ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

SUMS AND STRICT SUMS OF BIQUADRATES IN $\mathbf{F}_q[t]$, $q \in \{3, 9\}$

LUIS H. GALLARDO AND LEONID N. VASERSTEIN

ABSTRACT. Let q be a power of a prime number. Observe that just for $q \in \{3, 9\}$ some congruence obstructions occur to the representation of polynomials in $\mathbf{F}_q[t]$ as a sum (and so also as a strict sum) of biquadrates. We define $g(4, \mathbf{F}_q[t])$ as the least g such that every polynomial that is a strict sum of biquadrates is a strict sum of g biquadrates. We compare the set of sums of biquadrates with the set of strict sums of biquadrates for $q \in \{3, 9\}$. Our main result is that

$$g(4, \mathbf{F}_q[t]) \leq 14 \text{ when } q \in \{3, 9\}.$$

The set of sums of cubes in $\mathbf{F}_4[t]$ is also determined. This completes the study of the case of representation by sums of cubes (in which the congruence obstructions occur only for $q \in \{2, 4\}$).

1. Introduction. Let $\mathbf{F}_q$ be a finite field of characteristic p, with q elements. Let $k > 1$ be a positive integer. Let

$$A_k(q) = \{P \in \mathbf{F}_q[t] \mid P = A^k + \cdots, \ A \in \mathbf{F}_q[t]\}$$

be the set of all sums of kth powers in $\mathbf{F}_q[t]$. Let also:

$$SA_k(q) = \{P \in \mathbf{F}_q[t] \mid P = A^k + \cdots, A \in \mathbf{F}_q[t], \\ \deg(A^k) < k + \deg(P)\}$$

be the set of all strict sums of kth powers in $\mathbf{F}_q[t]$. Notice that one can never write P as a sum of kth powers with $\deg(A) < \lceil \deg(P)/k \rceil$, so that the condition for a strict sum of kth powers imposes the tightest possible constraint on the size of $\deg(A)$.

2010 AMS *Mathematics subject classification.* Primary 11T55 (11C08, 11T06, 11E76, 11P05).
Keywords and phrases. Waring's problem, polynomials, forms, biquadrates, cubes, Paley formulae, characteristic two, characteristic three, finite fields.
Received by the editors on January 16, 2008, and in revised form on May 23, 2008.
DOI:10.1216/RMJ-2010-40-6-1863

When k is even we consider also the corresponding sets

$$MA_k(q), \qquad MSA_k(q),$$

of mixed sum of kth powers in which we accept $+$ and $-$ signs in the sums above.

We deal in this paper mainly with the case $k = 4$ and $k = 3$ when these sets are not equal to the entire ring $\mathbf{F}_q[t]$. It is known that congruence obstructions occur exactly when $q \in \{2, 4\}$ for $k = 3$, and when $q \in \{3, 9\}$ for $k = 4$ (see [**2**, pages 2–3], [**9**, page 302]).

It is easy to identify the sets $C(q) = A_3(q)$ of polynomials $P \in \mathbf{F}_q[t]$ that are sums of cubes (see Theorem 1), respectively $SC(q) = SA_3(q)$ of polynomials $P \in \mathbf{F}_q[t]$ that are strict sums of cubes (see [**4, 6**]). When $q > 4$ the set $C(q)$ is the entire ring $\mathbf{F}_q[t]$.

For $q = 4$ the set $C(q)$ consists of polynomials $P \in \mathbf{F}_4[t]$ for which $P(r)$ lies in $\mathbf{F}_2$ for every $r \in \mathbf{F}_4$, while the set $SC(q)$ consists of polynomials $P \in C(q)$ such that P is monic when 3 divides $\deg(P)$. Finally $C(2) = A_3(2) = SA_3(2) = SC(2)$ is the set of $P \in \mathbf{F}_2[t]$ such that $P \equiv 0$ or $P \equiv 1 \pmod{t^2 + t + 1}$.

Let $v(k, \mathbf{F}_q[t]) = v \geq 0$ be the minimal integer such that every $P \in MA_k(q)$ is a mixed sum of v kth powers. We define as well: Let $w(k, \mathbf{F}_q[t]) = v \geq 0$ be the minimal integer such that every $P \in A_k(q)$ is a sum of v kth powers.

Let $g(k, \mathbf{F}_q[t]) = v \geq 0$ (in analogy to the definition of the "$g(k)$" for Waring's problem over the positive numbers) be the minimal integer such that every $P \in SA_k(q)$ is a sum of v kth powers.

Let $gm(k, \mathbf{F}_q[t]) = v \geq 0$ be the minimal integer such that every $P \in MSA_k(q)$ is a mixed sum of v kth powers.

Some results are known (mainly upper bounds) about these numbers (and only for $k = 3$) in the case we deal with here, i.e., when $q \in \{2, 4\}$ for $k = 3$ and $q \in \{3, 9\}$ for $k = 4$:

In 1933, (see [**8, 10**]) Paley proved that

$$v(3, \mathbf{F}_q[t]) \leq 5$$

for $q \in \{2, 4\}$. Later, in [**10, 11**], Vaserstein improved the result for $q = 4$, respectively for $q = 2$, to:

$$v(3, \mathbf{F}_q[t]) \leq 4.$$

Recently, (see [**6**]) Gallardo and Heath-Brown (for $q = 2$) proved that

$$5 \leq g(3, \mathbf{F}_q[t]) \leq 6,$$

and Gallardo (for $q = 4$) (see [**5**]) proved that

$$g(3, \mathbf{F}_q[t]) \leq 6.$$

The actual values of $v(3, \mathbf{F}_q[t])$, $g(3, \mathbf{F}_q[t])$, for $q \in \{2, 4\}$ are unknown.

It is interesting to observe that $\mathbf{F}_4$ is the only finite field $\mathbf{F}_q$ for which the set of sums of cubes in $\mathbf{F}_q[t]$ contains strictly the set of strict sums of cubes. In [**10**] Vaserstein characterizes (as a special case) for any commutative ring A the set A_3 of sums of cubes in A in terms of ring homomorphisms from A to the finite field $\mathbf{F}_4$. We show here that specializing these results to the ring $A = \mathbf{F}_4[t]$ we get the set $C(4)$ described above. See Theorem 1 for details. This set can also be described (but we do not do it here) in terms of remainders of the division by the polynomial $t^4 - t$ by using Paley's formulae; (these formulae are in, e.g., [**10**]).

In this paper we study the analogue problem for biquadrates instead of cubes.

More precisely, the main object of this paper is to prove, (see the last section of the paper), the following upper bounds:

$$v(4, \mathbf{F}_q[t]) \leq 6, \text{ for } q \in \{3, 9\}, \tag{1}$$

$$w(4, \mathbf{F}_q[t]) \leq 8, \text{ for } q \in \{3, 9\}, \tag{2}$$

$$gm(4, \mathbf{F}_q[t]) \leq 10, \text{ for } q \in \{3, 9\}, \tag{3}$$

and

$$g(4, \mathbf{F}_q[t]) \leq 14, \text{ for } q \in \{3, 9\}. \tag{4}$$

Remark 1. Concerning the latter bound: Observe that every element of $\mathbf{F}_3$ and of $\mathbf{F}_9$ is a sum of two biquadrates. By using (see [**7**, Corollary

1.3]) we obtain only the weaker result $g(4, \mathbf{F}_q[t]) \leq 2(4^3 - 2\cdot 4^2 - 4 + 1) = 58$, for $q \in \{3, 9\}$.

Some computations are used to get our results. We shall describe the (simple) algorithms used in the proofs.

The classical method of "descent" used in previous work (see, e.g., [**1, 3, 4**]) does not work well here (but it is still necessary as a first step of our new method). We get better results by adapting the method used for cubes in [**5, 6**] to biquadrates.

The resulting method (see next section) comes essentially from a refinement of the (trivial) observation that every element of $\mathbf{F}_q$ is a cube when q is a multiple of 3.

We denote by i a root of the (prime) polynomial $t^2 + 1 \in \mathbf{F}_3[t]$ in a fixed algebraic closure of $\mathbf{F}_3$ so that $\mathbf{F}_9 = \mathbf{F}_3[i]$.

All rings are assumed commutative and with 1.

2. Identities and descent. The following lemmas are the keys in order to obtain our main results. First of all we introduce some notation:

Lemma 1. *Let T be a ring of characteristic 3. Let $R = T[t]$ be the polynomial ring in one indeterminate t over T. Let $L : R \to R$ be defined by $L(y) = y^3 + y$. Let $C : R \times R \to R$ be defined by $C(r, s) = rs(r^2 + s^2)$. Let also $L_1 : R \to R$ be defined by $L_1(y) = C(y, t) = y^3 t + y t^3$, and let $L_2 : R \to R$ be defined by $L_2(y) = C(y, t^2) = y^3 t^2 + y t^6$. Then L, L_1 and L_2 are $\mathbf{F}_3$-linear functions.*

Secondly, by using the same notations as in Lemma 1 we present three simple identities that hold when every element of the ground ring T is a perfect cube (i.e., when T is perfect).

Lemma 2. *Let T be a perfect ring of characteristic 3. Let $R = T[t]$ be the polynomial ring in one indeterminate t over T. Let $a \in T$ be written as $a = s^3$ with $s \in T$, and let $n \geq 0$ be a non-negative integer.*

One has

$$at^{3n} = (at^{3n} + st^n) - st^n = L(st^n) - st^n, \tag{5}$$

$$at^{3n+1} = (st^n)^3 t + (st^n)t^3 - st^{n+3} = L_1(st^n) - st^{n+3}, \tag{6}$$

and

$$at^{3n+2} = (st^n)^3 t^2 + (st^n)t^6 - st^{n+6} = L_2(st^n) - st^{n+6}. \tag{7}$$

Let us recall the following identities, the second one being a generalization of a formula of Paley (see [**8**]) for cubes:

Lemma 3. *Let T be a ring of characteristic 3. Let $R = T[t]$ be the polynomial ring in one indeterminate t over B. Let $x, y \in R$. Then*

(i) $xy(x^2 + y^2) = (x - y)^4 - (x + y)^4$.

(ii) $y(x^9 - x) = (x^3 - y)^4 + (xy + 1)^4 - (x^3 + y)^4 - (xy - 1)^4$.

Our first (key) result is:

Lemma 4. *Let $n > 0$ be a positive integer. Let $q = 3^n$, and let $P \in \mathbf{F}_q[t]$ be a polynomial. Then there exist $R \in \mathbf{F}_q[t]$ with* $\deg(R) < 9$ *and $A, Q_1, Q_2 \in \mathbf{F}_q[t]$ such that*

$$P = A^3 + A + Q_1 t(Q_1^2 + t^2) + Q_2 t^2(Q_2^2 + t^4) + R, \tag{8}$$

where the following condition on degrees holds:

$$\max\{\deg(A^3), \deg(Q_1^3 t), \deg(Q_2^3 t^2)\} \le \deg(P).$$

Proof. When $\deg(P) \le 9$ we choose $A = Q_1 = Q_2 = 0$. While, when $\deg(P) > 9$ this follows immediately, by induction, from the reduction formulae of Lemma 2, used to remove the leading term of P, together with the addition properties proved in Lemma 1. More precisely: We can collect all terms containing the function L together. Moreover, by doing the same for all terms where the functions L_1, L_2 appear we obtain the result.

We need the definition:

Definition 1. Let $q \in \{3, 9\}$. We set:

$$B_4(q) = \{P(t) \in \mathbf{F}_q[t] \mid P(r) \in \mathbf{F}_3 \text{ for all } r \in \mathbf{F}_9\}$$

and for specific degrees:

$$\begin{aligned} B_4(q,8) &= \{P(t) \in B_4(q) \mid \deg(P(t)) \leq 8\}. \\ A_4(q,8) &= \{P(t) \in A_4(q) \mid \deg(P(t)) \leq 8\}. \\ SA_4(q,8) &= \{P(t) \in SA_4(q) \mid \deg(P(t)) \leq 8\}. \end{aligned}$$

Now, a lemma follows concerning memberships of some polynomials of small degree:

Lemma 5. *One has*:

(i) $B_4(3,8) = SA_4(3,8) = A_4(3,8) = S_3$. *Moreover this set* S_3 *has exactly* 729 *elements. Furthermore, every element of* S_3 *is a strict sum of two biquadrates.*

(ii) $B_4(9,8) = SA_4(9,8) = A_4(9,8) = S_9$. *Moreover, every element of* S_9 *is a strict sum of two biquadrates.*

Proof. A short computation gives immediately the first assertion of (i). The second assertion follows by applying Lemma 2 to the 729 elements of S_3. More precisely the following computer program was run in order to write a polynomial $P \in S_3$ as a (strict) sum of (at most) two biquadrates:

```
# reduce all polys of degree <= 8 that are sums of 4
# biquadrates in F3[x]
# under the forms r**3+r, y**2*x+y*x**3, z**3*x**2+z*x**6
# observe that now we can take s = a to satisfy a = s**3
# while for F9, we took: s = a**3. Accordingly, we have new rules
# to get reduced a poly of degree at most 8 in F3[x]
# that is a sum of biquadrates:

reduce3 := proc(Po)
local P,Q,lis,a0,a1,a2,a3,a4,a5,a6,a7,a8,
b0,b1,b2,b3,b4,b5,b6,b7,b8;
P := Po;
a0 := coeff(P,x,0);
a1 := coeff(P,x,1);
a2 := coeff(P,x,2);
a3 := coeff(P,x,3);
a4 := coeff(P,x,4);
a5 := coeff(P,x,5);
a6 := coeff(P,x,6);
a7 := coeff(P,x,7);
a8 := coeff(P,x,8);
if a8=0 then b8 := 0; fi;
if member(a8,{1,-1}) then b8 := 1; fi;
if not(member(a8,{1,-1,0})) then b8 := a8; fi;
b7 := 0;
b6 := 0;
b5 := mods(a5 - a7**3,3);
if a4 = 0 then b4 := 0; fi;
if member(a4,{1,-1}) then b4 := 1; fi;
if not(member(a4,{1,-1,0})) then b4 := a4; fi;
b3 := 0;
b2 := mods(a2 -a6,3);
b1 := mods(a1 - a3,3);
if member(a0,{0,1,-1}) then b0 := 0; fi;
if not(member(a0,{0,1,-1})) then b0 := a0; fi;
Q := b0+b1*x+b2*x**2+b3*x**3+b4*x**4+b5*x**5+b6*x**6+b7*x**7+b8*x**8;
end;
```

We prove assertion (ii) similarly. No attempt was done here to precompute the (huge) list of all elements of S_9. The computer (an eighth processor machine) took some time (22408 seconds) to do the reduction.

The following descent result is key.

Lemma 6. *Let $n > 1$ be an integer. Let q be a power of 3. Let $P \in \mathbf{F}_q[t]$ be a monic polynomial, of degree $d = 4n$. Then there exist polynomials $A, R \in \mathbf{F}_q[t]$ such that:*

(a) $P = A^4 + R$,

(b) $\deg(A) = n$,

(c) $\deg(R) \leq 3n$.

Proof. Set $A = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ with unknown coefficients $a_j \in \mathbf{F}_q$. Now fix any $a_0 \in \mathbf{F}_q$ and choose $a_{n-1}, \ldots, a_1 \in \mathbf{F}_q$ in such a manner that $R = P - A^4$ has degree at most equal to $3n$. This results in a soluble system of $n-1$ equations in $n-1$ unknowns. More precisely, observe that a_{n-1} must equal the coefficient of t^{d-1} in P (we are working modulo 3). The coefficient of t^{d-2} in A^4 is a sum of a_{n-2} and a monomial involving a_{n-1}. So, we can compute a_{n-2}. In general, for i up to $n-2$, the coefficient of $t^{d-(i+1)}$ in A^4 is a sum of $a_{n-(i+1)}$ and monomials involving $a_{n-1}, \ldots, a_{n-i}$ so that knowing $a_{n-1}, \ldots, a_{n-i}$ we can compute $a_{n-(i+1)}$.

This proves a), b) and c).

We are ready to establish the content of our main sets:

Lemma 7. *One has*:

(i) $A_4(3) = MA_4(3) = B_4(3)$.

(ii) $SA_4(3) = MSA_4(3) = B_4(3)$.

(iii) $A_4(9) = MA_4(9) = B_4(9)$.

(iv) $SA_4(9) = MSA_4(9) =$

$$\{P \in B_4(9) \mid \text{ the leading coefficient of } P \text{ is in } \mathbf{F}_3\}.$$

Proof. Observe that $-1 = 1^4 + 1^4$ in $\mathbf{F}_3$ so that the set of mixed sums of biquadrates equals the set of sums of biquadrates. It is also clear that the A's sets are contained in the B's sets. Let $P \in B_4(3)$. Write P in the form

$$P = y(t^9 - t) + z$$

for some polynomials $y, z \in \mathbf{F}_3[t]$, with either $z = 0$ or $\deg(z) < 9$. By Lemmata 3 and 5 we obtain that $P \in A_4(3)$. So we have proved (i). The other proofs are similar. Just observe that when managing strict sums we should use Lemma 4, once, instead of Lemma 3. More precisely, take $q \in \{3, 9\}$, and take $P \in B_4(q)$ with $\deg(P) \in \{4n, 4n-1, 4n-2, 4n-3\}$. Assume that $P = ct^{4n} + \cdots$ for some $c \in \mathbf{F}_q$. We distinguish two cases:

Case 1: $c = 0$. We write $P = (-1)t^{4n} + (t^{4n} + P)$. Since -1 is a sum of two biquadrates in $\mathbf{F}_q$ and since $P_2 = t^{4n} + P$ is monic of degree $4n$, we get from Lemma 6 that $P_2 = A^4 + R_2$ with $\deg(R_2) \leq 3n$. Now apply Lemma 4 to represent R_2 as a strict sum of biquadrates plus a new remainder R_3 with degree at most equal to 8. Finally, by Lemma 5 we conclude that P is indeed a strict sum of biquadrates.

Case 2: $c \neq 0$. Here we write $P = (c-1)t^{4n} + t^{4n} + S$ with S being the sum all monomials of degree less than $4n$ that appear in P. By the preceding argument we see that $t^{4n} + S$ is a strict sum of biquadrates. So, $t^{4n} + S \in B_4(q)$. Therefore, the monomial $M(t) = (c-1)t^{4n}$ belongs also to $B_4(q)$. Substituting 1 for t into $M(t)$ we get immediately that $c - 1 \in \mathbf{F}_3$. Thus, $M(t) \in \{t^{4n}, t^{4n} + t^{4n}\}$. So, P is a strict sum of biquadrates, thereby finishing the proof of the lemma.

3. The case of cubes. The object of this section is to prove the theorem:

Theorem 1. *Let* $S = \{P \in \mathbf{F}_4[t] \mid P(r) \in \mathbf{F}_2$ *for all* $r \in \mathbf{F}_4\}$. *Then*

$$C(4) = A_3(4) = S.$$

Proof. Clearly, $C(4)$ is a subset of S. Take now $P \in S$. We claim that $P \in C(4)$. In order to prove the claim we apply Vaserstein result [**9**, Theorem 1] in the special case when $k = 3$. This reduces the problem to prove that the following subsets of $\mathbf{F}_4[t]$ are equal.

$$A = \{P \in \mathbf{F}_4[t] \mid h(P) \in \mathbf{F}_2 \text{ for all } h \in H\}$$

and

$$B = \{P \in \mathbf{F}_4[t] \mid h(P) \in \mathbf{F}_2 \text{ for all } h \in K\}$$

where $H = \text{Hom}(\mathbf{F}_4[t], \mathbf{F}_4) = \{h : \mathbf{F}_4[t] \to \mathbf{F}_4 \mid h \text{ is a ring homomorphism}\}$ and $K = \{ev_0, ev_1, ev_\alpha, ev_{\alpha^2}\}$ in which $ev_\gamma \in H$ satisfy $ev_\gamma(P) = P(\gamma)$ for given $\gamma \in \mathbf{F}_4$. Here we write $\mathbf{F}_4 = \mathbf{F}_2[\alpha]$ in which $\alpha^2 = \alpha+1$, where α lives on some fixed algebraic closure of $\mathbf{F}_2$.

Observe that the multiplicative set J generated by K equals the subset H_1 of H formed by the ring homomorphisms that are $\mathbf{F}_4-$linear, i.e.,

$$J = \{h \in H \mid h = ev_0^a ev_1^b ev_\alpha^c ev_{\alpha^2}^d \text{ with } a,b,c,d \text{ integers } \geq 0\}$$

equals

$$H_1 = \{h \in H \mid h \text{ is } \mathbf{F}_4 - \text{linear}\} :$$

A simple computer program run some milliseconds to establish the assertion. Moreover, we obtain that $J \subsetneq H$ and $\text{card}(J) = 12 < \text{card}(H) = 16$. More precisely the following four homomorphisms are the elements of $H \backslash K = \{h \in H \mid h \notin K\}$: $h_1 : 0 \to 0, 1 \to 1$, $\alpha \to 0$, $\alpha^2 \to 0$, $t \to 0$; $h_2 : 0 \to 0$, $1 \to 1$, $\alpha \to 0$, $\alpha^2 \to 0$, $t \to 1$; $h_3 : 0 \to 0$, $1 \to 1$, $\alpha \to 0$, $\alpha^2 \to 0$, $t \to \alpha$; and $h_4 : 0 \to 0$, $1 \to 1$, $\alpha \to 0$, $\alpha^2 \to 0$, $t \to \alpha^2$.

It is now clear (since trivially $j(P) \in \mathbf{F}_2$ for $j \in J$) that it suffices to prove that $h_i(P) \in \mathbf{F}_2$ for all i. This is clear for h_1 and for h_2. From the definition of h_3 and h_4 it follows that we can assume that $P \in \mathbf{F}_2[t]$. But then $h_3(P) = ev_\alpha(P) \in \mathbf{F}_2$. Analogously we get that $h_4(P) = ev_{\alpha^2}(P) \in \mathbf{F}_2$. This proves the claim thereby finishing the proof of the theorem.

4. The case of biquadrates: Main results.

Our first result is

Theorem 2. *Let $q \in \{3, 9\}$. Then*

$$v(4, \mathbf{F}_q[t]) \leq 6.$$

Proof. Let $P \in \mathbf{F}_q[t]$ to decompose as a mixed sum of biquadrates. Write

$$P = y(t^9 - t) + R$$

for some polynomials y, R with R of degree < 9. Now, by Lemma 6 the polynomial $y(t^9 - t)$ is a mixed sum of four biquadrates with two minus signs. But, from Lemma 5, R is a sum of two biquadrates. The result follows.

The second result is:

Theorem 3. *Let* $q \in \{3, 9\}$. *Then*

$$w(4, \mathbf{F}_q[t]) \leq 8.$$

Proof. The proof is the same as the proof of Theorem 2. Just observe that the polynomial $y(t^9 - t)$ is a mixed sum of four biquadrates with two minus signs, i.e., it is a sum of six biquadrates.

Our third result is:

Theorem 4. *Let* $q \in \{3, 9\}$. *Then*

$$gm\,(4, \mathbf{F}_q[t]) \leq 10.$$

Proof. Let $P \in \mathbf{F}_q[t]$ decompose as a mixed sum of biquadrates. By Lemmata 6 and 7, we can write P as a strict mixed sum of two biquadrates plus a remainder R of the right degree. From Lemma 4 together with Lemma 3 we see that R is a mixed sum of six biquadrates plus a remainder S of degree ≤ 8. But, S is a sum of two biquadrates by Lemma 5. The result follows.

Our fourth result is:

Theorem 5. *Let* $q \in \{3, 9\}$. *Then*

$$g(4, \mathbf{F}_q[t]) \leq 14.$$

Proof. Let $P \in \mathbf{F}_q[t]$ to decompose as a mixed sum of biquadrates. The proof is the same that of Theorem 4. Observe that at the beginning we get P as a strict sum of three biquadrates plus a remainder R of the right degree. Just observe that we get exactly three minus signs in the representation of R so that R is a strict sum of nine biquadrates. So P is a strict sum of $3 + 9 + 2 = 14$ biquadrates.

Acknowledgments. We thank the referee for fruitful criticism.

REFERENCES

1. M. Car and L.H. Gallardo, *Waring's problem for polynomial biquadrates over a finite field of odd characteristic*, Funct. Approx., Comment. Math. **37** (2007), 39–50.

2. G. Effinger and D. Hayes, *Additive number theory of polynomials over a finite field*, Oxford Math. Mono., Clarendon Press, Oxford, 1991.

3. L. Gallardo, *Une variante du problème de Waring sur* $\mathbf{F}_{2^n}[t]$, (An $\mathbf{F}_{2^n}[t]$-*variant of Waring's problem*), C.R.A.S **327** (1998), 117–121.

4. ———, *Waring's problem for Polynomial cubes and squares over a finite field of even characteristic*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), 349–362.

5. ———, *Every strict sum of cubes in* $\mathbf{F}_4[t]$ *is a strict sum of* 6 *cubes*, Portugal Math. **65** (2008), 227–236.

6. L.H. Gallardo and D.R. Heath-Brown, *Every sum of cubes in* $\mathbf{F}_2[t]$ *is a strict sum of* 6 *cubes*, Finite Fields Appl. **13** (2007), 648–658.

7. L.H. Gallardo and L.N. Vaserstein, *The strict Waring problem for polynomial rings*, J. Number Theory **128** (2008), 2963–2972.

8. R.E.A.C. Paley, *Theorems on polynomials in a Galois field*, Quart. J. Math. **4** (1933), 52–63.

9. L.N. Vaserstein, *Waring's problem for commutative rings*, J. Number Theory **26** (1987), 299–307.

10. ———, *Sums of cubes in polynomial rings*, Math. Comp. **56** (1991), 349–357.

11. ———, *Ramsey's theorem and the Waring's problem for algebras over fields*, Proc. Workshop "Arithmetic of Function Fields," (Ohio State Univ., 1991), Walter de Gruyter Verlag, Berlin, 1992, 435–442.

Department of Mathematics, University of Brest, 6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest Cedex 3, France
Email address: Luis.Gallardo@univ-brest.fr

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
Email address: vstein@math.psu.edu