that a proper submodule N of M is called a *primary submodule* if for each $r \in R$ and $m \in M$, the condition $rm \in N$ implies either $m \in N$ or $r \in \sqrt{(N : M)}$, where $\sqrt{(N : M)} = \{t \in R \mid \exists n \in \mathbf{N},\ t^n \in (N : M)\}$.

If N is primary submodule, then $P = \sqrt{(N : M)}$ is a prime ideal of R: for $st \in P$ with $t \notin P$, there is an integer $k \geq 1$ such that $(st)^k M \subseteq N$,

2010 AMS *Mathematics subject classification.* Primary 13C99, 13C13, 13E05, 13F05, 13F15.
Keywords and phrases. Prime and primary submodule, radical of submodules, secondary representable modules, special modules.
Received by the editors on January 2, 2008, and in revised form on October 27, 2008.

DOI:10.1216/RMJ-2011-41-4-1045 Copyright ©2011 Rocky Mountain Mathematics Consortium

but for each $n \geq 1$ there is an $m \in M$ such that $t^n m \notin N$. Thus for some $b \in M$, $t^k b \notin N$ which implies $s^k \in P$. Hence $s \in P$.

For a submodule B of M, the intersection of all prime submodules of M containing B is called the *radical* of B and it is denoted by $\operatorname{rad} B$ (or $\operatorname{rad}_M B$). If no prime submodule of M contains B, then $\operatorname{rad} B = M$.

In this paper, we will try to establish the conditions by which the radical of a primary submodule Q is a prime submodule, if $\operatorname{rad} Q \neq M$. This subject has been studied in [**9, 12**].

In Section 2, we will study the rings R such that for every R-module M and every primary submodule Q of M, $\operatorname{rad} Q$ is a prime submodule whenever $\operatorname{rad} Q \neq M$.

In Section 3, we put the conditions on modules to get the same result.

2. When the radical of every primary R-module is prime. In [**9**, Theorem 1.3], it is proved that if R is an integral domain of Krull dimension one, then the radical of any primary submodule Q of M is a prime submodule of M or $\operatorname{rad} Q = M$; particularly this property holds for every Dedekind domain. In this section we will show this property for some generalizations of Dedekind domains such as ZPI-rings, almost multiplication rings and arithmetical rings with ACC on principal ideals.

Lemma 2.1. *Let M be an R-module and N a proper submodule of M. Then*

(i) *If $(N : M)$ is a maximal ideal of R, then N is a prime submodule of M.*

(ii) *If there exists a prime ideal P of R such that $(T : M) = P$, for all prime submodules T of M containing N, then $\operatorname{rad} N$ is a prime submodule of M or $\operatorname{rad} N = M$.*

(iii) *Let M be a finitely generated R-module. If $(N : M) \subseteq P$, where P is a prime ideal of R, then there exists a P-prime submodule of M containing N.*

(iv) $\sqrt{(N : M)} \subseteq (\operatorname{rad} N : M)$.

Proof. The proofs of parts (i) and (ii) are clear.

(iii) See [**3**, Lemma 4], or [**8**, Theorem 3.3].

(iv) Let $t \in \sqrt{(N:M)}$. Then for some positive integer n, $t^n \in (N:M)$. If no prime submodule of M contains B, then $\operatorname{rad} B = M$, and therefore $\sqrt{(N:M)} \subset R = (\operatorname{rad} N : M)$.

Otherwise $t^n M \subseteq N \subseteq T$, for every prime submodule T of M containing N. Then $t^n \in (T:M)$, and since $(T:M)$ is a prime ideal, $t \in (T:M)$. So $tM \subseteq T$, for every prime submodule T containing N. Thus $tM \subseteq \operatorname{rad} N$, that is, $t \in (\operatorname{rad} N : M)$. □

Note. Let M be an R-module.

(a) According to Lemma 2.1 (iii), if M is finitely generated, then for any primary submodule Q of M, $\operatorname{rad} Q \neq M$.

(b) If we consider $M = \mathbf{Q}$, the set of rational numbers as a $\mathbf{Z}$-module, then it is easy to see that the only prime submodule of M is the zero submodule (see [**6**, Theorem 1]). So for any non-zero submodule B of M, we have $\operatorname{rad} B = M$.

(c) There exists an R-module M such that the zero submodule is a primary submodule, but $\operatorname{rad} 0 = M$ (see [**12**, Example 1.6]).

(d) Let R be the polynomial ring $\mathbf{Z}[X]$ and consider $M = R \oplus R$, $Q = R(2,X) + R(X,0)$. Then by [**12**, Example 1.11], Q is a primary submodule of M and $\operatorname{rad} Q = (R \oplus RX) \cap ((R2+RX) \oplus (R2+RX))$. It is easy to see that $2(1,X) \in \operatorname{rad} Q$, but $(1,X) \notin (R2+RX) \oplus (R2+RX)$, and $2(1,1) \notin R \oplus RX$, which implies that $\operatorname{rad} Q$ is not a prime submodule of M.

Proposition 2.2. *Let Q be a submodule of an R-module M. If one of the following holds, then* $\operatorname{rad} Q$ *is a prime submodule of M or* $\operatorname{rad} Q = M$.

(i) $\sqrt{(Q:M)}$ *is a maximal ideal of R.*

(ii) *Q is a primary submodule of M, and $\sqrt{(Q:M)} = 0$ or $\sqrt{(Q:M)}$ is a maximal ideal of R.*

Proof. (i) Let $\operatorname{rad} Q \neq M$. By Lemma 2.1 (iv), $\sqrt{(Q:M)} \subseteq (\operatorname{rad} Q : M) \subseteq (N:M)$, for each prime submodule N of M containing Q. So

$\sqrt{(Q:M)} = (N:M)$. Therefore, $\operatorname{rad} Q$ is a prime submodule of M, by Lemma 2.1 (ii).

(ii) Suppose that $\operatorname{rad} Q \neq M$. If $\sqrt{(Q:M)}$ is a maximal ideal of R, then by part (i), $\operatorname{rad} Q$ is a prime submodule of M.

Now let $\sqrt{(Q:M)} = 0$. In this case we show that Q is a prime submodule of M, and then $\operatorname{rad} Q = Q$ is a prime submodule. Consider $rx \in Q$, where $r \in R$ and $x \in M \setminus Q$. Note that Q is a primary submodule, then $r \in \sqrt{(Q:M)} = 0$. So $r = 0 \in (Q:M)$. □

Lemma 2.3. *Let M be an R-module and S a multiplicatively closed subset of R.*

(i) *If W is a Q-prime submodule of M_S as an R_S-module, then $W^c = \{x \in M \mid x/1 \in W\}$ is a Q^c-prime submodule of M, $(W^c)_S = W$ and $Q^c \cap S = \varnothing$.*

(ii) *If N is a P-prime submodule of M such that $P \cap S = \varnothing$, then N_S is a P_S-prime submodule of M_S as an R_S-module and $(N_S)^c = N$.*

Proof. See [**6**, Proposition 1]. □

Lemma 2.4. *Let M be an R-module, Q a primary submodule of M, and suppose that $rx \in \operatorname{rad} Q$, where $r \in R$ and $x \in M \setminus Q$. If P is a prime ideal of R containing $(Q:M)$ and $r.1 \in \sqrt{(Q_P : x/1)}$, then $r \in (\operatorname{rad} Q : M)$.*

Proof. Suppose that $(r^n/1)(x/1) \in Q_P$, where n is a positive integer. We have $(r^n x/1) = (q/s)$, where $q \in Q$ and $s \in R \setminus P$. Then for some $s' \in R \setminus P$, $s'sr^n x = s'q \in Q$. Note that $s's \notin \sqrt{(Q:M)}$, since $\sqrt{(Q:M)} \subseteq P$. Since Q is primary and $x \in M \setminus Q$, $r^n x \in Q$ and consequently, $r \in \sqrt{(Q:M)}$. Thus $r \in (\operatorname{rad} Q : M)$, by Lemma 2.1 (iv). □

Theorem 2.5. *Let R be a ring such that for each non-minimal prime ideal P of R, the ring R_P is a domain of Krull dimension one, and let M be an R-module. Then for every primary submodule Q of M, $\operatorname{rad} Q = M$ or $\operatorname{rad} Q$ is a prime submodule of M.*

Proof. Assume that $\operatorname{rad} Q \neq M$ and $rx \in \operatorname{rad} Q$, where $r \in R$ and $x \in M \setminus \operatorname{rad} Q$. Since $x \notin \operatorname{rad} Q$, there exists a prime submodule N of M containing Q such that $x \notin N$. So $rx \in \operatorname{rad} Q \subseteq N$ implies that $r \in (N : M)$. Put $(N : M) = P$. If P is a minimal prime ideal of R, then $(Q : M) \subseteq P$ implies that $P = \sqrt{(Q : M)}$. Then by Lemma 2.1 (iv), $r \in P = \sqrt{(Q : M)} \subseteq (\operatorname{rad} Q : M)$.

Now suppose that P is a non-minimal prime ideal of R. Since $(Q : M) \subseteq P$, Q_P is a primary submodule of M_P.

We have $(r/1)(x/1) \in (\operatorname{rad} Q)_P \subseteq N_P$ and by Lemma 2.3 (ii), N_P is a prime submodule; then $x/1 \in N_P$ or $r/1 \in (N_P : M_P)$.

If $x/1 \in N_P$, then $x \in (N_P)^c = N$, by Lemma 2.3 (ii), which is impossible. Thus $r/1 \in (N_P : M_P)$.

According to our assumption R_P is an integral domain of dimension one, so $\sqrt{(Q_P : M_P)} = 0$ or $\sqrt{(Q_P : M_P)}$ is a maximal ideal of R_P. If $\sqrt{(Q_P : M_P)} = 0$, then Q_P is a prime submodule of M_P. Hence $(Q_P)^c$ is a prime submodule of M, by Lemma 2.3 (ii). Also since Q is a primary submodule of M with $(Q : M) \subseteq P$, we have $Q = (Q_P)^c$. Then $\operatorname{rad} Q = Q$ is a prime submodule of M. So $r \in (\operatorname{rad} Q : M)$.

If $\sqrt{(Q_P : M_P)}$ is a maximal ideal of R_P, then since $\sqrt{(Q_P : M_P)} \subseteq (N_P : M_P)$, we have $r/1 \in (N_P : M_P) = \sqrt{(Q_P : M_P)}$.

Now from $r/1 \in \sqrt{(Q_P : M_P)} \subseteq \sqrt{(Q_P : x/1)}$ and Lemma 2.4, we get $r \in (\operatorname{rad} Q : M)$, which completes the proof. □

In [**5**, Chapters VI and IX], some generalizations of Dedekind domains such as ZPI-rings and almost multiplication rings are studied. Recall that a ring R is said to be a *ZPI-ring*, if every proper ideal of R can be written as a product of prime ideals of R.

Corollary 2.6. *If R is one of the following rings, then for every primary submodule Q of M, $\operatorname{rad} Q = M$ or $\operatorname{rad} Q$ is a prime submodule of M.*

(a) *R is a ZPI-ring.*

(b) *R is an almost multiplication ring.*

Proof. (a) According to the proof of [**1**, Theorem 3.7(ii)], for each prime ideal P of R, R_P is a field or every non-zero prime ideal of R_P is maximal.

Let P be a non-minimal prime ideal of R. Then $\dim R_P = \operatorname{ht} P \geq 1$, and so R_P is not a field; consequently, every non-zero prime ideal of R_P is maximal. Therefore if R_P is not an integral domain, then every prime ideal of R_P is maximal, that is $\dim R_P = 0$, which is a contradiction. This shows that for any non-minimal prime ideal P of R, the ring R_P is an integral domain of dimension one. Thus the proof is given by Theorem 2.5.

(b) By [**5**, Theorem 9.23], for every prime ideal P of R, the ring R_P is a ZPI-ring. So by the above argument, for any non-minimal prime ideal P of R, the ring R_P is an integral domain of dimension one. Now the proof is given by Theorem 2.5. □

Recall that a ring R is said to be an *arithmetical ring*, if for all ideals I, J and K of R, we have $I + (J \cap K) = (I + J) \cap (I + K)$ (see [**4**]). Obviously Prüfer domains, valuation rings, and Dedekind domains are arithmetical.

Lemma 2.7. *A ring R is arithmetical if and only if for each prime ideal P of R, every two ideals of the ring R_P are comparable.*

Proof. See [**4**, Theorem 1]. □

Lemma 2.8. *Let R be a valuation domain with a height one prime ideal $P = \sqrt{Rr}$, where $r \in R$. If M is an R-module and $x \in M$, then the following are equivalent.*

(i) $r \notin \sqrt{\operatorname{Ann} x}$.

(ii) $\operatorname{Ann} rx = 0$.

Proof. (i) $\Rightarrow$ (ii). Consider $s \in \operatorname{Ann} rx$. If for some $k \in \mathbf{N}$, $Rr^k \subseteq Rs$; then $r^{k+1}x = 0$, which is impossible. Hence $Rs \subseteq I = \cap_{n \in \mathbf{N}} Rr^n$.

According to [**2**, Lemma 2.3 (ii)], I is a prime ideal or r is a nilpotent element of R. Note that R is an integral domain; then I is a prime ideal contained in $\sqrt{Rr} = P$, and since $\operatorname{ht} P = 1$, we have $I = 0$. Consequently $Rs = 0$. □

In the following, we will say that R is a ring *with locally ACC on principal ideals*, if for each maximal ideal m of R, the ring R_m has ACC on principal ideals.

Theorem 2.9. *Let Q be a primary submodule of an R-module M, where R is an arithmetical ring with locally ACC on principal ideals. Then $\operatorname{rad} Q = M$ or $\operatorname{rad} Q$ is a prime submodule of M.*

Proof. One can easily prove the following.

(1) *Q/Q is a primary submodule of the R-module M/Q.*

(2) $\operatorname{rad}_{M/Q} Q/Q = (\operatorname{rad}_M Q)/Q$.

(3) *$\operatorname{rad}_M Q$ is a prime submodule of the R-module M if and only if $(\operatorname{rad}_M Q)/Q$ is a prime submodule of the R-module M/Q.*

Hence by passing from the module M to the module M/Q, we may suppose that $Q = 0$ is a primary submodule of M. Then let $rx \in \operatorname{rad} 0$, where $r \in R$ and $x \in M \setminus \operatorname{rad} 0$. Let P be a maximal ideal of R containing $(0 : M)$.

Put $I = \cap_{n\in\mathbf{N}} R_P(r^n/1)$. First we show that $I = 0$.

According to our assumption R_P has ACC on principal ideals. We will show that R_P is a Noetherian ring.

Let $I_1 \subset I_2 \subset I_3 \subset \cdots$ be a chain of ideals of R_P. For each $j \geq 2$, consider $x_j \in I_j \setminus I_{j-1}$.

By Lemma 2.7, every two ideals of R_P are comparable, and if $R_P x_{j+1} \subseteq R_P x_j$, then $x_{j+1} \in I_j$, which is impossible, so $R_P x_j \subset R_P x_{j+1}$. Since the chain $R_P x_2 \subset R_P x_3 \subset R_P x_4 \subset \cdots$ stops, the chain $I_1 \subset I_2 \subset I_3 \subset \cdots$ must stop.

Now since R_P is a local Noetherian ring, by Krull intersection theorem, $I = 0$.

Consider the R_P-module M_P. Obviously the zero submodule 0_P is a primary submodule of M_P and $(rx)/1 \in \operatorname{rad}_{M_P} 0_P$. If $r/1$ is a nilpotent element of R_P, then obviously $r/1 \in \sqrt{(0_P : x/1)}$, and the proof is given by Lemma 2.4. Now suppose that $r/1$ is a non-nilpotent element of R_P. We show that R_P is a valuation domain with ht $\sqrt{R_P(r/1)} = 1$.

Every two ideals of R_P are comparable; then the radical of each proper ideal of R_P is a prime ideal. Particularly the nilradical ideal of R_P, $\mathcal{N}(R_P) = \sqrt{0}$ is a prime ideal. Note that $r/1 \notin \mathcal{N}(R_P)$, then $\mathcal{N}(R_P) \subseteq \cap_{n\in\mathbf{N}} R_P r^n/1 = 0$, that is, R_P is an integral domain. Also $\sqrt{R_P r/1}$ is a prime ideal. Let P'' be a prime ideal of R_P with $P'' \subsetneq \sqrt{R_P(r/1)}$. Then for each $n \in \mathbf{N}$, $r^n/1 \notin P''$, and so $P'' \subseteq \cap_{n\in\mathbf{N}} R_P(r^n/1) = 0$. This shows that ht $\sqrt{R_P(r/1)} = 1$.

Now if $r/1 \in \sqrt{(0_P : x/1)} = \sqrt{\mathrm{Ann}\,(x/1)}$, then the proof is given by Lemma 2.4. Otherwise from Lemma 2.8, we get $\mathrm{Ann}\,(rx/1) = 0$. Consequently $(0_P : M_P) = 0$, and since 0_P is a primary submodule of M_P, 0_P is a prime submodule of M_P. Hence $0 = (0_P)^c$ is a prime submodule of M, and so in this case $\mathrm{rad}\,0 = 0$ is a prime submodule of M. □

3. Modules for which radical of primary submodules are prime. According to [**10**], an R-module M is called *special* if for any maximal ideal $\mathbf{m}$ of R and any $r \in \mathbf{m}$, $x \in M$, there exist a positive integer n and an element $c \in R \setminus \mathbf{m}$ such that $cr^n x = 0$.

Proposition 3.1. *Let M be a special R-module and Q a primary submodule of M. Then*

(i) $\sqrt{(Q : M)}$ *is a maximal ideal of R.*

(ii) $\mathrm{rad}\,Q$ *is a prime submodule of M or* $\mathrm{rad}\,Q = M$.

(iii) *If R is a local ring, then for every submodule B of M,* $\mathrm{rad}\,B$ *is a prime submodule of M or* $\mathrm{rad}\,B = M$.

Proof. (i) Let $\mathbf{m}$ be a maximal ideal of R containing $\sqrt{(Q : M)}$ and $x \in M \setminus Q$. Since M is a special module, for any arbitrary element $r \in \mathbf{m}$ there exist a positive integer n and an element $c \in R \setminus \mathbf{m}$ such that $cr^n x = 0$. Now $cr^n x = 0 \in Q$ and Q is a primary submodule of M, so $r \in \sqrt{(Q : M)}$, that is, $\mathbf{m} = \sqrt{(Q : M)}$.

(ii) The proof is given by part (i) and Proposition 2.2 (i).

(iii) Let $\mathbf{m}$ be the only maximal ideal of R. If $\operatorname{rad} B \neq M$, then there exists at least one prime submodule of M containing B. According to part (i), for every prime submodule N of M containing B, we have $(N : M) = \mathbf{m}$. Thus by Lemma 2.1 (ii), $\operatorname{rad} B$ is a prime submodule of M. $\square$

Definition. An R-module M will be called strongly special if for any maximal ideal $\mathbf{m}$ of R and any $r \in \mathbf{m}$, there exist a positive integer n and an element $c \in R \setminus \mathbf{m}$ such that $cr^n M = 0$.

Obviously every strongly special module is a special module.

Proposition 3.2. *Let M be a non-zero R-module.*

(i) *If $R/(\operatorname{Ann} M)$ is a zero dimensional ring, then M is a strongly special R-module.*

(ii) *Let M be a finitely generated R-module. Then the following are equivalent.*

(1) *$R/(\operatorname{Ann} M)$ is a zero dimensional ring.*

(2) *M is a strongly special R-module.*

(3) *M is a special R-module.*

Proof. (i) Let $\mathbf{m}$ be a maximal ideal of R and $r \in \mathbf{m}$. First let $\operatorname{Ann} M = 0$. Consider the localization ring $R_{\mathbf{m}}$. Since $\dim R = 0$, the ideal $(\mathbf{m})_{\mathbf{m}}$ is the only prime ideal of the ring $R_{\mathbf{m}}$, and so $\mathcal{N}(\mathcal{R}_{\mathbf{m}}) = (\mathbf{m})_{\mathbf{m}}$, where $\mathcal{N}(\mathcal{R}_{\mathbf{m}})$ is the set of nilpotent elements of the ring $R_{\mathbf{m}}$. Note that $r/1 \in (\mathbf{m})_{\mathbf{m}} = \mathcal{N}(\mathcal{R}_{\mathbf{m}})$. Then there exists a positive integer n such that $r^n/1 = 0$. Thus there exists an element $c \in R \setminus \mathbf{m}$ such that $cr^n = 0$, and hence $cr^n M = 0$.

Now consider the general case. If $\operatorname{Ann} M \not\subseteq \mathbf{m}$, then for each $c \in \operatorname{Ann} M \setminus \mathbf{m}$ and any $r \in \mathbf{m}$, we have $crM = 0$. Otherwise $\mathbf{m}/(\operatorname{Ann} M)$ is a maximal ideal of the ring $R/(\operatorname{Ann} M)$ and $r + \operatorname{Ann} M \in [\mathbf{m}/(\operatorname{Ann} M)]$. Consider M as an $R/(\operatorname{Ann} M)$-module. By the first case, there exists an element $c + \operatorname{Ann} M \in [R/(\operatorname{Ann} M)] \setminus [\mathbf{m}/(\operatorname{Ann} M)]$ such that $(c + \operatorname{Ann} M)(r + \operatorname{Ann} M)^n M = 0$. Consequently $c \in R \setminus \mathbf{m}$, and $cr^n M = 0$.

(ii) (3) $\Rightarrow$ (1). Let P be the prime ideal of R containing $\operatorname{Ann} M$, and consider $\mathbf{m}$ to be a maximal ideal of R containing P. We show that $\mathbf{m} = P$.

Suppose that M is generated by $x_1, x_2, x_3, \dots, x_k$. Since M is a special module, for any arbitrary element $r \in \mathbf{m}$ there exist a positive integer n_i and an element $c_i \in R \setminus \mathbf{m}$ such that $c_i r^{n_i} x_i = 0$, for each $1 \leq i \leq k$. Therefore $c_1 c_2 c_3 \cdots c_k r^n x_i = 0$, where $n = \max\{n_1, n_2, n_3, \dots, n_k\}$, for each $1 \leq i \leq k$. Then $c_1 c_2 c_3 \cdots c_k r^n M = 0$, that is, $c_1 c_2 c_3 \cdots c_k r^n \in \operatorname{Ann} M \subseteq P$. Now since P is a prime ideal and for each i, $c_i \notin P$, $r \in P$. That is, $\mathbf{m} = P$. □

Corollary 3.3. *Let M be a finitely generated special R-module and Q a submodule of M such that $(Q : M)$ is a primary ideal of R. Then*

(i) $\sqrt{(Q : M)}$ *is a maximal ideal of R.*

(ii) $\operatorname{rad} Q$ *is a prime submodule of M.*

(iii) *If R is a local ring, then for every proper submodule B of M, $\operatorname{rad} B$ is a prime submodule of M.*

Proof. (i) By Proposition 3.2 (ii), $R/(\operatorname{Ann} M)$ is a zero dimensional ring and since $\sqrt{(Q : M)}$ is a prime ideal of R containing $\operatorname{Ann} M$, $\sqrt{(Q : M)}$ is a maximal ideal of R.

(ii) Note that $(Q : M) \subseteq \sqrt{(Q : M)}$, where $\sqrt{(Q : M)}$ is a maximal ideal of R. Then by Lemma 2.1 (iii), there exists a prime submodule of M containing Q, that is, $\operatorname{rad} Q \neq M$. Since $\sqrt{(Q : M)}$ is a maximal ideal of R, Q is a primary submodule of M. Now the proof is completed by Proposition 3.1 (ii).

(iii) Let P be a prime ideal of R containing $(B : M)$. By Lemma 2.1 (iii), there exists a P-prime submodule of M containing B. Then, $\operatorname{rad} B \neq M$. Now the proof is completed by Proposition 3.1 (iii). □

Recall that an R-module $0 \neq S$ is said to be a *P-secondary module*, if for each $r \in R$, $rS = S$ or $r \in P = \sqrt{(0 : S)}$. A minimal secondary representation of an R-module M is an expression of M as a finite sum of P_i-secondary submodules S_i, that is, $M = S_1 + S_2 + S_3 + \cdots + S_n$ such that $P_1, P_2, P_3, \cdots, P_n$ are all distinct. If M has a secondary representation, then it is said that M is a *secondary representable module* (see [**7**, Section 6]).

Lemma 3.4. *Let Q be a primary submodule of M such that $M = \sum_{i=1}^{k} S_i + Q$, where S_i is a submodule of M, for each i. If S_1 is a P_1-secondary module with $S_1 \nsubseteq Q$, then $P_1 = \sqrt{(Q : M)}$.*

Proof. Let $t \in P_1 = \sqrt{(0 : S_1)}$. Then, for some positive integer m, $t^m S_1 = 0 \subseteq Q$. We know that $S_1 \nsubseteq Q$, so $t \in \sqrt{(Q : M)}$.

Now assume that $r \in \sqrt{(Q : M)}$. Then, for some positive integer n, $r^n(\sum_{i=1}^{k} S_i + Q) = r^n M \subseteq Q$, and this implies that $r^n S_1 \subseteq Q$; and note that $S_1 \nsubseteq Q$ and S_1 is a secondary module, then $r^n S_1 \neq S_1$. Thus $r^n \in \sqrt{(0 : S_1)} = P_1$, and evidently $r \in P_1$. □

Theorem 3.5. *Let M be a secondary representable R-module. Then, for every primary submodule Q of M, $\operatorname{rad} Q = M$ or $\operatorname{rad} Q$ is a prime submodule of M.*

Proof. Assume that $\operatorname{rad} Q \neq M$ and $rx \in \operatorname{rad} Q$, where $r \in R$ and $x \in M$.

Let $M = \sum_{i=1}^{k} S_i + \sum_{i=k+1}^{n} S_i$, where for each i, $1 \leq i \leq n$, S_i is P_i-secondary, and for $1 \leq i \leq k$, $rS_i = S_i$ and for $k+1 \leq i \leq n$, $r \in \sqrt{(0 : S_i)}$ and assume that $P_1, P_2, \dots, P_n$ are all distinct. Then for each $k+1 \leq i \leq n$, there exists a positive integer n_i such that $r^{n_i} S_i = 0 \in Q$, and since Q is a primary submodule, $r \in \sqrt{(Q : M)}$ or $S_i \subseteq Q$.

If, for some i, $k+1 \leq i \leq n$, $S_i \nsubseteq Q$, then $r \in \sqrt{(Q : M)}$, and in this case Lemma 2.1 (iii) applies to show that $r \in \sqrt{(\operatorname{rad} Q : M)}$, which completes the proof. Therefore, we may suppose that $S_i \subseteq Q$, for each $k+1 \leq i \leq n$.

Hence, $M = \sum_{i=1}^{k} S_i + Q$. Suppose that $k' \leq k$ is a positive integer such that for each i, $1 \leq i \leq k'$, $S_i \nsubseteq Q$, and for each i, $k'+1 \leq i \leq k$, $S_i \subseteq Q$. Then $M = \sum_{i=1}^{k'} S_i + Q$, and Lemma 3.4, shows that $P_1 = \sqrt{(Q : M)}$.

Applying Lemma 3.4, will also show that for each i, $1 \leq i \leq k'$, $P_i = \sqrt{(Q:M)}$ and since $P_1, P_2, \dots, P_{k'}$ are distinct, $k' = 1$. Hence, $M = S_1 + Q$.

Now let N be an arbitrary prime submodule of M containing Q. We will show that $(N : M) = P_1$.

If $S_1 \subseteq N$, then $M = S_1 + Q \subseteq N$, which is impossible, so $S_1 \not\subseteq N$. Also $M = S_1 + Q \subseteq S_1 + N$, that is, $M = S_1 + N$. Putting $Q = N$, in Lemma 3.4, implies that $P_1 = \sqrt{(N:M)} = (N:M)$.

Now according to Lemma 2.1 (ii), $\operatorname{rad} Q$ is a prime submodule of M. □

Theorem 3.6. *Let M be an R-module and Q a primary submodule of M. Then* $\operatorname{rad} Q = M$ *or* $\operatorname{rad} Q$ *is a prime submodule of M, if one of the following holds.*

(i) *M has DCC on cyclic submodules.*

(ii) *For any $r \in R$, the chain $\{r^n M \mid n \in \mathbf{N}\}$ stops.*

(iii) *For any $r \in R$, the chain $\{Rr^n \mid n \in \mathbf{N}\}$ stops.*

Proof. (i) Suppose that $\operatorname{rad} Q \neq M$ and $rx \in \operatorname{rad} Q$, where $r \in R$ and $x \in M \setminus \operatorname{rad} Q$. Then there exists a prime submodule N of M containing Q such that $x \notin N$. Put $P = (N : M)$ and consider the R_P-module M_p.

Evidently $(r/1)(x/1) \in (\operatorname{rad} Q)_P \subseteq N_P$. If $(r/1) \notin P_P$, then $r/1$ is a unit in the ring R_P, and so $x/1 \in N_P$. Then $x \in (N_P)^c$, and $(N_P)^c = N$, by Lemma 2.3 (ii). So $x \in N$, which is a contradiction.

Therefore $r/1 \in P_P$. It is easy to see that the R_P-module M_P also has DCC on cyclic submodules (see [**2**, Lemma 2.6]). Now consider the following chain of cyclic submodules of M_P,

$$\cdots \subseteq R_P \frac{r^3 x}{1} \subseteq R_P \frac{r^2 x}{1} \subseteq R_P \frac{rx}{1}.$$

Then there exists a positive integer n such that

$$\frac{r^n x}{1} \in R_P \frac{r^n x}{1} = R_P \frac{r^{n+1} x}{1}.$$

So there exist $t \in R$ and $s' \in R \setminus P$, with

$$\frac{r^n x}{1} = \frac{tr^{n+1}x}{s'},$$

and hence

$$\frac{r^n}{1}\left(1 - \frac{rt}{s'}\right)\frac{x}{1} = 0.$$

Note that $(rt)/s' \in P_P$; then $1 - (rt)/s'$ is a unit in R_P, and so $(r^n/1)(x/1) = 0 \in Q_P$, that is, $r/1 \in \sqrt{(Q_P : x/1)}$. Hence, by Lemma 2.4, $r \in \sqrt{(\operatorname{rad} Q : M)}$.

(ii) and (iii) For the proofs of parts (ii) and (iii), let N be an arbitrary prime submodule of M containing Q. Obviously $\sqrt{(\operatorname{rad} Q : M)} \subseteq \sqrt{(N : M)} = (N : M)$. By Lemma 2.1 (ii), it is enough to show that $(N : M) \subseteq \sqrt{(\operatorname{rad} Q : M)}$.

On the contrary, suppose that $r \in (N : M) \setminus \sqrt{(\operatorname{rad} Q : M)}$, and let $m \in M \setminus N$.

If the chain $\{r^n M \mid n \in \mathbf{N}\}$ stops, then there exists a positive integer k with $r^k m \in r^k M = r^{k+1} M$. So there exists an $m' \in M$ such that $r^k(m - rm') = 0 \in Q$. Then $m - rm' \in Q \subseteq N$ and $rm' \in N$; thus, $m \in N$, which is a contradiction.

If the chain $\{Rr^n \mid n \in \mathbf{N}\}$ stops, then there exist a positive integer k' and an element $t \in R$ such that $r^{k'} = tr^{k'+1}$. Thus, $r^{k'}(m - rtm) = 0 \in Q$. So $m - rtm \in Q \subseteq N$ and, since $rtm \in N$, we have $m \in N$, which is impossible. □

Acknowledgments. I would like to thank the referee for many helpful suggestions and especially for his comments on Corollary 2.6, Lemma 2.8 and Theorem 2.9.

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