ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 4, 2011

CHARACTERIZING MINIMAL RING EXTENSIONS

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ABSTRACT. Given a pair of commutative rings $R \subsetneq T$ with the same identity, T is a minimal ring extension of R if there are no rings properly between R and T. Such an extension is said to be closed if R is integrally closed in T; otherwise, T is integral over R and the extension is a minimal integral extension. An extension $R \subsetneq T$ is a closed minimal extension if and only if there is a maximal ideal M of R such that (R, M) is a rank 1 valuation pair of T (equivalently, for each $t \in T \backslash R$, M is the radical of $(R :_R t)$ and there is an element $m \in M$ such that $mt \in R \backslash M$). Also, for a pair of rings $R \subsetneq T$ and element $u \in T \backslash R$, the pair $R \subsetneq R[u]$ is a closed minimal extension if and only if for each $t \in R[u] \backslash R$, there are elements $c, d \in \sqrt{(R :_R u)}$ such that $ct + d = 1$. For a minimal integral extension $R \subsetneq T$, the conductor $M = (R : T)$ is a maximal ideal of R. In this case, if M has no nonzero annihilators in T, then there is an R-algebra isomorphism between T and a ring extension S of R in the complete ring of quotients of R. Moreover, M is regular if and only if S is in the total quotient ring of R, and M is semiregular but not regular if and only if S is in the ring of finite fractions over R but not in the total quotient ring of R.

1. Introduction. All rings and algebras considered below are commutative with identity and all ring/algebra homomorphisms and subrings are unital. The set of prime (respectively, maximal) ideals of R is denoted by Spec(R) (respectively, Max(R)). A *regular element* is one that is not a zero divisor, and a *regular ideal* is one that contains a regular element. An ideal that has no nonzero annihilators is said to be *dense* and an ideal that contains a finitely generated dense ideal is *semiregular*. For a pair of rings $R \subsetneq T$, an element b of R may be regular in R but a zero divisor in T. Similarly, an ideal of R may be

2010 AMS *Mathematics subject classification.* Primary 13B02, Secondary 13A15, 13A18, 13B30, 13B21.
Keywords and phrases. Minimal ring extension, valuation pair, ring of finite fractions.
The first author was on sabbatical leave and the second author was supported in part by a University of Tennessee Faculty Leave Award.
Received by the editors on July 2, 2008, and in revised form on October 28, 2008.
DOI:10.1216/RMJ-2011-41-4-1081 Copyright ©2011 Rocky Mountain Mathematics Consortium

dense in R but have a nonzero annihilator in T. Also, a regular ideal can only be either semiregular or dense with respect to T.

If R is a subring of T, we say $R \subsetneq T$ is a *minimal* (*ring*) *extension* if there are no rings properly between R and T. Clearly, this is the case if and only if T is of the form $T = R[u]$ for each $u \in T \backslash R$. Since the integral closure of R in T is a ring, if $R \subsetneq T$ is a minimal extension, then either T is integral over R (equivalently, T is a finite R-module) or R is integrally closed in T. For the former we say $R \subsetneq T$ is a *minimal integral extension*, and for the latter we say $R \subsetneq T$ is a *closed minimal extension*. By way of [**6**, Theorem 2.7] and [**24**, page 1738, lines 8–13], if $D \subsetneq T$ is a closed minimal extension with D an integral domain but not a field, then T is an overring of D; that is, T sits between D and the quotient field of D. Moving beyond the context of domains, one can find generalizations of the above-cited result from [**24**] concerning "overrings" in contexts involving nontrivial zero divisors in work of Picavet and Picavet-L'Hermitte [**23**, Proposition 3.9] and Dobbs [**3**, Theorem 2.2].

One way to form a minimal integral extension is by using the idealization of a module. Given a ring R and an R-module B, one can form a new ring $R(+)B$ from the Abelian group $R \oplus B$ by setting $(r, a)(s, b) = (rs, rb + sa)$; this is the *idealization* (*of* B *over* R). The ring R, viewed as $\{(r, 0) \mid r \in R\}$, is a subring of $R(+)B$. In many instances, some rings naturally take the form of an idealization. For example, if X is an indeterminate over R, then the ring $R[X]/(X^2)$ is naturally isomorphic to $R(+)R$, the idealization of R as a module over itself. With regard to minimal extensions, Ferrand and Olivier showed that $K(+)K$ is one of only three types of minimal (necessarily integral) extensions of a field K [**8**, Lemma 1.2]. More generally, if M is a maximal ideal of R, then $R(+)R/M$ is a minimal integral extension of R with conductor $M(+)(0)$ [**2**, Corollary 2.5].

In Section 2 we develop some tools we need for later use. In particular, revisiting some of the elementary results of Ferrand and Olivier [**8**], we show that the conductor $I = (R : T)$ of a minimal extension $R \subsetneq T$ is a maximal ideal of R for a minimal integral extension, and that it is a prime ideal of both R and T but not maximal in R for a closed minimal extension. We also show that a necessary condition for a pair $R \subsetneq T$ to be a minimal extension is that there is a (maximal) *critical ideal* M, that is, $M = \sqrt{(R :_R t)}$ for each $t \in T \backslash R$. Ayache proved that if D is an integrally closed domain but not a field, then $D \subsetneq T$ is a closed minimal extension if and only if $T = \Omega(M)$ is the *Kaplansky ideal transform* of some maximal ideal M such that M is the radical of a finitely generated ideal and there is a prime $P \subsetneq M$ such that $PD_P = PD_M$ and $(D/P)_{M/P}$ is a one-dimensional valuation domain [**1**, Theorem 2.4]. This was extended to reduced rings with von Neumann regular total quotient rings by Dobbs and Shapiro [**7**, Theorem 3.7]. In order to develop a general characterization for closed minimal extensions, we need to revisit the notion of ideal transforms and valuation pairs. For a nonzero ideal I of an integral domain D with quotient field K, the *Kaplansky ideal transform* of I is the overring $\Omega(I) = \{x \in K \mid I \subseteq \sqrt{(D :_D x)}\}$. An equivalent form is $\Omega(I) = \bigcap\{D_Q \mid I \not\subseteq Q \in \operatorname{Spec}(D)\}$ [**14**, Theorem 1.7]. If in addition, I is a finitely generated ideal, then the *Nagata ideal transform* $\mathcal{T}(I) = \bigcup(R : I^n)$ is equal to the Kaplansky transform $\Omega(I)$ [**14**, Theorem 1.3]. In this case we refer to $\mathcal{T}(I)$ simply as the *ideal transform of* I. The work in [**7**] was accomplished in part with the help of a generalization of $\Omega(I)$ that was developed in [**25**] for the context of base rings having von Neumann regular total quotient rings. For our purposes, it is convenient to extend Kaplansky transforms to arbitrary pairs $R \subsetneq T$. For a prime ideal P of R, let $R_{\langle P,T\rangle} = \{t \in T \mid rt \in R$ for some $r \in R\backslash P\}$. The ring $R_{\langle P,T\rangle}$ is referred to as the *large quotient ring of R with respect to P and T* (see, for example, [**13**]). If $t \in \bigcap\{R_{\langle M,T\rangle} \mid M \in \operatorname{Max}(R)\}$, no maximal ideal of R contains $(R :_R t)$. Hence $R = \bigcap\{R_{\langle M,T\rangle} \mid M \in \operatorname{Max}(R)\}$. For a nonzero ideal I of R not contained in the nilradical, we define the *generalized Kaplansky transform* $\Psi_T(I)$ of I with respect to T to be the intersection $\Psi_T(I) = \bigcap\{R_{\langle Q,T\rangle} \mid I \not\subseteq Q \in \operatorname{Spec}(R)\}$.

For a pair of rings $R \subsetneq T$ and a prime ideal P of R, (R,P) is said to be a *valuation pair of T* if there is a surjective *valuation map* $\mathrm{v} : T \mapsto G\bigcup\{\infty\}$, with G a totally ordered Abelian group (that is, v is such that $\mathrm{v}(st) = \mathrm{v}(s) + \mathrm{v}(t)$ and $\mathrm{v}(s+t) \geq \min\{\mathrm{v}(s), \mathrm{v}(t)\}$) such that $R = \{s \in T \mid \mathrm{v}(s) \geq 0\}$, $P = \{s \in T \mid \mathrm{v}(s) > 0\}$, and $\mathrm{v}^{-1}(\infty) = (R : T) =: P_\infty$ is a prime ideal of both R and T (called the *prime at infinity*). Clearly, for each $t \in T\backslash R$, there is an element $p \in P$ such that $pt \in R\backslash P$–simply choose any p such that $\mathrm{v}(p) = -\mathrm{v}(t)$. The *rank of the valuation pair* is the rank of the group G (the length of the chain of isolated subgroups) and it is the same as the height of P over P_∞.

In Section 3, we generalize Ayache's result on closed minimal extensions of domains, showing that $R \subsetneq T$ is a closed minimal extension if and only if $T = \Psi_T(M)$ for some maximal ideal M of R such that (R, M) is a rank one valuation pair of T (Theorem 3.5). We also consider the notion of a closed minimal extension from a different perspective: given a pair of rings $R \subsetneq T$, what algebraic properties of a given element $u \in T \backslash R$ are necessary and sufficient for $R \subsetneq R[u]$ to be a closed minimal extension? One characterization is that, for each $t \in R[u] \backslash R$, there are elements $c, d \in \sqrt{(R :_R u)}$ such that $ct + d = 1$. Another is that $R \subsetneq R[u]$ is a closed minimal extension if and only if $M = \sqrt{(R :_R u)}$ is a maximal ideal of R and for each $a \in R \backslash (R :_R u)$, there is an element $b \in M$ such that $bau \in R \backslash M$. Both of these characterizations appear in Theorem 3.4. We also give a characterization in terms of invertibility. For a pair of rings $R \subsetneq T$, we say that an R-submodule B of T is *T-invertible* if $B(R :_T B) = R$. As with invertible (fractional) ideals of an integral domain, if an R-submodule is T-invertible, then it is finitely generated over R. In addition, such a module has no nonzero annihilators in T. For an ideal I of R and rings $R \subseteq S \subseteq T$, it may be that I is T-invertible but not S-invertible. (For example, the ideal YR in the ring $R = K[X, Y]$ is T-invertible for $T = K[X, Y, 1/Y]$ but not S-invertible for $S = K[X, Y, X/Y]$.) However, if the R-submodule B of T contains R, then $(R :_T B)$ is contained in R and the ring T's relevance is then simply that it contains B. In particular, for an element $t \in T \backslash R$, the R-module $J = R + tR$ is "T-invertible" if and only if there are elements $m, n \in R$ such that $m + nt = 1$ and both mt and nt are in R. In Theorem 3.5 we show that $R \subsetneq T$ is a closed minimal extension if and only if, for each $t \in T \backslash R$, $\sqrt{(R :_R t)}$ is a maximal ideal of R and $(R + tR)$ is invertible.

Section 4 is devoted to the embedding in various quotient rings. For a ring R, we denote by $Q_{cl}(R)$ the total quotient ring, by $Q(R)$ the complete ring of quotients and by $Q_0(R)$ the ring of finite fractions (definitions recalled below within said section). An *overring* is simply a ring T between R and $Q_{cl}(R)$, while a *Q-overring* is one between R and $Q(R)$ and a *Q_0-overring* is one between R and $Q_0(R)$. Extending results in [**7, 24**], we show that if $R \subsetneq T$ is a closed minimal extension, then there is a natural R-algebra isomorphism from T into a Q_0-overring of R (Theorem 4.1). In Theorem 4.4 we consider the case of a minimal integral extension $R \subsetneq T$ such that $M = (R : T)$ has no nonzero annihilator in T. In the case that M is regular in T (and in R), there is an R-algebra isomorphism from T to an overring of R. If M is semiregular in T and not regular in R, then there is an R-algebra isomorphism from T to a Q_0-overring of R, but there cannot be one to an overring. Finally, for M dense in T but not semiregular in R, there is an R-algebra isomorphism from T to a Q-overring of R, but there cannot be one to a Q_0-overring. In Theorem 4.6 we show that if M has an (nonzero) annihilator in $R \backslash M$, then $M = eR$ is both minimal and maximal with e idempotent and T takes one of three forms: (i) R-algebra isomorphic to a ring of the form $eR \oplus L$ where L is a minimal algebraic field extension of the field R/M, (ii) R-algebra isomorphic to the ring $R \times R/M$, or (iii) R-algebra isomorphic to the idealization ring $R(+)R/M$.

In Section 5 we consider the Nagata ring $R(X) = R[X]_{\mathcal{U}(R)}$, where $\mathcal{U}(R)$ denotes the set of polynomials in $R[X]$ with unit content; that is, those whose coefficients generate R as an ideal. For a pair of rings $R \subsetneq T$, if T is integral over R and the coefficients of $f \in T[X]$ generate T as an ideal, then there is a polynomial $g \in T[X]$ such that the coefficients of $fg \in R[X]$ generate R as an ideal (see the proof of [**12**, Theorem 3]). It follows that in this case $\mathcal{U}(T)$ is the saturation of $\mathcal{U}(R)$ in $T[X]$. A key step in the proof of Theorem 5.4 is showing that the same conclusion holds in the case that $R \subsetneq T$ is a closed minimal extension. The end result is that $R \subsetneq T$ is a closed minimal extension if and only if $R(X) \subsetneq T(X)$ is a closed minimal extension.

We devote Section 6 to examples illustrating the diversity of minimal ring extensions in the presence of nontrivial zero divisors. In particular, we give a pair of examples of closed minimal extensions $R \subsetneq S$ such that each element of $S \backslash R$ is a zero divisor (Examples 6.2 and 6.4). In Example 6.2, the ring S is a subring of $Q_0(R)$ that cannot be embedded into $Q_{cl}(R)$, while in Example 6.4, S is contained in $Q_{cl}(R)$. With regard to minimal integral extensions, Example 6.5 shows that the conductor of the minimal integral extension $R \subsetneq S$ into R can be a semiregular ideal of S that is not regular. The last example is of a minimal integral extension $R \subsetneq S$ where the conductor of S into R is a dense ideal of S that is not semiregular (Example 6.6).

2. Conductors, valuation pairs and critical ideals. In this section we revisit some of the elementary results of Ferrand and Olivier

[**8**] as well as some tools which will be used later. The section is divided into three subsections.

Conductors. For a ring extension $R \subsetneq T$, we first recall a few facts about the conductor $I = (R : T)$. Clearly I is an ideal shared by R and T, and hence, we have a pullback:

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & T/I \end{array}$$

This can be considered trivial if $I = (0)$. But in any case, $R \subsetneq T$ is a minimal extension if and only if $R/I \subsetneq T/I$ is a minimal extension (as every ring S between R and T shares the ideal I with them): cf. [**5**, Lemma II.3].

First, the following lemmas about minimal extensions are immediate.

Lemma 2.1. *Let $R \subsetneq T$ be a ring extension with conductor $I = (R : T)$. If T is a minimal extension of R, then for each $t \in T \backslash I$, $T = R + tT$.*

Lemma 2.2 (cf. [**8**, Lemme 1.3]). *Let $R \subsetneq T$ be a minimal extension and S a multiplicatively closed set in R. Then either $R_S \subsetneq T_S$ is a minimal extension or $R_S = T_S$. If S meets the conductor $I = (R : T)$ nontrivially, then $R_S = T_S$.*

Recall from [**8**, Lemme 1.2] that if K is a field and $K \subsetneq L$ is a ring extension, then $K \subsetneq L$ is a minimal ring extension if and only if L satisfies one of the following:

(i) L is a minimal (necessarily algebraic) field extension of K,

(ii) $L \cong K \times K$, or

(iii) $L \cong K[X]/(X^2) \cong K(+)K$.

In all three cases, it is clear that L is a finite K-module.

Theorem 2.3. *Let $R \subsetneq T$ be a minimal extension with conductor $I = (R : T)$. Then exactly one of the following holds.*

(1) *I is a maximal ideal of R, and T is a finite R-module and therefore integral over R.*

(2) *I is not a maximal ideal of R and R is integrally closed in T.*

Proof. Since $R \subsetneq T$ is minimal, either T is integral over R or R is integrally closed in T.

Assume first that $I \in \mathrm{Max}\,(R)$. Then R/I is a field and $R/I \subsetneq T/I$ is a minimal extension of one of the three types (i)–(iii) recalled above. In any case, T/I is a finite R/I-module and it follows that T is a finite R-module.

Next, assume that T is integral over R. Let P be any maximal ideal of R such that $P \neq I$. If $p \in P \backslash I$, then $T = R + pT = R + PT$. As T is an integral minimal extension of R, it is a finite R-module. As $T_P = R_P + PT_P$, Nakayama's lemma gives $T_P = R_P$. Since $T \neq R$, it follows via globalization that $I \in \mathrm{Max}\,(R)$. □

In case (1) of Theorem 2.3, I is a maximal ideal, thus a prime ideal of R, but not necessarily a prime ideal of T (indeed T/I need not even be reduced). In case (2) of Theorem 2.3, it follows from Theorem 2.4 that I is a prime ideal of T (and hence, also of R).

Theorem 2.4. *Let $R \subsetneq T$ be a minimal extension with $I = (R : T)$. If R is integrally closed in T, then I is a prime ideal of T.*

Proof. Suppose $xy \in I$ with $x \in T$, $y \in T \setminus I$. Then $T = R + yT$. Thus, $x = a + by$ for some $a \in R$ and $b \in T$. Multiplying by x, we obtain $x^2 = ax + byx$, with $byx = b(xy) \in bI \subseteq R$. As R is integrally closed in T, we conclude that $x \in R$. Then $xT = xR + xyT \subseteq R + R = R$, whence $x \in I$. □

Valuation pairs. In the next section we characterize the minimal closed extensions. In particular, we generalize Ayache's result [**1**, Theorem 2.4] (cf. also [**7**, Theorem 3.7]) that if D is an integrally closed domain but not a field, a ring extension $D \subsetneq T$ is a minimal closed extension if and only if there is a maximal ideal M of D that is the radical of a finitely generated ideal with $T = \Omega(M)$ and there is a prime $P \subsetneq M$ such that $PD_P = PD_M$ and $(D/P)_{M/P}$ is a one-dimensional valuation domain. As we will see, the pair (D, M) is a rank one valuation pair of T with P the corresponding prime at infinity.

For our work, the following alternate (equivalent) definition for a valuation pair is quite useful. For a pair of rings $R \subsetneq T$ and a prime ideal P of R, the pair (R, P) is a *valuation pair* of T if, for each $t \in T \setminus R$, there is an element $c \in P$ such that $ct \in R \setminus P$. Such a pair gives rise to a valuation map v from T onto a totally ordered Abelian group (together with symbol ∞) with $R = \{t \in T \mid \mathrm{v}(t) \geq 0\}$ and $P = \{t \in T \mid \mathrm{v}(t) > 0\}$. (See, for example, [**17, 22**]). Note that one may consider the vacuous case of (R, P) being a valuation pair of R, but we shall always assume we are dealing with distinct rings $R \subsetneq T$.

We next record the following (see, for instance, [**19**, Section X.1]).

Lemma 2.5. *Let (R, P) be a valuation pair of T. Then*

(1) *If $t \in T \setminus R$ and $r \in R \setminus P$, then $tr \in T \setminus R$.*

(2) *$T \setminus R$ is multiplicatively closed (although $1 \notin T \setminus R$).*

(3) *If $b \in P$ is such that $bT \not\subseteq R$, then there is an element $q \in T$ such that $bq \in R \setminus P$.*

Proof. For (1), choose $p \in P$ such that $pt \in R \setminus P$. Then $p(rt) = r(pt) \in R \setminus P$, and so $rt \in T \setminus R$.

For (2), let $t, s \in T \setminus R$. Choose $p \in P$ such that $pt \in R \setminus P$. By (1), $p(ts) = s(pt) \in T \setminus R$, and so $ts \in T \setminus R$.

For (3), pick $w \in T$ such that $wb \notin R$. Then there is an element $c \in P$ such that $b(cw) = c(wb) \in R \setminus P$. Then the assertion follows, with $q := cw$. $\square$

Let (R, P) be a valuation pair of T. It follows easily from the above characterization of valuation pairs that no ideal of T can contract to P. Also, if S is any intermediate ring (that is, such that $R \subsetneq S \subseteq T$), (R, P) is also a valuation pair of S, and so no ideal of S can contract to P. It follows (from the Lying-over theorem [**10**, Theorem 11.5]) that R is integrally closed in T.

It follows immediately from Lemma 2.5 (1) (and the assumption that T properly contains R) that the conductor $P_\infty = (R : T)$ is contained in P. Consider $a, b \in T \backslash P_\infty$. There are elements $s, t \in T$ such that $as, bt \in T \backslash R$. Thus by Lemma 2.5(2), $abst = (as)(bt) \in T \backslash R$, and so $ab \in T \backslash P_\infty$. Hence P_∞ is a prime ideal of both R and T, called the *prime at infinity.*

The next result completes the derivation of the properties of a valuation pair that were given in the Introduction.

Lemma 2.6. *Let (R, P) be a valuation pair of T (with $R \neq T$). Then the set $\{(P :_T t) \mid t \in T \setminus P_\infty\}$ forms a totally ordered Abelian group $\langle G, * \rangle$ under the operation $(P :_T t) * (P :_T s) = (P :_T ts)$, with $0_G = (P :_T 1) = P$. Moreover, the surjective function $\mathrm{v} : T \mapsto G \bigcup \{\infty\}$, given by $\mathrm{v}(t) = (P :_T t)$ if $t \notin P_\infty$ and $\mathrm{v}(t) = \infty$ otherwise, is a valuation map.*

Proof. We have seen that P_∞ is the conductor $(R : T)$. It follows easily that if $s \in T$, then $(P :_T s) = T$ if and only if $s \in P_\infty$.

To see that the set of conductors $\{(P :_T t) \mid t \in T \setminus P_\infty\}$ is totally ordered under containment, suppose $a, b \in T \backslash P_\infty$ where $(P :_T a)$ is not contained in $(P :_T b)$. We shall show that if $d \in (P :_T b)$, then $d \in (P :_T a)$, that is, $da \in P$. Pick an element $c \in (P :_T a) \backslash (P :_T b)$. As $bc \notin P$, there is an element $f \in R \backslash P_\infty$ such that $bcf \in R \backslash P$. (In detail, use $f = 1$ if $bc \in R$; otherwise, such an f can be found in P, since (R, P) is a valuation pair of T.) Then the product $(ad)(bcf) = (ac)(bd)f$ is in P with $bcf \in R \backslash P$. Hence $ad \in R$ by Lemma 2.5 (1), and so $ad \in P$ since $P \in \mathrm{Spec}\,(R)$.

That the operation is both well-defined and compatible with the order is from a basic property of the colon operation. Suppose $(P :_T b) \subseteq (P :_T a)$ and $(P :_T f) \subseteq (P :_T g)$, and let $u \in (P :_T bf)$. Then $(ub)f = u(bf) \in P$ implies $ub \in (P :_T g)$. Thus $ug \in (P :_T b) \subseteq (P :_T a)$ and we have $u \in (P :_T ag)$, as desired.

Next, to prove that the asserted group has inverses, we shall show that for each $t \in T \backslash P_\infty$, there is an element $q \in T \backslash P_\infty$ such that $qt \in R \backslash P$ and $(P :_T qt) = (P :_T 1) = P$. If $t \in R \setminus P$, it suffices to take $q = 1$, as Lemma 2.5 (2) and the primeness of P ensure that $(P :_T qt) = P$. If $t \in P$, Lemma 2.5 (3) supplies $q \in T$ such that $qt \in R \setminus P$, and one then shows $(P :_T qt) = P$ as above. Observe that $q \notin P_\infty$, for otherwise, $qt \in P_\infty T = P_\infty \subseteq P$, a contradiction. In the remaining case, $t \in T \setminus R$, and so our "valuation pair" definition yields $q \in P$ such that $qt \in R \setminus P$. The argument for this case can be completed as above.

Finally, it is straightforward to verify the assertion about v. $\square$

Recall that the *rank of the valuation pair* (R, P) is the rank of the group G (that is, the length of its chain of isolated subgroups). This is the same as the height of P over P_∞ [**19**, Theorem X.10.10]. In particular, (R, P) has rank 1 (that is, G is isomorphic to a nonzero subgroup of the real numbers) if and only there are no prime ideals strictly between P and P_∞. In the subsection **Critical ideals**, we show that a valuation pair (R, P) of a ring T has rank 1 if and only if P is the critical ideal of the extension $R \subsetneq T$. Subsequently, based on the rings in Remark 3.8 (2), we see that a critical ideal need not be maximal.

Remark 2.7. Let (R, P) be a valuation pair of a ring T, with the usual meanings attached to P_∞ and v. It is relatively straightforward to show that $(R/P_\infty, P/P_\infty)$ is a valuation pair of T/P_∞; and that, for a multiplicatively closed subset S of R that is disjoint from P, (R_S, PR_S) is a valuation pair of T_S [**17**, page 13]. It follows that $(R/P_\infty)_P$ is a valuation domain, having the same rank as (R, P) and having quotient field $(T/P_\infty)_P$. Indeed, if $P_\infty = (0)$, then T and R are integral domains; and if R is quasilocal, then for every $t \in T \setminus \{0\}$, there exists $t' \in T$ with $\mathrm{v}(tt') = 0$, that is, tt' is a unit in R (and so t is invertible). It follows that the prime ideals between P and P_∞ are totally ordered by inclusion.

In general, the prime P in a valuation pair (R, P) need not be a maximal ideal of R, even if (R, P) is a valuation pair of its total quotient ring: an example is given below in Remark 3.8 (2). Next, we characterize the valuation pairs (R, P) of T such that P *is* maximal and then we describe some of their properties.

Proposition 2.8. *Let $R \subsetneq T$ be rings and P an ideal of R. Then the following are equivalent.*

(1) *P is a maximal ideal of R and (R, P) is a valuation pair of T.*

(2) *For each $t \in T \setminus R$, there are elements $c, d \in P$ such that $ct + d = 1$.*

Proof. If (R, P) is a valuation pair, then, for each $t \in T\backslash R$, there is an element $c \in P$ such that $ct \in R\backslash P$. If P is maximal, then $ctR + P = R$ and (2) follows.

Conversely, if (2) holds then, for each $t \in T\backslash R$ and $r \in R\backslash P$, $tr \in T\backslash R$ (as we saw for valuation pairs in Lemma 2.5(1)). Indeed, we have elements $c, d \in P$ such that $ct + d = 1$. Hence $c(rt) + rd = r \in R\backslash P$, and therefore $rt \in T\backslash R$, as asserted. Since $rt \in T\backslash R$, it follows from (2) that there are elements $f, g \in P$ such that $frt + g = 1$. Thus $r(ft) \in R$, and so the above fact gives $ft \in R$. It follows that, for each $r \in R\backslash P$, we have $rR + P = R$. Thus P is maximal, *a fortiori* prime. Then (2) implies that (R, P) is a valuation pair of T (for $ct = 1 - d \in R \setminus P$). □

Proposition 2.9. *Let (R, M) be a valuation pair of T with M a maximal ideal of R. Then*

(1) *Each finitely generated M-primary ideal of R is both T-invertible and 2-generated.*

(2) *If R has at least one finitely generated M-primary ideal, then the intersection $P := \bigcap\{I \mid I$ is M-primary and finitely generated$\}$ is a prime ideal properly contained in M, and each prime ideal of R that is properly contained in M is contained in P.*

Proof. For (1), let v be the valuation map on T associated with the valuation pair (R, M), and let $I = (a_1, a_2, \ldots, a_n)$ be a finitely generated M-primary ideal. Since M properly contains the prime ideal $P_\infty = (R :_T T)$, $IT \not\subseteq R$. Thus $0 < \mathrm{v}(a_i) < \infty$ for some a_i. Since I is finitely generated and v is a valuation, we may relabel so that $\mathrm{v}(a_1) \leq \mathrm{v}(a)$ for each $a \in I$ and $\mathrm{v}(a_1) < \infty$. Since v is surjective, there is a $t \in T$ such that $\mathrm{v}(t) = -\mathrm{v}(a_1) < 0$. Thus $\mathrm{v}(ta_1) = \mathrm{v}(t) + \mathrm{v}(a_1) = 0 \leq \mathrm{v}(t) + \mathrm{v}(a) = \mathrm{v}(ta)$ for each $a \in I$. It follows that $t \in I^{-1}$. Also, since $M = \sqrt{I}$ is maximal and $ta_1 \notin M$, there exist $r \in R$ and a positive integer k such that $(rta_1 - 1)^k \in I$. It follows that $tI + I = R$, whence I is invertible, and there are elements $b, c \in I$ such that $tb + c = 1$. Thus $I = bR + cR$.

For (2), let Q be a prime ideal of R that is properly contained in M, and let I be a finitely generated M-primary ideal of R. Consider any $q \in Q$. Then $B := I + qR$ is M-primary and, by (1), is T-invertible. Hence, there exists a $w \in (R :_T B)$ such that $wB \not\subseteq M$. We have

$qw \in Q \subset M$, since $(qw)I = q(wI) \subseteq Q$ and $I \not\subseteq Q$. So we must have $wI \not\subseteq M$, and it follows easily that $wI + I = R$. Multiplying by q yields $qR = qwI + qI \subseteq I$. Hence I contains Q. It follows that $P = \bigcap\{A \mid A$ is M-primary and finitely generated$\}$ contains each prime ideal that is properly contained in M. Also P is properly contained in I^k for each positive integer k (since I^k is M-primary). By an easy localization argument, P is prime. $\square$

The final result of this subsection deals with characterizing when (R, P) is a valuation pair of a simple (proper) extension $R[u]$.

Proposition 2.10. *Let $R \subsetneq R[u]$ be a simple ring extension, and let P be a prime ideal of R. Then (R, P) is a valuation pair of $R[u]$ if and only if, for each $a \in R\backslash(R :_R u)$, there is an element $p \in P$ such that $pau \in R\backslash P$.*

Proof. The condition is clearly necessary; we have to show it is sufficient. Thus, assume that for each $a \in R\backslash(R :_R u)$, there is an element $p \in P$ such that $pau \in R\backslash P$. Specifically, let $b \in P$ be such that $bu \in R\backslash P$.

Let $f = f_n u^n + f_{n-1}u^{n-1} + \cdots + f_1 u + f_0 \in R[u]\backslash R$ with each f_i in R. We may assume $f_n u$ is not in R (if $f_n u \in R$, writing $f_n u^n + f_{n-1}u^{n-1} = (f_n u + f_{n-1})u^{n-1}$, one could write f as a polynomial in u of lesser degree). By hypothesis, there exists a $z \in P$ such that $zf_n u \in R\backslash P$. We also have $(bu)^{n-1} \in R\backslash P$ and therefore $zb^{n-1}f_n u^n = (zf_n u)(bu)^{n-1} \in R\backslash P$. On the other hand, $b^{n-1}f_i u^i \in R$ for each $i \leq n-1$. It follows that $zb^{n-1}f = zb^{n-1}f_n u^n + \sum_{i \leq n-1} zb^{n-1}f_i u^i \in R\backslash P$. As $zb^{n-1} \in P$, (R, P) is a valuation pair of $R[u]$. $\square$

In the next section we give an example to show that just having an element $p \in M$ such that $pu \in R\backslash M$ is not enough to make (R, M) a valuation pair of $R[u]$, even if pu is a unit of R (see Remark 3.8 (3)).

Critical ideals. We devote the first paragraph of this subsection to recalling a special type of ideal whose existence characterizes minimal ring extensions. A key result of Ferrand and Olivier [**8**, Théorème 2.2 (i)] shows that if $R \subsetneq T$ is a minimal ring extension, then there exists a (necessarily unique) $M \in \mathrm{Max}\,(R)$ such that, for each $P \in$

Spec(R), the canonical injective ring homomorphism $R_P \to T_P$ is an isomorphism if $P \neq M$ and a minimal ring extension if $P = M$. It has become customary to call this M the *crucial maximal ideal* of the minimal ring extension $R \subsetneq T$. It is easy to see via globalization that a ring extension $R \subsetneq T$ has a crucial maximal ideal (that is, an ideal $M \in \mathrm{Max}(R)$ with the above properties) if and only if $R \subsetneq T$ is a minimal ring extension. Having reviewed "crucial" ideals, we next proceed to introduce the "critical" ideals.

We will see that for any minimal extension $R \subsetneq T$, there is a maximal ideal M of R such that $M = \sqrt{(R :_R t)}$ for each $t \in T \backslash R$. We thus say an ideal J is *critical for an extension* $R \subsetneq T$ if $J = \sqrt{(R :_R t)}$ for each $t \in T \backslash R$. By definition, a critical ideal (if any) for a given $R \subsetneq T$ is unique and it is a radical ideal of R. In fact, if an extension has a critical ideal, we next show that this ideal is necessarily prime.

Lemma 2.11. *If a ring extension $R \subsetneq T$ has a critical ideal P, then P is a prime ideal of R.*

Proof. Suppose, by way of contradiction, that $J := \sqrt{(R :_R t)}$ is not a prime ideal of R for some $t \in T \backslash R$. There exist $a, b \in R \backslash J$ such that $ab \in J$. In particular, there is a positive integer n such that $(ab)^n \in (R :_R t)$. Since $a \notin J$, $a^n \notin (R :_R t)$, and hence $a^n t \notin R$. Then $b \in \sqrt{(R :_R a^n t)} = J$, a contradiction. □

The next result determines when a valuation pair leads to a critical ideal.

Lemma 2.12. *Let (R, P) be a valuation pair of a ring T. The extension $R \subsetneq T$ has a critical ideal if and only if the valuation pair (R, P) has rank 1. Moreover, under these conditions, P is the critical ideal of $R \subsetneq T$.*

Proof. Consider the valuation map $\mathrm{v} : T \mapsto G \bigcup \{\infty\}$ defined in Lemma 2.6. If $t \in T \backslash R$, then $v(t) < 0$ and thus, $(R :_R t) \subseteq P = \{s \in T \mid \mathrm{v}(s) > 0\}$. Hence $\sqrt{(R :_R t)} \subseteq P$. If (R, P) has rank 1 (that is, G is isomorphic to a nonzero subgroup of $\mathbf{R}$), then for each $s \in P$, the Archimedean property of $\mathbf{R}$ gives a positive integer n such that $nv(s) > -v(t)$, and so $s \in \sqrt{(R :_R t)}$. If, on the contrary, the rank of the valuation pair is not 1, there are isolated subgroups of G and it follows that there are elements $t_1, t_2 \in T \backslash R$ with $\sqrt{(R :_R t_1)} \neq \sqrt{(R :_R t_2)}$. (In detail, since G is nonarchimedean in this case, there are elements $g < h < 0$ in G such that $g < nh$ for each positive integer n. Since v is surjective, there are elements $t_1, t_2 \in T \backslash R$ and $r \in R$ such that $\mathrm{v}(t_2) = g$ and $\mathrm{v}(t_1) = h = -\mathrm{v}(r)$. Then $rt_1 \in R$ but $\mathrm{v}(r^n t_2) < 0$ for each $n > 0$, and so $r \in \sqrt{(R :_R t_1)} \backslash \sqrt{(R :_R t_2)}$). □

We will give four sufficient conditions for an extension to have a critical ideal, but first, we establish an easy lemma for a simple extension.

Lemma 2.13. *Let $R \subsetneq T = R[u]$ be a simple extension. Then for each $t \in R[u] \backslash R$, $(R :_R u) \subseteq \sqrt{(R :_R t)}$.*

Proof. Write $t = g(u)$ for some polynomial $g(X) \in R[X]$, say of degree n. Thus, for any $r \in (R :_R u)$, we have $r^n \in (R :_R t)$. □

Proposition 2.14. *Let $R \subsetneq T$ be a ring extension. Then $R \subsetneq T$ has a critical ideal in each of the following four cases.*

(1) *$T = R[u]$ for some $u \in T \backslash R$ (that is, $R \subsetneq T$ is a simple extension) and $\sqrt{(R :_R u)}$ is a maximal ideal of R.*

(2) *$R \subsetneq T$ is a minimal ring extension.*

(3) *For each $t \in T \backslash R$, $\sqrt{(R :_R t)}$ is a maximal ideal of R.*

(4) *The conductor $(R : T)$ is a maximal ideal M of R.*

Proof. It follows from Lemma 2.13 that (1) and (2) are each sufficient. (For (2), with $R \subsetneq T = R[u]$ a minimal extension, let $t \in T \backslash R$, note $T = R[t]$, and exchange the roles of u and t in Lemma 2.13.) For (3), let $s \neq t$ be elements of $T \backslash R$ and set $r := s + t$. If $r \in R$, then clearly $(R :_R t) = (R :_R s)$. If $r \notin R$, then $\sqrt{(R :_R r)}$, $\sqrt{(R :_R s)}$, and $\sqrt{(R :_R t)}$ are each maximal ideals, and each one contains the intersection of the other two. It follows easily that $\sqrt{(R :_R s)} = \sqrt{(R :_R t)}$ $(= \sqrt{(R :_R r)})$. Finally, if (4) holds, then it is clear that $M = (R :_R t) = \sqrt{(R :_R t)}$ for each $t \in T \backslash R$. □

Under conditions (1), (3) or (4) of Proposition 2.14, the critical ideal is obviously maximal. We will show this is also the case for (2), namely, when $R \subsetneq T$ is a minimal extension. (At least, we already know from Theorem 2.3 that in the case of a minimal integral extension, the conductor $M = (R : T)$ is a maximal ideal.) Examples abound to show that extensions with a maximal critical ideal are not necessarily minimal extensions: simply take a suitable pair of quasilocal rings $R \subsetneq T$ with the same maximal ideal. Also, it follows from Lemma 2.12 and Remark 3.8 (2) that a critical ideal need not be a maximal ideal.

Finally, recall that, for a pair of rings $R \subsetneq T$ and an ideal I of R not contained in the nilradical, we defined the generalized Kaplansky transform $\Psi_T(I)$ of I with respect to T to be the intersection $\Psi_T(I) = \bigcap\{R_{\langle Q,T\rangle} \mid I \not\subseteq Q \in \operatorname{Spec}(R)\}$, where $R_{\langle Q,T\rangle} := \{t \in T \mid rt \in R$ for some $r \in R\backslash Q\}$ is the large quotient ring of R with respect to Q and T. The last result of this section links the ideal transform of a maximal ideal with the notion of a critical ideal.

Lemma 2.15. *Let $R \subsetneq T$ be a pair of rings, and let M be a maximal ideal of R. If $R \neq \Psi_T(M)$, then M is a critical ideal for the extension $R \subsetneq \Psi_T(M)$. Conversely, if M is a critical ideal for the extension $R \subsetneq T$, then $T = \Psi_T(M)$.*

Proof. For each $t \in \Psi_T(M)\backslash R$ (if any) and each prime $P \neq M$, we have $t \in R_{\langle P,T\rangle}$, and so there exists an $r \in R\backslash P$ such that $rt \in R$; that is, $(R :_R t)$ is not contained in P. Hence $\sqrt{(R :_R t)} = M$. Conversely, if $\sqrt{(R :_R t)} = M$ for each $t \in T\backslash R$, then it is easy to see that $T \subseteq R_{\langle P,T\rangle}$ for each prime $P \neq M$, and so $T = \Psi_T(M)$. □

3. Closed minimal extensions. In this section, we give several characterizations of closed minimal extensions, either in terms of specific properties related to u for $R[u]$ to be such a minimal extension, or in terms of valuation pairs. Before doing so, we need a few lemmas.

Closed minimal extensions and Kaplansky ideal transforms.

Lemma 3.1. *Let $R \subsetneq T$ be rings and $u \in T$. If R is integrally closed in $R[u]$ and $h(X) \in R[X]$ is a polynomial of degree $n > 0$ such that $h(u) = 0$, then the leading coefficient of h multiplies u into R.*

Proof. Let $h(X) = h_nX^n + \cdots + h_0 \in R[X]$ be such that $h(u) = 0$. Then

$$h_n^{n-1}h(u) = (h_nu)^n+h_{n-1}(h_nu)^{n-1}+h_nh_{n-2}(h_nu)^{n-2}+\cdots+h_n^{n-1}h_0 = 0.$$

Hence the monic polynomial $g(X) := X^n + h_{n-1}X + h_nh_{n-2}X^{n-2} + \cdots + h_n^{n-1}h_0$ is such that $g(h_nu) = 0$. As R is integrally closed in $R[u]$, we have $h_nu \in R$. □

Lemma 3.2. *Let $R \subsetneq T$ be rings and $u \in T$. If R is integrally closed in $R[u]$ and $R[u] = R[u^2]$, then $(R + uR)(R :_R u) = R$.*

Proof. Since $R[u] = R[u^2]$, there is a degree n polynomial ($n \geq 1$) $f(X) \in R[X]$ such that $u = f(u^2)$. Setting $g(X) := f(X^2)$ gives a polynomial g of degree $2n$ with no term in X such that $u = g(u)$. Thus there is a polynomial $h(X) \in R[X]$ of minimal degree with no X term such that $h(u) = u$. Write $h(X) = h_0 + h_2X^2 + \cdots + h_kX^k$. Without loss of generality, $u \notin R$, and so $k > 0$. By Lemma 3.1, $h_ku \in R$. If $k > 2$, we can rewrite $h(X)$ as

$$b(X) = h_0 + h_2X^2 + \cdots + (h_{k-1} + h_ku)X^{k-1},$$

obtaining a polynomial $b(X) \in R[X]$ of smaller degree, again with $b(u) = u$ and no X term, thus reaching a contradiction. Hence $k = 2$, and we have $u = h_0 + h_2u^2$ with $h_2u \in R$. Rearranging yields $h_0 = u(1 - h_2u)$. Thus, both h_2 and $(1 - h_2u)$ are in $(R :_R u)$. Hence $1 = (1-h_2u)1+h_2u \in (R+uR)(R :_R u)$. Thus, $(R+uR)(R :_R u) = R$; also, both $R + uR$ and $(R :_R u)$ are $R[u]$-invertible. □

We are ready for a first characterization. Given rings $R \subsetneq T$ and a maximal ideal M of R, we next give a condition for the generalized Kaplansky ideal transform of M to be a closed minimal extension of R. In fact, we shall later see (in Theorem 3.5) that, for any closed minimal extension $R \subsetneq T$, there is a maximal ideal M of R such that $T = \Psi_T(M)$.

Lemma 3.3. *Let $R \subsetneq T$ be a ring extension, and let M be a maximal ideal of R. The following assertions are equivalent.*

(1) $\Psi_T(M)$ *is a closed minimal extension of* R.

(2) (R, M) *is a valuation pair of* $\Psi_T(M)$.

(3) (R, M) *is a rank* 1 *valuation pair of* $\Psi_T(M)$.

Proof. Assume $\Psi_T(M)$ is a closed minimal extension of R, and let $u \in \Psi_T(M) \backslash R$. By Lemma 2.15, M is a critical ideal for the extension $R \subsetneq \Psi_T(M)$; in particular, $\sqrt{(R :_R u)} \subseteq M$. Since R is integrally closed in $\Psi_T(M)$, no power of u is in R. Hence $\Psi_T(M) = R[u^n]$ for each positive integer n. By Lemma 3.2, $(R + uR)(R :_R u) = R$. As $(R :_R u) \subseteq M$, it follows there is an element $m \in M$ such that $mu \in R \backslash M$. By definition, (R, M) is thus a valuation pair of $\Psi_T(M)$. By Lemma 2.12, this valuation pair is rank 1.

Conversely, assume that (R, M) is a valuation pair of $\Psi_T(M)$. It follows (by considering the associated valuation or via the Lying-over theorem) that R is integrally closed in $\Psi_T(M)$. Let $S \neq R$ be any ring between R and $\Psi_T(M)$. Since (R, M) is a valuation pair of S, no (prime) ideal of S contracts to M. Thus each maximal ideal N of S contracts to a prime Q of R distinct from M, and so $R_{\langle Q,T \rangle} \subseteq S_{\langle N,T \rangle}$. Then $S = \bigcap \{S_{\langle N,T \rangle} \mid N \in \mathrm{Max}\,(S)\}$ contains, and hence is equal to, $\Psi_T(M) = \bigcap \{R_{\langle Q,T \rangle} \mid Q \in \mathrm{Spec}\,(R) \backslash \{M\}\}$ Thus, if $\Psi_T(M) \neq R$, (1) holds. □

When is R[u] a closed minimal extension of R? Given an element $u \in T \backslash R$, we now give a list of necessary and sufficient conditions for $R \subsetneq R[u]$ to be a closed minimal extension. One necessary condition is that $M = \sqrt{(R :_R u)}$ is a maximal ideal of R and we note that, by Proposition 2.14 (condition 1), this is equivalent to say that $\sqrt{(R :_R t)} = M$ for each $t \in R[u] \backslash R$ (that is, M is the critical ideal of the extension $R \subsetneq R[u]$). Another necessary condition is that, for each $t \in R[u] \backslash R$, $R + tR$ is invertible. Note that, as $R + tR$ contains R, its T-invertibility is the same as its $R[u]$-invertibility and simply means that $(R + tR)(R :_R t) = R$ (while the S-invertibility of an ideal depends in general on the ring S containing R relative to which it is considered).

Theorem 3.4. *Let* $R \subsetneq T$ *be rings and* $u \in T \backslash R$. *Then the following are equivalent.*

(1) $R \subsetneq R[u]$ *is a closed minimal extension.*

(2) *For each* $t \in R[u] \backslash R$, *there are elements* c, d *in the ideal* $\sqrt{(R :_R u)}$ *such that* $ct + d = 1$.

(3) $M = \sqrt{(R :_R u)}$ *is a maximal ideal of* R, *and* (R, M) *is a valuation pair of* $R[u]$.

(4) *There is a maximal ideal* M *of* R *such that* (R, M) *is a rank* 1 *valuation pair of* $R[u]$.

(5) $M = \sqrt{(R :_R u)}$ *is a maximal ideal of* R *and, for each* $t \in R[u] \backslash R$, $R + tR$ *is invertible.*

(6) $M = \sqrt{(R :_R u)}$ *is a maximal ideal of* R, $(R :_R u)$ *is finitely generated, and each finitely generated* M*-primary ideal is* $R[u]$*-invertible.*

(7) $M = \sqrt{(R :_R u)}$ *is a maximal ideal of* R, *and* $R + auR$ *is invertible for each* $a \in R$.

(8) $M = \sqrt{(R :_R u)}$ *is a maximal ideal of* R *and, for each* $a \in R \backslash (R :_R u)$, *there is an element* $b \in R$ *such that* $bau \in R \backslash M$.

Proof. Assume (1). Then for each $t \in R[u] \backslash R$, we have $R \subsetneq R[t^2] = R[u] = R[t]$, with R integrally closed in $R[t]$. Thus, by Lemma 3,2, $R + tR$ is invertible. On the other hand, by Proposition 2.4 (condition 2), $(R :_R t) \subseteq \sqrt{(R :_R t)} = \sqrt{(R :_R u)}$. Therefore, there are elements $c, d \in \sqrt{(R :_R u)}$ such that $ct + d = 1$. This is (2).

That (2) implies (3) follows from Proposition 2.8.

If $M = \sqrt{(R :_R u)}$ is a maximal ideal of R, then M is the critical ideal of the extension $R \subsetneq R[u]$ by Proposition 2.14 (condition 1). If moreover (R, M) is a valuation pair of $R[u]$, then this valuation pair is rank 1 by Lemma 2.12. Hence (3) implies (4).

If M is a maximal ideal of R such that (R, M) is a rank 1 valuation pair of $R[u]$, then M is the critical ideal of the extension $R \subsetneq R[u]$ by Lemma 2.12. Thus, it follows from Lemma 2.15 that $R[u] = \Psi_{R[u]}(M)$. Hence, (4) implies (1) by Lemma 3.3.

The equivalent conditions (1)–(3) clearly imply (5). Conversely, if $M = \sqrt{(R :_R u)}$ is maximal, then it is the critical ideal of the extension $R \subsetneq R[u]$ by Proposition 2.14 (condition 1); and if $R + tR$ is invertible with $t \in R[u] \setminus R$, there are elements $c, d \in (R :_R t) \subseteq \sqrt{(R :_R u)}$ such that $ct + d = 1$. Thus (5) implies (2).

The equivalent conditions (3) and (5) imply (6): If (R, M) is a valuation pair of $R[u]$ with M maximal, it follows from Proposition 2.9 that each finitely generated M-primary ideal is $R[u]$-invertible. If $R + tR$ is invertible for each $t \in R[u]\backslash R$, then in particular, $R + uR$ is $R[u]$-invertible, and so its inverse $(R :_R u)$ is finitely generated.

(6) implies (5). Let $t \in R[u]\backslash R$. Since $t \in R[u]$, there is an integer $n \geq 1$ such that $t(R :_R u)^n \subseteq R$. Since $(R :_R u)$ is finitely generated and M-primary, $(R + tR)(R :_R u)^n$ is finitely generated and either M-primary or equal to R. In either case, $R + tR$ is invertible.

Obviously, (5) implies (7).

If $M = \sqrt{(R :_R u)}$ is a maximal ideal of R and $a \in R\backslash(R :_R u)$ is such that $(R + auR)(R :_R au) = R$, then Proposition 2.14 (1) yields that M is the critical ideal of $R \subsetneq R[u]$, and so there is an element $b \in (R :_R au) \subseteq \sqrt{(R :_R au)} = M$ such that $bau \in R\backslash M$. Hence (7) implies (8).

By Propositions 2.10 and 2.14 (condition 1) and Lemma 2.12, (8) implies (R, M) is a rank 1 valuation pair of $R[u]$. Thus (8) implies (4). □

When is $R \subsetneq T$ a closed minimal extension? We next obtain necessary and sufficient conditions for a pair of rings $R \subsetneq T$ to be a closed minimal extension.

Theorem 3.5. *Let $R \subsetneq T$ be a pair of rings. Then the following are equivalent.*

(1) *$R \subsetneq T$ is a closed minimal extension.*

(2) *The extension $R \subsetneq T$ has a critical ideal M such that M is a maximal ideal of R, and for each $t \in T\backslash R$, $R + tR$ is invertible.*

(3) *For each $t \in T\backslash R$, $\sqrt{(R :_R t)}$ is a maximal ideal of R and $R + tR$ is invertible.*

(4) *There is a maximal ideal M of R such that (R, M) is a rank 1 valuation pair of T.*

Moreover, under these conditions, $T = \Psi_T(M)$.

Proof. Assume (1). Then $T = R[u]$ for some (in fact, each) $u \in T \backslash R$). Thus it follows from the proof of Theorem 3.4 that (1) implies (2) (and (4)).

Obviously (2) implies (3); conversely, (3) implies (2) by Proposition 2.14 (condition 3).

Assume (2). Let $t \in T \setminus R$. By (2), there are elements $c, d \in M$, such that $ct + d = 1$. By Proposition 2.8, it follows that M is maximal and (R, M) is a valuation pair of T. Since M is the critical ideal, this valuation pair is rank 1 by Lemma 2.12. Thus (2) implies (4).

Finally, if (R, M) is a rank 1 valuation pair of T with M maximal, then M is the critical ideal of the extension $R \subsetneq T$ by Lemma 2.12. Hence $T = \Psi_T(M)$ by Lemma 2.15, and so $R \subsetneq T$ is a closed minimal extension by Lemma 3.3. Thus, (4) implies (1). □

It follows from the classification of the minimal ring extensions of a field [**8**, Lemme 1.2] that condition (1) of Theorem 3.5 cannot hold if R is a field. It is instructive to verify directly that the same is true for conditions (2)–(4), by using the following facts: if a field K is a proper subring of a ring T, then $\{0\}$ is a critical ideal for $K \subsetneq T$ and $(K, \{0\})$ is not a valuation pair of T.

We next give two corollaries. The first of these is proved by combining Proposition 2.9 (1) with Theorem 3.5. The second corollary combines Lemma 3.3 with Theorems 3.4 and 3.5.

Corollary 3.6. *Let $R \subsetneq T$ be a closed minimal extension. Then, for each $t \in T$, $(R :_R t)$ is T-invertible and 2-generated.*

Corollary 3.7. *Let $R \subsetneq T$ be a pair of rings and M be a maximal ideal of R. Then the following are equivalent.*

(1) *$R \subsetneq \Psi_T(M)$ is a closed minimal extension.*

(2) *M is the radical of a finitely generated ideal of R, and each finitely generated M-primary ideal of R is T-invertible.*

Proof. That (1) implies (2) follows from Lemmas 2.12, 2.15 and 3.3 and Theorem 3.4. Conversely, assuming (2), there exists a finitely generated M-primary ideal; let us denote it by I. Since M is the only prime that contains I, $(R:_T I)$ is contained in $R_{\langle P,T\rangle}$ for each prime $P \neq M$. Thus $(R:_T I) \subseteq \Psi_T(M)$. As I is T-invertible, $(R:_T I) \nsubseteq R$; hence $R \subsetneq \Psi_T(M)$. By Lemma 2.15, M is the critical ideal for this extension. Let $t \in \Psi_T(M)\backslash R$. Then $(R:_R t)$ is an M-primary ideal since $\sqrt{(R:_R t)} = M$. As I is finitely generated and contained in M, $(R:_R t)$ contains some power I^n of I. Hence $I^n + tI^n = I^n(R+tR)$ is a finitely generated ideal of R which is either M-primary or equal to R. At any rate, $R+tR$ is T-invertible. Thus $R \subsetneq \Psi_T(M)$ is a closed minimal extension by Theorem 3.5. □

Remark 3.8. (1) We have seen in Lemma 3.3 that if $T = \Psi_T(M)$ for some maximal ideal M, then it is enough to know that (R,M) is a valuation pair of T to conclude that $R \subsetneq T$ is a closed minimal extension (and the valuation pair is rank 1). In general however, if (R,M) is a valuation pair of T, then $R \subsetneq T$ is not necessarily a closed minimal extension. For instance, if V is a valuation domain with maximal ideal M and quotient field K, then $V \subsetneq K$ is not a minimal extension and $\Psi_T(M) \neq K$, unless the valuation is rank 1.

(2) If (R,P) is a rank 1 valuation pair of $\Psi_T(P)$ with P not maximal in R, then R is integrally closed in $\Psi_T(P)$ but the extension $R \subsetneq \Psi_T(P)$ is not minimal. For example, let $D := K[Y,Z]$, $T := K(Y,Z)$ its quotient field, and $P := ZD$. Then the ideal transform $\Omega(P) = \Psi_T(P)$ is the ring $K[Y,Z,1/Z]$. It is easy to check that (D,P) is a valuation pair of $\Psi_T(P)$ and, since P is a height 1 prime of D, this valuation pair is obviously rank 1. However, the ring $S := D[Y/Z] = K[Y,Z,Y/Z]$ sits strictly between D and $\Omega(P)$. Note also that while $P = ZD$ is $\Omega(P)$-invertible, it is not S-invertible.

(3) Given a pair of rings $R \subsetneq T$ and a maximal ideal M of R, it is not enough to simply have $\Psi_T(M) = R[u]$ for some $u \in T\backslash R$ with R integrally closed in $\Psi_T(M)$ and an element $p \in M$ such that $pu \in R\backslash M$ for $\Psi_T(M)$ to be a closed minimal extension of R. For example, let $D := K[Y,Z]$, $T := K(Y,Z)$ and $P := ZD$ as above. Then set $R := K + P = K[\{Y^nZ \mid n \geq 0\}]$. Clearly, the rings $R \subsetneq D$ share the ideal P, and P is maximal in R; we thus call it M as an ideal of R. For each prime ideal $Q \neq M$ of R, one has $R_Q = D_Q$: hence the ideal transform $\Psi_T(M)$ of the ideal M of R is the same as the ideal transform of the ideal P of D. In other words, $\Psi_T(M) = K[Y,Z,1/Z]$. It follows

that $\Psi_T(M)$ is not a minimal extension of R and, by Lemma 3.3, that (R, M) is not a valuation pair of $\Psi_T(M)$. On the other hand, R is integrally closed in D (since $R/P = K$ is integrally closed in $D/P = K[Y]$) and thus also integrally closed in $\Psi_T(M)$ (since D is integrally closed in $\Psi_T(M)$). Moreover, $\Psi_T(M) = R[u]$ with $u := 1/Z$; and for $p := Z \in M$, one has $pu = 1 \notin M$.

(4) The converse of Corollary 3.6 does not hold, even if $R + tR$ is a cyclic R-module for each t. For example, let R be a principal ideal domain with quotient field T. Then R is integrally closed in T and, for each $t \in T$, $(R :_R t)$ is a principal ideal (thus invertible, that is, T-invertible). If R is semilocal, then $T = R[u]$ for some u. Yet, if there is more than one maximal ideal in R, $R \subsetneq T$ is not a minimal extension. What is missing is that $\sqrt{(R :_R u)}$ is not maximal in this case.

(5) Although we have not made use of the theory of the "crucial maximal ideal" as developed by Ferrand and Olivier [**8**], it is interesting to note that if $R \subsetneq T$ is a closed minimal extension with crucial maximal ideal M, then M is also the critical ideal of $R \subsetneq T$. To see this, let N be any maximal ideal of R other than M. Then $R_N = T_N$ canonically. Thus, if $t \in T \setminus R$, one has $(R :_R t) \not\subseteq N$ and, *a fortiori*, $\sqrt{(R :_R t)} \not\subseteq N$. It then follows from the proof of Theorem 3.4 that N cannot be the critical ideal of $R \subsetneq T$, and so M must be that critical ideal.

Using Proposition 2.9, we can easily describe the conductor $(R : T)$ when $R \subsetneq T$ is a closed minimal extension.

Corollary 3.9. *If $R \subsetneq T$ is a closed minimal extension with critical maximal ideal M, then the conductor of T into R is the prime ideal $P = \bigcap\{I \subseteq M \mid I$ is M-primary and finitely generated$\}$. Moreover, P contains all the prime ideals of R that are properly contained in M.*

Proof. By Theorem 3.5, (R, M) is a valuation pair of T. Thus by Proposition 2.9, the intersection $P = \bigcap\{I \subseteq M \mid I$ is M-primary and finitely generated$\}$ is a prime ideal contained in M and it contains all the other prime ideals of R that are properly contained in M. (Notice that P must be the prime at infinity since the valuation pair (R, M) has rank 1; in particular, $P = (R : T)$.) □

Note that, for a domain D with quotient field K, if M is the radical of a finitely generated ideal I of D, then the ideal transform $\mathcal{T}(I)$ of I (which is both the Kaplansky and the Nagata transform of I) is the same as $\Omega(M) = \Psi_K(M)$. Thus, a closed minimal extension of D is always of the form $\mathcal{T}(I)$ for some finitely generated ideal I whose radical is a maximal ideal M. By Corollary 3.7, the extra condition to characterize when $\mathcal{T}(I)$ is "closed minimal" is that each finitely generated M-primary ideal is invertible (that is, K-invertible). In the case of a Prüfer domain, every nonzero finitely generated ideal is invertible, and so this extra condition is automatically satisfied in this case. We thus recover an alternate form of [**1**, Theorem 2.4].

Corollary 3.10. *Let D be a Prüfer domain. An overring S of D is a (necessarily closed) minimal extension of D if and only if S is the ideal transform of an invertible ideal of D whose radical is a maximal ideal of D.*

Closed minimal extensions of the form $\mathbf{R[t^{-1}]}$. If R is a quasilocal ring and $R[u]$ is a closed minimal extension of R, then $R[u] = R[t^{-1}]$ for some t in the maximal ideal of R (this is part of [**7**, Theorem 3.1]). In fact, the next result shows that one can say a great deal more (cf. [**8**, Lemma 2.1]).

Lemma 3.11. *Let (R, M) be a quasilocal ring and $R \subsetneq T$ a closed minimal extension with conductor $P := (R : T)$. Then each element $t \in T \backslash P$ is a unit of T. More precisely, if $t \in T \backslash R$, then $t^{-1} \in M \backslash P$; if $t \in R \backslash M$, then $t^{-1} \in R \backslash M$; and if $t \in M \backslash P$, then $t^{-1} \in T \backslash R$.*

Proof. By Theorem 3.5, (R, M) is a valuation pair of T. Since the image of the associated valuation map is a group, we see that if $t \in T \backslash P$, there exists $t' \in T$ such that $\mathrm{v}(tt') = \mathrm{v}(t) + \mathrm{v}(t') = 0$. Thus $tt' \in R \backslash M$; that is, tt' is a unit of R. As $\mathrm{v}(t) = -\mathrm{v}(t') = -\mathrm{v}(t^{-1})$, the "more precise(ly)" assertions follow from the fact that $T \setminus R$, $R \setminus M$ and $M \setminus P$ are, respectively, the sets of elements t such that $\mathrm{v}(t) < 0$, $\mathrm{v}(t) = 0$ and $\mathrm{v}(t) > 0$. □

Remark 3.12. If R is a quasilocal domain that shares its maximal ideal M with a proper overring T of R, then R has no closed minimal extensions. (This applies, for instance, in case R is a pseudo-valuation domain which is distinct from its canonically associated valuation overring T.) For a proof, note by Lemma 3.11 (or [**7**, Theorem 3.1]) that such a closed minimal extension would be an overring of the form $R[t^{-1}]$ for some $t \in M$. Then $R[t^{-1}] = T[t^{-1}]$ contains (and, by minimality, must coincide with) T. However, no proper ideal of $R[t^{-1}]$ can contract to M (since $t \in M$), contradicting the fact that M is an ideal of T.

The result that was just proved stands in contrast to the non-closed (that is, integral) case. Indeed, *any* nonzero ring R, having a maximal ideal M, leads to the integral minimal extension $R \subsetneq R(+)R/M$ [**2**, Corollary 2.5] and $R(+)R/M$ shares the ideal M with R.

Outside the quasilocal case, it is not particularly difficult to find an example of a closed minimal extension $R \subsetneq R[u]$ for which u is not a unit in $R[u]$. For example, $\mathbf{Z} \subsetneq \mathbf{Z}[2/3]$ is a minimal extension such that $2/3$ is not a unit of $\mathbf{Z}[2/3]$. Of course, in this case $\mathbf{Z}[2/3] = \mathbf{Z}[1/3]$. A natural question to ask is, "For which rings R is each closed minimal extension $R \subsetneq S$ of the form $S = R[u]$ for some unit u of S?"

Our next two results answer the above question for two special cases, Dedekind domains and Prüfer domains. Note that related results appear in [**1**, Proposition 2.3] and [**7**, Theorem 3.1]. In fact, alternate proofs of the next two results can be obtained via [**1**, Proposition 2.3 and Theorem 5.2].

Theorem 3.13. *The following are equivalent for a Dedekind domain R which is not a field.*

(1) *Each minimal overring of R is of the form $R[u]$ for some element $u^{-1} \in R$.*

(2) *R has torsion class group.*

(3) *Each minimal overring of R is the ideal transform of a (nonzero) principal ideal of R.*

Proof. By Corollary 3.10, the minimal overrings of R are the rings of the form $\Psi_T(M)$ where T is the quotient field of R and $M \in \operatorname{Max}(R)$ (where M is necessarily the radical of a finitely generated ideal of R).

(1) $\Rightarrow$ (2). Assume (1). Let $M \in \operatorname{Max}(R)$. Then $\Psi_T(M) = R[u]$ where $u^{-1} \in R$. By Theorem 3.5, $M = \sqrt{(R :_R u)}$. Since $(R :_R u) = Ru^{-1}$, it follows that M is the only prime ideal of R which contains

Ru^{-1}. Since R is a Dedekind domain, the factorization of Ru^{-1} as a product of prime ideals of R must take the form $Ru^{-1} = M^n$ for some positive integer n. In particular, some power of M is principal and (2) follows.

(2) $\Rightarrow$ (1). Assume (2). Let $M \in \mathrm{Max}\,(R)$. Then $M^n = Rs$ for some positive integer n and nonzero element $s \in R$. As M is the only maximal ideal of R that contains s, it follows (cf. [**10**, Theorem 26.1 (2)]) that $R[s^{-1}] = \bigcap\{R_N \mid N \in \mathrm{Max}\,(R) \text{ and } s \notin N\}$. In other words, $R[s^{-1}] = \Psi_T(M)$. As $M = \sqrt{Rs}$ and Rs is invertible, (1) follows, with $u := s^{-1}$.

(2) $\Rightarrow$ (3). Using the notation of the proof that (2) $\Rightarrow$ (1), we have that $\Psi_T(M) = \Psi_T(\sqrt{Rs}) = \Psi_T(Rs) = \mathcal{T}(Rs)$.

(3) $\Rightarrow$ (1). Assume that $M \in \mathrm{Max}\,(R)$ and some nonzero element $s \in R$ is such that $\Psi_T(M) = \mathcal{T}(Rs)$. This minimal overring of R is

$$\Psi_T(Rs) = \bigcap\{R_N \mid N \in \mathrm{Max}\,(R),\ s \notin N\} = R[s^{-1}],$$

where the last equality holds via [**10**, Theorem 26.1 (2)] as above. Then (1) follows, with $u := s^{-1}$. □

In Theorem 3.14, we provide a companion for Theorem 3.13 where R is an arbitrary Prüfer domain which is not a field.

A maximal ideal M of a domain R is *sharp* if R_M does not contain the intersection of the R_N's as N ranges over the set $\mathrm{Max}\,(R)\backslash\{M\}$ (see, for example, [**9**]). If R is a Prüfer domain, M is sharp if and only if R has a finitely generated ideal $J \subseteq M$ that is contained in no other maximal ideal [**11**, Corollary 2]. Also, a nonzero prime P of a Prüfer domain is said to be *branched* if it contains a proper P-primary ideal. By [**10**, Theorem 23.3], P is branched if and only if it is minimal over a finitely generated ideal. Combining the two concepts, we see that a maximal ideal M of a Prüfer domain R is both sharp and branched if and only if M is the radical of a finitely generated ideal of R [**11**, Theorem 2].

Theorem 3.14. *The following are equivalent for a Prüfer domain R which is not a field.*

(1) *Each minimal overring of R is of the form $R[u]$ for some element $u^{-1} \in R$.*

(2) *Each sharp branched maximal ideal of R is the radical of a* (*nonzero*) *principal ideal of R.*

(3) *Each minimal overring of R is the ideal transform of a* (*nonzero*) *principal ideal of R.*

Proof. Let T denote the quotient field of R. By Corollary 3.10, an overring of R is a minimal overring of R if and only if it is the ideal transform $\mathcal{T}(J) = \Psi_T(M)$ of a finitely generated ideal J such that the radical of J is a maximal ideal M. As noted above, such M are the maximal ideals which are both sharp and branched. Hence (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3). Assume (1). Let $M \in \operatorname{Max}(R)$ be sharp and branched. Then $\Psi_T(M) = R[u]$ with $u^{-1} \in R$. As $u \in \Psi_T(M) \setminus R$, we have $M = \sqrt{(R :_R u)}$, and so $u^{-1} \in M$. Since each prime of R distinct from M survives in $\Psi_T(M)$, no other prime of R can contain u^{-1}. Thus $M = \sqrt{Ru^{-1}}$ and $\Psi_T(M) = \Psi_T(Ru^{-1}) = \mathcal{T}(Ru^{-1})$.

(2) $\Rightarrow$ (1). If $M \in \operatorname{Max}(R)$ is the radical of a nonzero principal ideal Rs of R, then $\Psi_T(M) = \Psi_T(Rs) = R[s^{-1}]$, the last equality following from [**10**, Theorem 26.1 (2)] as above. □

To close this section, we note that it is possible for a domain R which is not a Prüfer domain to have the property that each minimal overring of R is the ideal transform of a principal ideal (and thus is of the form $R[u]$ for some element u such that $u^{-1} \in R$). For instance, this is the case for the ring $R := D + (X,Y)K[[X,Y]]$ whenever D is a Prüfer domain with quotient field K ($\neq D$) such that each maximal ideal of D is the radical of a principal ideal. By Theorem 3.4 and Theorem 3.5, each minimal overring of R is the ideal transform of a principal ideal whose radical is maximal.

4. Embedding in a quotient ring. In this section, we discuss the embedding of a minimal ring extension into various quotient rings.

The simple way to form the total quotient ring of a ring R is to say $Q_{cl}(R)$ consists of the fractions a/b where $a, b \in R$ with b regular, with the same equivalence relation used to form the quotient field of an integral domain. For an alternate approach, note that if J is a nonzero ideal of an integral domain D, then each element of $\operatorname{Hom}_D(J, D)$ can be viewed as multiplication by some fixed element of the quotient field. Indeed, if $f \in \mathrm{Hom}_D(J, D)$, choose any nonzero element $b \in J$; then for each $c \in J$, $f(c) = (a/b)c$ where $a := f(b)$. A similar identification is possible when I is a regular ideal of a ring R (with nonzero zero divisors), although one must choose $b \in I$ to be regular. On the other hand, such a simple interpretation is not possible if I is not regular. Since a product of dense ideals is dense, a product of semiregular ideals is semiregular. Thus for a pair of dense (respectively, semiregular) ideals I and J, the product and sum of a pair of homomorphisms $f \in \mathrm{Hom}_R(I, R)$ and $g \in \mathrm{Hom}_R(J, R)$ restrict to R-module homomorphisms on the dense (respectively, semiregular) ideal IJ. Also, there is a dense ideal B such that $f(x) = g(x)$ for all $x \in B$ if and only if $f(y) = g(y)$ for all $y \in I \bigcap J$. Setting f equivalent to g when this occurs gives an equivalence relation that is compatible with sums and products. The complete ring of quotients of R, $Q(R)$, consists of the equivalence classes of homomorphisms over dense ideals and the ring of finite fractions of R, $Q_0(R)$, consists of those equivalence classes of homomorphisms over semiregular ideals. In this setting, one can identify $Q_{cl}(R)$ with the equivalence classes of those homomorphisms defined on regular ideals. In general, $R \subseteq Q_{cl}(R) \subseteq Q_0(R) \subseteq Q(R)$ with one or both of the last two containments being proper. (See [**18**, Chapter 2] for details on this construction for $Q(R)$; for an alternate way of constructing $Q_0(R)$ see [**20**]). Also, for each nonzero $t \in Q(R)$, there is a dense ideal I of R such that $tI \subseteq R$ and $tI \neq (0)$. The same can be said of $Q_0(R)$, replacing "dense ideal" by "semiregular ideal," and of $Q_{cl}(R)$ by replacing "dense" by "regular."

Embedding a closed minimal extension into $\mathbf{Q_0(R)}$. We wish to study the embedding of a minimal extension of R into the rings $Q(R)$, $Q_0(R)$ or $Q_{cl}(R)$. Extending results in [**7, 24**], we first show that any closed minimal extension is R-isomorphic to a Q_0-overring of R. The ring R in Example 6.2 shows that such an extension need not be isomorphic to an overring of R (that is, a $Q_{cl}(R)$-overring). Moreover, the particular extension T in that example is such that each element of $T \backslash R$ is a zero divisor in T. In Example 6.4, we present a closed minimal extension $R \subsetneq S$ where $S \subsetneq Q_{cl}(R)$ and each element of $S \backslash R$ is a zero divisor.

Theorem 4.1. *Let $R \subsetneq T$ be a closed minimal extension. Then there is a natural R-algebra isomorphism from T into a Q_0-overring of R.*

Proof. For each $t \in T$, multiplication by t gives an R-module homomorphism $\varphi(t) : (R :_R t) \to R$. Since Theorem 3.4 provides elements $a, b \in \sqrt{(R :_R t)}$ such that $at + b = 1$, the ideal $aR + bR$ has no nonzero annihilators in T. Indeed, $u = 0$ is the only element $u \in T$ such that $ua = 0 = ub$, as $u = uat + ub$ for each $u \in T$. Choose a positive integer ν such that $J := (aR + bR)^\nu \subseteq (R :_R t)$. As the set of dense ideals of R is closed under finite products, J is a finitely generated dense ideal of R. Therefore, $(R :_R t)$ is a semiregular ideal of R.

We claim that the kernel of φ is $\{0\}$. To see this, consider any $t \in \ker(\varphi)$, and pick positive integers m, n such that $a^m t, b^n t \in R$. As $(R :_R t)t = 0$, we have $a^m t = 0 = b^n t$. Raising $at + b = 1$ to the exponent $m + n$ shows that $1 = b^{m+n} \in Rb^n \subseteq (R :_R t)$, whence $t = 1t = 0$, thus proving the above claim.

Obviously, for each $r \in R$, multiplication by rt also defines an R-module homomorphism on the semiregular ideal $(R :_R t)$. Since $\varphi(rt)$ and $r\varphi(t)$ agree on this ideal, $\varphi(rt) = r\varphi(t)$.

Let $s, t \in T \backslash \{0\}$. Then both $\varphi(t)$ and $\varphi(s)$ are defined as R-module homomorphisms on the semiregular ideal $(R :_R t)(R :_R s)$. Both the product $\varphi(t)\varphi(s)$ and the sum $\varphi(t) + \varphi(s)$ are defined on this ideal, and ts and $t + s$ multiply each element of $(R :_R t)(R :_R s)$ into R. Since neither $(R :_R t)$ nor $(R :_R s)$ has a nonzero annihilator in T, $\varphi(ts) = \varphi(t)\varphi(s)$ and $\varphi(t + s) = \varphi(t) + \varphi(s)$. Therefore φ is an R-algebra isomorphism from T to a Q_0-overring of R. □

Minimal integral extensions. We now turn to the case of a minimal integral extension $R \subseteq T$. These have been classified into three non-overlapping classes, identified in terms of the algebras $T/(R : T)$ (cf. [**5**, Corollary II.2]). These classes were later each characterized via generator-and-relations in [**7**, Proposition 2.12]. Our purpose here is to use embeddings into $Q(R)$ to shed new light.

By Theorem 2.3 or [**8**, Théorème 2.2], the conductor $(R : T)$ is a maximal ideal M of R. More generally we first consider a pair of rings $R \subseteq T$ sharing an ideal J. If J has a nonzero annihilator in T, it is obvious that no ideal of R which is contained in J can extend to a dense ideal in T. In Lemma 4.2, we establish a strong version of the converse; namely, if $\mathrm{Ann}_T(J) = (0)$ where $J = (R : T)$, then every ideal of R behaves well when extended to T, regardless of whether or not it is contained in J.

Lemma 4.2. *Let $R \subseteq T$ be rings with conductor $J = (R : T)$. If $\mathrm{Ann}_T(J) = (0)$, then every regular element of R is regular in T and for every regular (respectively, semiregular; respectively, dense) ideal I of R, the extension IT of I is a regular (respectvely, semiregular; respectively, dense) ideal of T.*

Proof. Suppose that some dense ideal I of R is such that IT is not dense in T, and pick $t \in T\backslash\{0\}$ such that $tI = (0)$. Then $tIJ = (0)$ with $IJ \subseteq R$. Since I is dense, we then have $tJ = 0$. As $\mathrm{Ann}_T(J) = (0)$, we have $t = 0$, the desired contradiction. Thus the extension of every dense ideal of R is a dense ideal of T.

The remaining two assertions will follow from the "dense" case. If I is semiregular in R, then I contains a dense finitely generated ideal H. Then IT contains HT, which is dense and finitely generated in T, and so IT is semiregular.

Finally, any regular element can be seen as a generator of a principal dense ideal, and so it follows that every regular element of R is regular in T. Thus if I is regular in R, then IT is regular in T. □

The key to the proof given for Theorem 4.1 was that $(R :_R t)$ had no nonzero annihilator in T. Using Lemma 4.2, we establish similar conclusions when we know that the conductor $(R : T)$ has no nonzero annihilators in R. The main difference is that in the context of Theorem 4.1, the conductor $(R : T)$ may have a nonzero annihilator in T, and perhaps in R as well.

Lemma 4.3. *Let $R \subseteq T$ be a pair of rings with conductor $J = (R : T)$ such that $\mathrm{Ann}_T(J) = (0)$. Then the following hold.*

(1) *J is dense in both R and T, and there is an R-algebra isomorphism from T to a Q-overring of R.*

(2) *J is semiregular in R if and only if J is semiregular in T. If this is the case, there is an R-algebra isomorphism from T to a Q_0-overring of R.*

(3) *J is regular in R if and only if J is regular in T. If this is the case, there is an R-algebra isomorphism from T into $Q_{cl}(R)$, that is, onto an overring of R.*

Proof. Since $tJ \subseteq R$ for each $t \in T$, multiplication by t defines an R-module homomorphism from J into R. Denote this homomorphism by $\varphi(t)$. As $tJ \subseteq J$, $\varphi(t+s) = \varphi(t) + \varphi(s)$, $\varphi(ts) = \varphi(t)\varphi(s)$ and $\varphi(rt) = r\varphi(t)$ for all $s, t \in T$ and $r \in R$–in this case all maps have "domain" J. Moreover, if $t \neq 0$, then $\varphi(t) \neq 0$ since $\text{Ann}_T(J) = (0)$. It follows that φ is an R-algebra isomorphism from T into $Q(R)$, that is, from T onto a Q-overring of R. As J is evidently dense in both R and T, (1) holds.

By Lemma 4.2, each dense (respectively, semiregular; respectively, regular) ideal of R extends to a dense (respectively, semiregular; respectively, regular) ideal of T. Since J is a common ideal of R and T, an element of J is either regular in both R and T or a zero divisor of both. Thus J is regular in R if and only if J is regular in T. Hence if J is regular, $\varphi(t) \in Q_{cl}(R)$ for each $t \in T$, and so (3) holds.

For any finite nonempty subset $A = \{a_1, \dots, a_n\} \subseteq J$, Lemma 4.2 shows that AR is semiregular in R if and only if AT is semiregular in T. Thus J is semiregular in R if and only if J is semiregular in T. Hence if J is semiregular, $\varphi(t) \in Q_0(R)$ for each $t \in T$. Thus (2) holds. □

For an integral minimal extension, the conductor $(R : T)$ is a maximal ideal of R. Thus, we next focus on rings $R \subsetneq T$ sharing an ideal M which is maximal in R. In this situation, M is critical for the extension $R \subsetneq T$. Indeed, more can be said: $(R :_R t) = M$ for each $t \in T \setminus R$, as was noted in the proof of Proposition 2.14 (4).

Theorem 4.4. *Let $R \subseteq T$ be rings sharing an ideal $M = (R : T)$ which is maximal in R. Then T is R-algebra isomorphic to a Q-overring of R if and only* $\text{Ann}_T(M) = (0)$. *Moreover, under these equivalent conditions,*

(1) *T is R-algebra isomorphic to a Q_0-overring of R if and only if M is semiregular*; *and*

(2) *T is R-algebra isomorphic to an overring of R if and only if M is regular.*

Note that if T is R-algebra isomorphic to either an overring of R or a Q_0-overring of R, then T is, *a fortiori*, isomorphic to a Q-overring of R and then $\mathrm{Ann}_T(M) = (0)$ (that is, M is dense in both R and T). When this is the case, M is regular (respectively, semiregular) in R if and only if it is so in T (by Lemma 4.3). Thus we need not be precise in stating in which ring M has the given property.

Proof. As noted above, if $t \in T \backslash R$, then $(R :_R t) = M$. It follows from Lemma 4.3 (1) and the above comments that T is R-algebra isomorphic to a Q-overring of R if and only if $\mathrm{Ann}_T(M) = (0)$. Certainly, if T is R-algebra isomorphic to either an overring of R or a Q_0-overring of R, then T is R-algebra isomorphic to a Q-overring of R. The equivalences in (1) and (2) now follow from Lemma 4.3. □

For a pair of rings $R \subseteq T$ sharing an ideal M that is maximal in R, four distinct cases may occur (the last one being divided into two subcases), as follows.

(1) $\mathrm{Ann}_T(M) = (0)$ and M is a regular ideal (in both R and T). Then T is R-algebra isomorphic to an overring of R.

(2) $\mathrm{Ann}_T(M) = (0)$ and M is semiregular but not regular (in both R and T). Then T is R-algebra isomorphic to a Q_0-overring of R but not to an overring of R.

(3) $\mathrm{Ann}_T(M) = (0)$ (thus M is dense in both R and T), but M is not semiregular. Then T is R-algebra isomorphic to a Q-overring of R but not to a Q_0-overring of R.

(4) $\mathrm{Ann}_T(M) \neq (0)$ (that is, M is not dense in T). Then T is not R-algebra isomorphic to any of the above kinds of overrings of R. This may happen regardless of whether M is dense in R or not dense in R.

In the final section, we give examples of integral minimal extensions illustrating each of the above four cases.

Conductors with nonzero annihilators. We now turn to a subcase of case (4), namely, the "worst" case where $M = (R : T)$ has a nonzero annihilator in R. Since M is maximal, either $M + \mathrm{Ann}_R(M) = R$ or $\mathrm{Ann}_R(M) \subseteq M$. If $\mathrm{Ann}_R(M) \subseteq M$ and $0 \neq x \in \mathrm{Ann}_R(M)$, then $x^2 = 0$, whence R is not reduced. On the other hand, if R is reduced, then one sees similarly that $M \bigcap \mathrm{Ann}_R(M) = (0)$, and so $R = M \oplus \mathrm{Ann}_R(M)$. More generally, we have the following.

Lemma 4.5. *The following are equivalent for a ring R which is not a field.*

(1) *There exist a maximal ideal M of R and an element $x \in R \backslash M$ such that $xM = (0)$.*

(2) *$R \cong S \times K$, an R-algebra direct product, for some ring S and some field K.*

(3) *There exists a maximal ideal M of R such that $R = M \oplus \mathrm{Ann}_R(M)$ as an additive group.*

Assume the above conditions hold. Then M is a minimal prime ideal of R; M is principal, in fact, generated by an idempotent; and R is reduced if and only if S is reduced.

Proof. Assume (1). Since R is not a field, $M \neq (0)$, and so $\mathrm{Ann}_R(M) \neq R$. Since M is maximal, there are nonzero elements $r \in R$ and $e \in M$ such that $rx + e = 1$. If $b \in M$, then $b = be$. Hence $M = Me = Re$. In particular, $e^2 = e$, and so e is a nontrivial idempotent (that is, $e \neq 0, e \neq 1$). We can thus write $R = Re \times R(1-e) = S \times K$, a ring direct product, where the ring $S := Me = M = Re$ and $K := R(1-e) \cong R/M \cong R_M/MR_M \cong R_M$ is a field. It is obvious that R is reduced if and only if S is reduced. Note also that $M \in \mathrm{Max}\,(R)$, M is a minimal prime ideal of R, and M is generated by an idempotent.

It is clear that (3) $\Rightarrow$ (1). It remains to show that (2) $\Rightarrow$ (3). Assume (2). Without loss of generality, $R = S \times K$ is an internal ring direct product (with K a field). Note that $S \neq 0$ since R is not a field. Then $M := S \times \{0\} \in \mathrm{Max}\,(R)$ and $\mathrm{Ann}_R(M) = \{0\} \times K$, and so (3) follows. □

Consider any ring direct product $R = S' \times S''$. Then $e', e'' = (1,0), (0,1)$ are orthogonal idempotents; in fact, $e'e'' = 0$ and $e' + e'' = 1$. If T is a ring containing R, then e' and e'' are also orthogonal idempotents in T, and $T = T' \times T''$ is itself a direct product, with $T' = S'e'$ an extension of S' and $T'' = S''e''$ an extension of S''. We can now give the following result.

Theorem 4.6. *Let $R \subsetneq T$ be a minimal integral extension such that the conductor $M = (R : T) \neq (0)$ has an annihilator $x \in R\backslash M$. Put $K := R/M$. Then T can be described, up to R-algebra isomorphism, in one of the following three ways*:

(i) $T \cong M \times L$ *where L is a minimal* (*necessarily algebraic*) *field extension of K*,

(ii) $T \cong M \times (K \times K) \cong R \times K$,

(iii) $T \cong M \times K[X]/(X^2) \cong M \times (K(+)K) \cong R(+)K$.

Proof. By Lemma 4.5, we can write $R = S \times K$, with $M := S \times \{0\}$ and K a field. Then the R-algebra T is a product $T = T' \times T''$, where T' is an extension of S and T'' is an extension of K. In terms of the above notation, with $e' = (1, 0)$ and $e'' = (0, 1)$, we have $T' = Te' = M = S$ and $T'' = Te'' \cong T/M$ is a minimal ring extension of $R/M \cong K$. As noted by Ferrand and Olivier [**8**, Lemme 1.2], T'' is then, up to K-algebra isomorphism, one of three types: (i) a minimal field extension L of K, (ii) the K-algebra direct product $K \times K$, or (iii) $K[X]/(X^2)$, which is isomorphic to the idealization $K(+)K$. To conclude, note, in case (iii), that the assignment $(a, (b, c)) \mapsto ((a, b), c)$ gives an R-algebra isomorphism $M \times (K(+)K) \to (M \times K)(+)K$. □

Note that the conclusion of Theorem 4.6 shares some of the features of the classification of minimal ring extensions in [**7**, Corollary 2.5]. However, the two results have non-overlapping hypotheses, since Lemma 4.5 shows that the hypotheses of Theorem 4.6 lead to an ideal M that is both a maximal and a minimal prime of R.

The above case analysis, in conjunction with the examples in Section 6 and the work in Section 3, gives an extensive picture of minimal ring extensions. Writing nearly 40 years ago, Ferrand and Olivier noted that the study of integral minimal extensions $R \subsetneq T$ for which $\mathrm{Ann}_R((R : T)) = 0$ "est moins trivial ... nous n'avons pas abordé ce problème." We would suggest that our work on the cases (1), (2) and (3) can be viewed as answering the question that was implicit in the preceding quotation from [**8**]. Note that Ferrand and Olivier pursued their work in [**8**] via tools such as crucial maximal ideals, a concept that figured significantly in much of the subsequent literature on minimal extensions. In contrast, the present work has benefited from the new concept of critical ideals. To close this section, it seems fitting to honor the pioneering work in [**8**] by noting that Theorem 4.6 (uses and) retains some of the flavor of the characterization by Ferrand and Olivier of the minimal ring extensions of a field [**8**, Lemme 1.2].

5. The ring $R(X)$. Even if $T = R[u]$ is a minimal extension of a ring R, the polynomial ring $T[X]$ is never a minimal extension of $R[X]$. As noted in [**4**], there is a very simple infinite (descending) chain between $R[X]$ and $T[X]$, namely,

$$\begin{aligned} T[X] &\supsetneq R + XT[X] \supsetneq R + XR + X^2T[X] \\ &\supsetneq R + XR + X^2R + X^3T[X] \supsetneq \cdots . \end{aligned}$$

However, localizing at the set $\mathcal{U}(R)$ of unit content polynomials of R "collapses" this chain. In fact, for the Nagata ring $R(X) = R[X]_{\mathcal{U}(R)}$, $R(X)[u]$ is both a minimal overring of $R(X)$ and equal to $R[u](X)$ as we shall see in Theorem 5.4. This will take care of the transfer of the "minimal extension" property to Nagata rings for the case of a closed minimal extension. (Transfer for the case of an integral minimal extension was handled in [**5**, Theorem II.10].)

For $R \subseteq Q_{cl}(R)$ a valuation ring, Hinkle and Huckaba [**15**] showed $R[X]$ is a valuation ring of $Q_{cl}(R)[X]$. The proof of statement (1) in Lemma 5.1 is an (very) abbreviated version of what they did.

Lemma 5.1. *Let $R \subsetneq T$ be rings such that (R, P) is a valuation pair of T, and let* v *be the corresponding valuation map.*

(1) *For an indeterminate X, $(R[X], P[X])$ is a valuation pair of $T[X]$ with corresponding valuation the extension of* v *using* min; *that is, for each polynomial $g(X) = \sum_{i=0}^{n} g_i X^i \in T[X]$, $\mathrm{v}(g(X)) = \min\{\mathrm{v}(g_i) \mid 0 \le i \le n\}$.*

(2) *$(R(X), P(X))$ is a valuation pair of $T[X]_{\mathcal{U}(R)}$, with the same value group, using the natural extension of* v.

Proof. For (1), the extension of v using min is a valuation map on $T[X]$ with the same value group, $R[X]$ is clearly the set of elements of valuation $\mathrm{v}(f) \ge 0$, and $P[X]$ is the set of elements f such that $\mathrm{v}(f) > 0$.

Assertion (2) is simply a combination of (1) together with the fact that if (R, P) is a valuation pair of T and S a multiplicatively closed subset of R that is disjoint from P, then (R_S, PR_S) is a valuation pair of T_S with the same value group (the valuation of a/s being such that $\mathrm{v}(a/s) = \mathrm{v}(a)$ [**17**, page 13]). □

The following pair of examples shows that $\mathcal{U}(T)$ is not necessarily the saturation of $\mathcal{U}(R)$ in $T[X]$ and hence, that the conclusion in Lemma 5.1 (2) need not extend to having $(R(X), P(X))$ a valuation pair of $T(X) = T[X]_{\mathcal{U}(T)}$.

Example 5.2. Let K be a field. Then

(1) Let $R := K[Y, Z]$, $P := ZR$ and $S := K[Y, Z, 1/Z]$. Then (R, P) is a rank 1 valuation pair of S with P not maximal in R, but $f := X + (Y/Z) \in \mathcal{U}(S)$ is not in the saturation of $\mathcal{U}(R)$ in $S[X]$ and $(R(X), P(X))$ is not a valuation pair of $S(X)$.

(2) Let $D := K[Z] + YK[Y, Z, 1/Z]$, $M := ZD$ and $T := K[Y, Z, 1/Y, 1/Z]$. Then (D, M) is a rank 2 valuation pair of T with M a maximal ideal of D, but $X + (1+Z)/Y \in \mathcal{U}(T)$ is not in the saturation of $\mathcal{U}(D)$ in $T[X]$ and $(D(X), M(X))$ is not a valuation pair of $T(X)$.

Proof. (1) As noted in Remark 3.8 (2), (R, P) is a rank 1 valuation pair of S with P a nonmaximal height 1 prime of R. Consider the polynomial $f := X + (Y/Z) \in S[X]$. If $g \in S[X]$ is such that $gf \in R[X]$, then $g = hZ$ for some polynomial $h \in R[X]$. But this puts the R-content of gf inside the maximal ideal (Y, Z). Hence $X + (Y/Z)$ is not in the saturation of $\mathcal{U}(R)$ in $S[X]$.

We thus have $1/f \in S(X) \backslash R(X)$. If $(R(X), P(X))$ were a valuation pair of $S(X)$, there would be an element $r/u \in P(X)$, with $r \in P[X]$ and $u \in \mathcal{U}(R)$, such that $r/uf \in R(X) \backslash P(X)$. Thus $r/uf = b/w$, with $b \in R[X]$ and $w \in \mathcal{U}(R)$; that is, $rw = buf$. Multiplying through by Z yields $rwZ = bu(ZX+Y)$, with $rwZ \in P[X]$, and $bu(ZX+Y) \in R[X]$. Since P is prime and both u and $ZX + Y$ are not in $P[X]$, we would have $b \in P[X]$, that is, $r/uf \in P(X)$, a contradiction.

(2) For $D = K[Z] + YK[Y, Z, 1/Z]$, note that $ZD = M$ is maximal (since $D/ZD \cong K$). Consider the ring W of the standard rank 2

valuation on $K(Y, Z)$ where $\mathrm{v}(Y^n Z^m) = (n, m)$ for all pairs of integers n and m using lexicographic order on $\mathbf{Z} \times \mathbf{Z}$; denote its maximal ideal by N. Restricting this valuation to $T = K[Y, Z, 1/Y, 1/Z]$, we then have $D = W \bigcap T$ (the set of elements in T with nonnegative valuation) and $M = N \bigcap T$ (the set of elements in T with positive valuation). Hence (D, M) is a rank 2 valuation pair of T.

Let $f := X + (1 + Z)/Y$. As above, if $g \in T[X]$ is such that $gf \in D[X]$, then $g = hY$ for some polynomial $h \in D[X]$. Hence $gf = h(YX + (1 + Z))$. The ideal $(Y, 1 + Z)$ is proper. Thus gf is not in $\mathcal{U}(D)$ and f is not in the saturation of $\mathcal{U}(D)$ in $T[X]$. It follows that $1/f \in T(X)\backslash D(X)$. If $(D(X), M(X))$ were a valuation pair of $T(X)$, there would be an element $s/u \in M(X)$, with $s \in M[X]$ and $u \in \mathcal{U}(D)$, such that $s/uf \in T(X)\backslash M(X)$. Thus $s/uf = d/w$, with $d \in D[X]$ and $w \in \mathcal{U}(D)$; that is, $sw = duf$. Multiplying through by Y yields $swY = du(YX + (Z + 1))$, with $swY \in M[X]$. Since M is prime and both u and $YX + (Z + 1)$ are not in $M[X]$ (since $Z + 1 \notin M$), it follows that $s/uf \in M(X)$. Hence $(D(X), M(X))$ is not a valuation pair of $T(X)$. □

One consequence of Theorem 5.4 below is that if (R, M) is a rank 1 valuation pair of T with M maximal in R, then $U(T)$ is the saturation of $\mathcal{U}(R)$ in $T[X]$, and hence $(R(X), M(X))$ is a (rank 1) valuation pair of $T(X)$. The first statement in the next lemma is trivial but sometimes very useful; for instance, it enables a simple proof of the second. Both statements are used in the proof of Theorem 5.4.

Lemma 5.3. *Let S and W be rings. Then*

(1) *If $f(X) \in \mathcal{U}(S)$, then there is a polynomial $g(X) \in S[X]$ such that at least one of the coefficients of the product $f(X)g(X)$ is 1.*

(2) *If $S \subseteq W$, then $S(X) \bigcap W = S$.*

Proof. (1) If $f(X) = f_n X^n + \cdots + f_0$ has unit content in S, then there are elements $g_0, \ldots, g_n \in S$ such that $\sum f_i g_{n-i} = 1$. The polynomial $g(X) = \sum g_j X^j$ is such that the coefficient of X^n in $f(X)g(X)$ is 1.

(2) If $b \in S(X) \bigcap W$, one can write $b = h(X)/f(X)$ with $h(X) \in S[X]$ and $f(X) \in \mathcal{U}(S)$. Then $bf(X)g(X) = h(X)g(X) \in S[X]$ and some coefficient of $f(X)g(X)$ is 1. Thus $b \in S$. □

Theorem 5.4. *Let $R \subsetneq T$ be rings with R integrally closed in T. Then T is a minimal extension of R if and only if $T(X)$ is a minimal extension of $R(X)$. Moreover, under these conditions, $R[u](X) = R(X)[u] = T(X)$ for each $u \in T\backslash R$.*

Proof. If T is a minimal extension of R, it follows from Theorem 3.4 that (R, M) is a rank 1 valuation pair of T with M a maximal ideal of R. Showing next that $\mathcal{U}(T)$ is the saturation of $\mathcal{U}(R)$ in $T[X]$, it follows that $T(X) = T[X]_{\mathcal{U}(R)}$, and hence, by Lemma 5.1, that $(R(X), M(X))$ is a rank 1 valuation pair of $T(X)$. Again from Theorem 3.4, it finally follows that $T(X)$ is a minimal extension of $R(X)$.

Let $f(X) \in \mathcal{U}(T)$. Then by Lemma 5.3 there is a polynomial $g(X) \in T[X]$ such that some coefficient of gf is 1. Write $gf = h_nX^n + \cdots h_0$ with some $h_i = 1$. If $g \in R[X]$, then $gf \in \mathcal{U}(R)$ and we are done. If not, there is a coefficient h_j with minimum value under the valuation v associated with the valuation pair (R, M). As $g \notin R[X]$, it must be that $\mathrm{v}(h_j) < 0$. By Theorem 3.4, $R + h_jR$ is invertible and there are elements $r, s \in (R :_R h_j)$ such that $rh_j + s = 1$. As $\mathrm{v}(h_j) < 0$, we have $\mathrm{v}(r) > 0$ and $\mathrm{v}(s) > 0$. From $0 = \mathrm{v}(1) \geq \min\{\mathrm{v}(rh_j), \mathrm{v}(s)\}$, we obtain $0 = \mathrm{v}(rh_j)$. It follows that $rh_k, sh_k \in R$ for each coefficient h_k of the product fg. Setting $b(X) := rX^i + sX^j$, we then have $bfg \in R[X]$ and, since $h_i = 1$, the coefficient of X^{i+j} in bfg is $rh_j + s = 1$. Thus $bfg \in \mathcal{U}(R)$.

Conversely, if $T(X)$ is a minimal extension of $R(X)$, then $T(X) = T[X]_{\mathcal{U}(R)}$. Also, if a ring S is intermediate between R and T, then $S[X]_{\mathcal{U}(R)}$ is intermediate between $R(X)$ and $T(X)$ and thus equal to one of them. It then follows from Lemma 5.3 that S is equal to either R or T.

Finally, if $u \in T\backslash R$ and $R \subsetneq T$ is minimal, then $T = R[u]$ and (from above) $R(X) \subsetneq T(X) = R[u](X)$ is a minimal extension. Since $R(X) \bigcap T = R$ while $R(X)[u] \bigcap T$ contains u, we have $R(X) \subsetneq R(X)[u]$, and hence $R(X)[u] = T(X)$. □

It is worth noting that if (R, M) is a valuation pair of T with M maximal and R quasilocal, then $\mathcal{U}(T)$ is again the saturation of $\mathcal{U}(R)$ and thus $(R(X), M(X))$ is a valuation pair of $T(X)$. Essentially the above proof applies without explicitly using invertibility. Assuming the above notation, argue as follows. Simply select a coefficient h_j in the product fg with smallest value under the valuation v. Next, choose $r \in R$ such that $\mathrm{v}(r) = -\mathrm{v}(h_j)$. Then $rfg \in \mathcal{U}(R)$ with rh_j (now) a unit of R since R is quasilocal with maximal ideal M.

6. Examples. In this section, we construct the examples promised above. For some examples, the construction is based on (reduced) rings of the form $A + B$.

Let D be an integral domain, and let $\mathcal{P}$ be a nonempty subset of Spec(D). Let $\mathcal{A}$ be an index set for $\mathcal{P}$, and let $I = \mathcal{A} \times \mathbf{N}$ where $\mathbf{N}$ is the set of natural numbers. For each $i = (\alpha, n)$ in I, let K_i be the quotient field of D/P_α. Next let $B := \sum K_i$ and form the ring $R := D + B$ from the direct sum of D and B by defining multiplication as $(r, b)(s, c) = (rs, rc + sb + bc)$. We refer to R as the $A + B$ *ring corresponding to* D and $\mathcal{P}$. Two good sources for information about this construction are [**16**, Section 26] (albeit in a slightly different form) and [**21**, Section 8].

The following result collects many of the basic properties of the above construction. Except for the statements about the density of B and the characterization of $Q(R)$ in case B is dense, all can be found in [**21**, Theorems 8.3 and 8.4]. For each $i \in I$, we let e_i denote the element of B whose ith component is 1 and all other components are 0. For elements $r \in D$ and $b \in B$, we let $(r)_i$ and $(b)_i$ denote the image of r in K_i and the ith component of b, respectively.

Theorem 6.1. *Let D be an integral domain, and let $\mathcal{P}$ be a nonempty set of prime ideals of D. Let R be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then*

(1) *A finitely generated ideal $J = ((r_1, a_1), (r_2, a_2), \dots, (r_m, a_m))$ of R is semiregular if and only if no $P \in \mathcal{P}$ contains the ideal $J' = (r_1, r_2, \dots, r_m)D$. In case J is semiregular, $J = J'R = J' + B$.*

(2) *An ideal J of R is regular if and only if there is a nonzero ideal J' of D such that $J = J'R = J' + B$ with some element of J' not contained in the union of the primes $\bigcup_{P \in \mathcal{P}} P$.*

(3) *$Q_{cl}(R)$ can be identified with the ring $D_S + B$ where $S = D \setminus \bigcup_{P \in \mathcal{P}} P$.*

(4) $Q_0(R)$ *can be identified with the ring* $E + B$ *where* $E = \bigcup\{(D : J) \mid JR$ *is a semiregular ideal of* $R\}$.

(5) *For a semiregular ideal* J, *there is an ideal* J' *of* D *such that* $J = J'R = J' + B$. *Moreover,* $(R :_{Q_0(R)} J) = (D : J') + B$ *and* J *is* Q_0*-invertible if and only if* J' *is invertible.*

(4) *For each* $i \in I$, *the set* $M_i = \{(r, b) \in R \mid (r)_i = -(b)_i\}$ *is a principal ideal of* R *generated by the idempotent* $(1, -e_i)$ *and is both a maximal and a minimal prime of* R.

(5) *If* P' *is a prime ideal of* R, *then either* $P' = M_i$ *for some* $i \in I$ *or* $P' = P + B$ *where* P *is a prime ideal of* D.

(6) B *is a minimal prime of* R; B *is dense if and only if* $\bigcap_{P \in \mathcal{P}} P = (0)$.

(7) *If* B *is dense, then the natural map* $\psi : D \to \prod K_i$ *is injective and* $Q(R)$ *can be identified with* $\prod K_i$.

Proof. It is clear that $R/B \cong D$. Thus B is prime.

No nonzero element of B can annihilate B. Thus, for B to have a nonzero annihilator, there must be a nonzero element $s \in D$ such that $sB = (0)$. Clearly such an s must be in $\bigcap_{P \in \mathcal{P}} P$, and each nonzero $t \in \bigcap_{P \in \mathcal{P}} P$ will annihilate B.

For each $b \in B$ and $q \in \prod K_i$, $qb \in B$. Thus if B is dense, $Q(R)$ contains $\prod K_i$. Also D embeds in $\prod K_i$, since B dense implies $\bigcap_{P \in \mathcal{P}} P = (0)$. Hence $Q(R)$ can be identified with $\prod K_i$ when B is dense. □

In the first example, we present a reduced ring R with a pair of closed minimal extensions. One of them is $Q_{cl}(R)$ itself and is of the form $R[u]$ with $1/u \in R$; the other is a ring $S \subsetneq Q_0(R)$ that cannot be embedded in $Q_{cl}(R)$, while each $t \in S \backslash R$ is a zero divisor in S.

Example 6.2. Let D be a Dedekind domain with a principal maximal ideal $P = pD$ and a maximal ideal M such that no power of M is principal. Let $\mathcal{P} = \{N_\alpha\} = \text{Max}\,(D)\backslash\{M, P\}$, and let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$. Then

(1) M is contained in the union $\bigcup N_\alpha$.

(2) The regular ideals of R are the ideals of the form $P^jR = P^j + B$ for some integer $j \geq 0$. The semiregular ideals of R are the ideals of the form $M^kP^jR = M^kP^j + B$ for $k, j \geq 0$. In particular, $MR = M + B$ is a Q_0-invertible semiregular maximal ideal that is not regular.

(3) The total quotient ring of R is $Q_{cl}(R) = \mathcal{T}(P) + B = R[1/p]$; this is a closed minimal ring extension of R. The ring of finite fractions of R is $Q_0(R) = \mathcal{T}(MP) + B$ and R is integrally closed in $Q_0(R)$.

(4) The ring $S = \mathcal{T}(M) + B \subsetneq Q_0(R)$ is a closed minimal ring extension of R that is not contained in $Q_{cl}(R)$. Moreover, each element $s \in S \backslash R$ is a zero divisor of S.

Proof. (1) As no power of M is principal, each element of M is contained in some other maximal ideal of D. Let $x \in M$, and let n be the exponent (possibly zero) of P in the unique decomposition of xD as a product of maximal ideals. Then xp^{-n} is an element of M that is not contained in P, and is thus contained in some N_α. A fortiori, x belongs to the same N_α.

(2) From (1), the ideals of the form P^j are the only ideals of D with an element not contained in the union $\bigcup_{N_\alpha \in \mathcal{P}} N_\alpha$. Thus it follows from Theorem 6.1 (2) that the regular ideals of R are the ideals of the form $P^jR = P^j + B$. From Theorem 6.1 (2), the semiregular ideals are of the form $J = J'R = J' + B$, where the decomposition of J' in D as a product of maximal ideals contains only powers of M and P. In particular, $MR = M + B$ is a semiregular maximal ideal that is not regular. From Theorem 6.1 (5), $MR = M + B$ is Q_0-invertible (as obviously, M is invertible in D).

(3) It follows from Theorem 6.1 (3) that $Q_{cl}(R) = \mathcal{T}(P) + B = R[1/p]$. It follows from Theorem 6.1 (4) that $Q_0(R) = \mathcal{T}(MP) + B$, as $E = \bigcup\{(D : J) \mid JR \text{ is a semiregular ideal of } R\}$ is then the ring $E = \bigcup(D : M^kP^j)$, that is, $E = \mathcal{T}(MP)$. Since D is a Dedekind domain, both $\mathcal{T}(P) = D[1/p]$ and $\mathcal{T}(M)$ are closed minimal extensions of D between D and $\mathcal{T}(MP)$. Finally, R is integrally closed in $Q_0(R)$ since D is integrally closed.

(4) It remains to show that each element $s \in S \backslash R$ is a zero divisor of S. Since $Q_0(R)$ is its own total quotient ring, each regular element of S

is a unit in $Q_0(R)$. Thus it suffices to show that if $x = (t, b) \in Q_0(R)\backslash R$ is a unit of $Q_0(R)$, then t is not in $\mathcal{T}(M)$ and $1/t \in P$. Since $Q_0(R) = \mathcal{T}(MP) + B$ and x is a unit, t must be a unit of $\mathcal{T}(MP)$. Thus neither $(D :_D t)$ nor $(D :_D 1/t)$ is contained in a maximal ideal from the set $\mathcal{P}$, and therefore the ideals $(D :_D t)$ and $(D :_D 1/t)$ factor in (nonnegative) powers of M and P. Write $(D :_D t) = M^m P^j$ and $(D :_D 1/t) = M^n P^k$. Since $t(D :_D t) = (D :_D 1/t)$ (and M and P are invertible), we have $tP^{j-k} = M^{n-m}$. The left-hand side is a principal fractional ideal of D, but the only power of M that is principal is $M^0 = D$. It follows that $tD = p^{k-j}D$ with $k - j < 0$ (since $t \notin D$). Therefore $t \notin \mathcal{T}(M)$ and $1/t \in P$. □

In view of [**7**, Theorem 3.7 (a)], it follows from Example 6.2 (4) that if R is the ring in Example 6.2, then $Q_{cl}(R)$ is not a von Neumann regular ring. The reader is encouraged to verify this fact directly.

In Example 6.4, we exhibit a closed minimal extension $R \subsetneq S$ such that S is embedded in $Q_{cl}(R)$ but each $t \in S\backslash R$ is a zero divisor in S. First, we record a fairly trivial application of our original pullback diagram.

Lemma 6.3. *Let $R \subsetneq T$ be a minimal extension. Then $S = R + XT[X] \subsetneq T[X]$ is a minimal extension. Moreover, $S \subsetneq T[X]$ is closed minimal if and only if $R \subsetneq T$ is closed minimal.*

Proof. The rings S and $T[X]$ share the ideal $XT[X]$, and $S/XT[X] \cong R \subsetneq T \cong T[X]/XT[X]$ is a minimal extension. Hence $S \subsetneq T[X]$ is a minimal extension. The final assertion follows by considering the integral closure of a pullback. □

Example 6.4. Let $E = D + X\mathcal{T}(M)[X]$ where D is a Dedekind domain with a maximal ideal M such that no power of M is principal. Let $\mathcal{P}$ be the set of height 1 prime ideals of $\mathcal{T}(M)[X]$ excluding $X\mathcal{T}(M)[X]$, and set $B := \sum_{P \in \mathcal{P}} \mathcal{T}(M)[X]/P$. Thus B is a $\mathcal{T}(M)[X]$-module and hence also an E-module. We let $R := E(+)B$ and $S := \mathcal{T}(M)[X](+)B$ be the respective idealizations of B over E and $\mathcal{T}(M)[X]$. Then

(1) $R \subsetneq S$ is a closed minimal extension.

(2) $S = \mathcal{T}(M)[X] + B$ is contained in $Q_{cl}(R)$.

(3) Each element of $S\backslash R$ is a zero divisor.

Proof. (1) Since B is a common ideal of R and S, and $E \subsetneq \mathcal{T}(M[X])$ is a minimal closed extension by Lemma 6.3, $R \subsetneq S$ is a minimal closed extension.

(2) By construction, $(X, 0)$ is a regular element of S and, *a fortiori*, a regular element of R. Moreover $S[1/X] = R[1/X]$, and so S is contained in $Q_{cl}(R)$.

(3) Let (g, b) in $S \backslash R$. Then g is a polynomial $g(X) = \sum g_i X^i$ with each coefficient in $\mathcal{T}(M)$ and its constant term g_0 in $\mathcal{T}(M) \backslash D$; in particular, $g_0 \neq 0$. If g is non-constant, g is contained in a (height 1) prime ideal Q of $\mathcal{T}(M)[X]$ above (0) and distinct from $X\mathcal{T}(M)[X]$. Then $Q \in \mathcal{P}$ and (g, b) is a zero divisor in S (annihilated by $(0, c)$ where $c \in B$ is such that its component in $\mathcal{T}(M)[X]/Q$ is 1 and the other components are 0). If $g = g_0$ is a constant, then as in the proof of Example 6.2, g_0 cannot be a unit of $\mathcal{T}(M)$. In this case, g is in a height 1 prime of the form $N\mathcal{T}(M)[X]$ for some maximal ideal N of $\mathcal{T}(M)$ and we again have that (g, b) is a zero divisor of S. □

Our final examples are of minimal integral extensions $R \subsetneq T$. The conductor is a maximal ideal M and, as we have seen, there are four cases, according to the properties of M (in both R and T for the first three cases): 1) M is regular, 2) M is semiregular but not regular, 3) M is dense but not semiregular, 4) $\mathrm{Ann}_T(M) \neq (0)$ (that is, M is not dense in T, regardless of whether M is dense or not dense in R).

It is trivial to give examples of the extreme cases, that is, with M regular or, at the opposite extreme, M not dense in T.

• For M regular, we can consider a pair of quasilocal domains sharing their maximal ideal M. For instance, take T to be a valuation domain with residue field $\mathbf{C}$ and R to be the pullback formed by the elements in T with residue class modulo M in $\mathbf{R}$. Then R is a pseudo-valuation domain, and the extension $R \subsetneq T$ is a minimal integral extension since $\mathbf{R} \subsetneq \mathbf{C}$ is a minimal field extension. Clearly, T is contained in the quotient field $K = Q_{cl}(R)$ of R.

• For M not dense in T, we can simply start with any ring R, take any maximal ideal M of R and consider the idealization $T = R(+)R/M$ of R/M over R. As recalled in the Introduction, T is a minimal integral extension of R, and clearly $\mathrm{Ann}_T(M) \neq (0)$. Thus, T is not embedded in any kind of quotient ring of R. This may happen with M dense in

R (or even regular if, for instance, R is a domain) or not dense in R (if, for instance, M is nilpotent, as in $R = K[X]/(X^2)$).

We use the $A + B$ construction to provide an example where M is semiregular but not regular (in both R and T), and hence, by Theorem 4.4, T is embedded in $Q_0(R)$ but not in $Q_{cl}(R)$.

Example 6.5. Let $E := K[Y, Z]$ be the polynomial ring in two indeterminates over a field K, N the ideal $N := (Y^2, Y^3, YZ, Z)$, and $\mathcal{P}$ the set of height 1 primes of E that are contained in (Y, Z). Let $T := E + B$ be the $A + B$ ring corresponding to E and $\mathcal{P}$, $D := K[Y^2, Y^3, YZ, Z]$ and $R := D + B$ (the subring of T generated by D and B). Then T is a minimal integral extension of R. The conductor $(R : T)$ is $M = N + B = NT$, and it is semiregular but not regular (in both R and T).

Proof. The rings R and T share the ideal B, with $T/B \cong E$ and $R/B \cong D$. The rings D and E share the ideal N, with $E/N \cong K[Y]/(Y^2)$ and $D/N \cong K$. Since $K[Y]/(Y^2)$ is a minimal integral extension of K, it follows that T is a minimal integral extension of R. Clearly the conductor $(R : T)$ is the ideal $M = N + B$, since it is shared by R and T and maximal in R. Since N is a finitely generated ideal of E contained in no prime ideal of $\mathcal{P}$, it follows from the basic properties of the $A + B$ construction [Theorem 6.1] that $M = N + B$ is semiregular in T and that $M = NT$. Yet, every element of M is of the form (n, b) with $n \in N$ in some height 1 prime $P \in \mathcal{P}$, and so every element of M is a zero divisor in T; that is, M is not regular. □

Finally, we provide an example with M dense in T but not semiregular (in both R and T), and hence, T is embedded in $Q(R)$ but not in $Q_0(R)$. Such an example could have been realized in a rather trivial way as an $A + B$ ring by using a field for the base domain and the zero ideal for the set $\mathcal{P}$.

Example 6.6. Let $Q := \prod \mathbf{Z}_2$ be a countably infinite product of the integers mod 2. Let M be the ideal $M := \sum \mathbf{Z}_2$ and R the subring generated by 1_Q and M. Then

(1) The ideal M is maximal in R and dense in both R and $Q \cong Q(R)$.

(2) The only semiregular ideal of R is R. Thus $R = Q_{cl}(R) = Q_0(R)$.

(3) For each $t \in Q \backslash R$, $R[t]$ is a minimal integral extension of R.

Proof. By using Theorem 6.1 (9), we see that statements (1) and (2) follow by viewing R as the $A+B$ ring corresponding to $\mathbf{Z}_2$ and $\mathcal{P}=\{(0)\}$. From this point of view, M is the dense (maximal) ideal "B." Since each element of Q is idempotent, each such element is integral over R.

Let $t \in Q\backslash R$ and $s \in R[t]\backslash R$. Since $t^2=t$ and $tM \subseteq M$, we have $s=r+qt$ for some $r, q \in R$ with $q \notin M$. But this means $q=1_Q+b$ for some $b \in M$. Hence $s=(r+bt)+t$ with $r+bt \in R$. It follows that $R[t]=R[s]$, and therefore $R \subsetneq R[t]$ is a minimal integral extension. □

Acknowledgments. The first and second authors both thank the University of North Carolina Charlotte for its hospitality and partial support.

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