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SOME PROPERTIES OF THE MULTIPLICITY SEQUENCE FOR ARBITRARY IDEALS

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ABSTRACT. In this work we prove that the Achilles-Manaresi multiplicity sequence, like the classical Hilbert-Samuel multiplicity, is additive with respect to the exact sequence of modules. We also prove the associativity formula for this multiplicity sequence. As a consequence, we give new proofs for two results already known. First, the Achilles-Manaresi multiplicity sequence is an invariant up to reduction, a result first proved by Ciupercă. Second, $I \subseteq J$ is a reduction of (J, M) if and only if $c_0(I_\mathfrak{p}, M_\mathfrak{p}) = c_0(J_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$, a result first proved by Flenner and Manaresi.

1. Introduction. The integral closure of an ideal is an algebraic object which has had many applications in several aspects of algebraic geometry and commutative algebra. Apart from being studied for its own interest in the multiplicative theory of ideals, it has been extremely important in solving problems of singularity theory. In fact, Teissier [**13**] used the multiplicity and the integral closure of the product of two $\mathfrak{m}$-primary ideals in a local ring $(A, \mathfrak{m})$ in order to study the Whitney equisingularity of a 1-parameter family of isolated hypersurface singularities.

Let $(A, \mathfrak{m})$ be a local Noetherian ring, and let $I \subseteq J$ be two ideals in A. Recall that I is a reduction of J if $IJ^n = J^{n+1}$ for sufficiently large n. The notions of reduction and integral closure are related as follows: I is a reduction of J if and only if they have the same integral closure. If $I \subseteq J$ are $\mathfrak{m}$-primary and I is a reduction of J then it is well known and easy to prove that the Hilbert-Samuel multiplicities $e(J, A)$

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and $e(I, A)$ are equal. Rees proved his famous result, which nowadays has his name, that the converse also holds:

Theorem 1.1 (Rees's theorem [**12**]). *Let $(A, \mathfrak{m})$ be a quasi-unmixed local ring, and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals of A. Then, the following conditions are equivalent*:

(i) *I is a reduction of J*;

(ii) $e(J, A) = e(I, A)$.

The notion of Hilbert-Samuel multiplicities had been extended for arbitrary ideals, in the analytic case, by Gaffney and Gassler by means of the Segre numbers [**8**] and, in the algebraic case, by Achilles and Manaresi in [**1**]. Achilles and Manaresi introduced, for each ideal I of a d-dimensional local ring $(A, \mathfrak{m})$, a sequence of multiplicities $c_0(I, A), \ldots, c_d(I, A)$ which generalize the Hilbert-Samuel multiplicity in the sense that, for $\mathfrak{m}$-primary ideals I, $c_0(I, A)$ is the Hilbert-Samuel multiplicity of I and the remaining $c_k(I, A)$, $k = 1, \ldots, d$, are zero. Furthermore, Achilles and Rams in [**2**] proved that this multiplicity sequence coincides with the sequence consisting of the Segre numbers.

In this paper, by using the techniques developed by Flenner and Vogel in [**7**] (see also [**6**, Theorem 1.2.6]), we prove the additivity property for the Achilles-Manaresi multiplicity sequence (Theorem 1.2). This property for the coefficient c_0 is also a known result. It is proved for example in [**6**, Proposition 6.1.7] (in that book the coefficient j_d is the coefficient c_0 with the notation from this paper). As a consequence, we give new proofs for two results already known. First, the Achilles-Manaresi multiplicity sequence is an invariant up to reduction (Theorem 1.3), a result first proved by Ciupercă in [**4**]. Second, $I \subseteq J$ is a reduction of (J, M) if and only if $c_0(I_\mathfrak{p}, M_\mathfrak{p}) = c_0(J_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ (Theorem 1.4), a result first proved by Flenner and Manaresi in [**5**]. In the literature, the coefficient $c_0(I, M)$ is also referred to as the j-multiplicity of (I, M).

Precisely, we prove the following results:

Theorem 1.2 (**Additivity**). *Let $(A, \mathfrak{m})$ be a local ring, and let I be an ideal of A. Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence of finitely generated A-modules and D an integer with $D \geq d := \dim M_1$.*

Then we have that

$$c_k^D(I, M_1) = c_k^D(I, M_0) + c_k^D(I, M_2)$$

for all $k = 0, \ldots, D$.

Theorem 1.3. *Let* $(A, \mathfrak{m})$ *be a local ring, and let* M *be a* d*-dimensional finitely generated* A*-module. Let* $I \subseteq J$ *be proper arbitrary ideals of* A *such that* $\mathrm{ht}_M(I) > 0$. *If* I *is a reduction of* (J, M), *then* $c_k(I, M) = c_k(J, M)$ *for all* $k = 0, \ldots, d$.

Theorem 1.4. *Let* $(A, \mathfrak{m})$ *be a local Noetherian ring, and let* M *be a quasi-unmixed finitely generated* d*-dimensional* A*-module. Let* $I \subseteq J \subseteq \mathfrak{m}$ *be ideals of* A. *Then the following are equivalent.*

(i) I *is a reduction of* (J, M).

(ii) $c_0(I_\mathfrak{p}, M_\mathfrak{p}) = c_0(J_\mathfrak{p}, M_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,(A)$.

(iii) $c_0(I_\mathfrak{p}, M_\mathfrak{p}) \leq c_0(J_\mathfrak{p}, M_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\,(A)$.

2. Achilles-Manaresi multiplicity sequence. In this section we recall some well-known facts on Hilbert functions and Hilbert polynomials of bigraded modules, which will be essential for defining the Achilles-Manaresi multiplicity sequence associated to a pair (I, M).

Let $R = \oplus_{i,j=0}^{\infty} R_{i,j}$ be a bigraded ring, and let $T = \oplus_{i,j=0}^{\infty} T_{i,j}$ be a finitely generated d-dimensional bigraded R-module. Assume that $R_{0,0}$ is an Artinian ring and that R is finitely generated as an $R_{0,0}$-algebra by elements of $R_{1,0}$ and $R_{0,1}$. Since the length $\ell_{R_{0,0}}(T_{i,j})$ of $T_{i,j}$ is finite, we can consider the *Hilbert function* $h_T(i,j)$ of T, given by $h_T(i,j) := \ell_{R_{0,0}}(T_{i,j})$, as a function in two variables i and j. This function was first studied by van der Waerden [**14**] who proved that there is a polynomial $P_T(i,j)$ of total degree at most $d-2$ such that $\ell_{R_{0,0}}(T_{i,j}) = P_T(i,j)$ for i, j sufficiently large [**14**, Theorem 7, page 757; Theorem 11, page 759]. We can write $P_T(i,j)$ in the form

$$P_T(i,j) = \sum_{\substack{k,l \geq 0 \\ k+l \leq d-2}} a_{k,l}(T) \binom{i+k}{k} \binom{j+l}{l},$$

with $a_{k,l}(T) \in \mathbf{Z}$ and $a_{k,l}(T) \geq 0$ if $k + l = d - 2$.

Let $h_T^{(1,0)}(i,j) = \sum_{u=0}^{i} h_T(u,j)$ be the so-called *sum transform* of h with respect to the first variable, and let

$$h_T^{(1,1)}(i,j) = \sum_{v=0}^{j} h_T^{(1,0)}(i,v) = \sum_{v=0}^{j}\sum_{u=0}^{i} h_T(u,v).$$

From this description it is clear that, for i, j sufficiently large, $h_T^{(1,0)}$ and $h_T^{(1,1)}$ become polynomials with rational coefficients of degree at most $d-1$ and d, respectively. As usual, we can write these polynomials in terms of binomial coefficients

$$P_T^{(1,0)}(i,j) = \sum_{\substack{k,l\geq 0\\ k+l\leq d-1}} a_{k,l}^{(1,0)}(T)\binom{i+k}{k}\binom{j+l}{l}$$

with $a_{k,l}^{(1,0)}(T)$ integers and $a_{k,d-k-1}^{(1,0)}(T) \geq 0$, and

$$P_T^{(1,1)}(i,j) = \sum_{\substack{k,l\geq 0\\ k+l\leq d}} a_{k,l}^{(1,1)}(T)\binom{i+k}{k}\binom{j+l}{l}$$

with $a_{k,l}^{(1,1)}(T)$ integers and $a_{k,d-k}^{(1,1)}(T) \geq 0$.

Since

$$h_T(i,j) = h_T^{(1,0)}(i,j) - h_T^{(1,0)}(i-1,j),$$

we get $a_{k+1,l}^{(1,0)}(T) = a_{k,l}(T)$ for $k,l \geq 0$, $k+l \leq d-2$.

Similarly, we have

$$h_T^{(1,0)}(i,j) = h_T^{(1,1)}(i,j) - h_T^{(1,1)}(i,j-1),$$

which implies that $a_{k,l+1}^{(1,1)}(T) = a_{k,l}^{(1,0)}(T)$ for $k,l \geq 0$, $k+l \leq d-1$.

Definition 2.1. For the coefficients of the terms of highest degree in $P_T^{(1,1)}$, we introduce the symbols

$$c_k(T) := a_{k,d-k}^{(1,1)}(T), \quad k = 0, \dots, d$$

which are called the *multiplicity sequence* of T.

Let A be an arbitrary Noetherian ring, $I \subseteq A$ an ideal and M an A-module. Consider the Rees ring of I:

$$\mathcal{R}(I, A) := \bigoplus_{n \in \mathbf{N}} I^n t^n \subseteq A[t]$$

and the extended Rees ring of I

$$\mathcal{R}_e(I, A) := \bigoplus_{n \in \mathbf{Z}} I^n t^n \subseteq A[t, t^{-1}],$$

where we set $I^n = A$ for $n \leq 0$.

Analogously, consider the Rees module

$$\mathcal{R}(I, M) := \bigoplus_{n \in \mathbf{N}} I^n M t^n \subseteq M[t]$$

and the extended Rees module

$$\mathcal{R}_e(I, M) := \bigoplus_{n \in \mathbf{Z}} I^n M t^n \subseteq M[t, t^{-1}].$$

Notice that $\mathcal{R}(I, M)$ and $\mathcal{R}_e(I, M)$ are modules over $\mathcal{R}(I, A)$ and $\mathcal{R}_e(I, A)$, respectively. Obviously, $\mathcal{R}_e(I, M)/t^{-1}\mathcal{R}_e(I, M)$ is just the associated graded module $G_I(M)$.

We define next the c^D-multiplicity sequence associated to an ideal. Let $(A, \mathfrak{m})$ be a local ring, $S = \oplus_{j \in \mathbf{N}} S_j$ a standard graded A-algebra (with $S_0 = A$), $N = \oplus_{j \in \mathbf{N}} N_j$ a finitely generated graded S-module, and

$$T := G_{\mathfrak{m}}(N) = \bigoplus_{i,j \in \mathbf{N}} \frac{\mathfrak{m}^i N_j}{\mathfrak{m}^{i+1} N_j}$$

the bigraded R-module with

$$R := G_{\mathfrak{m}}(S) = \bigoplus_{i,j \in \mathbf{N}} \frac{\mathfrak{m}^i S_j}{\mathfrak{m}^{i+1} S_j}.$$

Notice that $R_{0,0} = A/\mathfrak{m}$ is a field.

Definition 2.2. Consider an integer D such that $D \geq \dim N$. For all $k = 0 \ldots, D$, we set

$$c_k^D(N) = \begin{cases} 0 & \text{if } \dim N < D \\ c_k(T) & \text{if } \dim N = D \end{cases}$$

which is called the c^D-*multiplicity sequence* of N. Moreover, we set $c_k(N) := c_k^{\dim N}(N)$.

First we show that this c^D-multiplicity sequence behaves well with respect to short exact sequences.

Proposition 2.3. *Let $0 \to N_0 \to N_1 \to N_2 \to 0$ be an exact sequence of finitely generated graded S-modules. Set $d := \dim N_1$. Then we have that*

$$h_{T_0}^{(1,1)}(i,j) + h_{T_2}^{(1,1)}(i,j) - h_{T_1}^{(1,1)}(i,j)$$

is, for $i, j \gg 0$, a polynomial of degree at most $d-1$, where $T_s := G_{\mathfrak{m}}(N_s)$. In particular, for $D \geq d$ we have that

$$c_k^D(N_1) = c_k^D(N_0) + c_k^D(N_2)$$

for all $k = 0, \ldots, D$.

Proof. Let $M_s := \mathcal{R}_e(\mathfrak{m}, N_s)$ be the extended Rees module associated to N_s, $s = 0, 1, 2$. For any bigraded module T we denote by $h_T(i,j)$ the Hilbert-Samuel function of T.

Set $M_0' := \ker(M_1 \to M_2) = \oplus_{i \in \mathbf{Z},\, j \in \mathbf{N}} (N_0)_j \cap \mathfrak{m}^i (N_1)_j \, u^i t^j$. We consider the natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0'(1,0) & \longrightarrow & M_1(1,0) & \longrightarrow & M_2(1,0) & \longrightarrow & 0 \\ & & \downarrow u^{-1} & & \downarrow u^{-1} & & \downarrow u^{-1} & & \\ 0 & \longrightarrow & M_0' & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

which gives an exact sequence of cokernels

$$0 \longrightarrow G' := \frac{M_0'}{u^{-1} M_0'} \longrightarrow G_{\mathfrak{m}}(N_1) \longrightarrow G_{\mathfrak{m}}(N_2) \longrightarrow 0. \tag{1}$$

Denote the cokernel of the natural injection $M_0 \hookrightarrow M_0'$ by L. Using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0(1,0) & \longrightarrow & M_0'(1,0) & \longrightarrow & L(1,0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u^{-1}} & & \downarrow{\scriptstyle u^{-1}} & & \downarrow{\scriptstyle u^{-1}} & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & M_0' & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

the snake-lemma yields an exact sequence

$$0 \longrightarrow V \longrightarrow G_{\mathfrak{m}}(N_0) \longrightarrow G' \longrightarrow W \longrightarrow 0, \tag{2}$$

where V and W are the kernel and cokernel of $u^{-1} : L(1,0) \to L$, respectively, i.e., the sequence

$$0 \longrightarrow V \longrightarrow L(1,0) \longrightarrow L \longrightarrow W \longrightarrow 0. \tag{3}$$

For $n \leq 1$, the coefficient modules of u^n in $M_0 = \mathcal{R}_e(\mathfrak{m}, N_0)$ and in M_0' coincide; hence ,the action of u^{-1} on L is nilpotent. Therefore, the dimension of L is at most that of G', which is bounded by d. Thus, all modules occurring in the exact sequence (3) have dimension at most d.

Now (1), (2) and (3) are exact sequences of finitely generated modules of dimension at most d.

Set $T_s := G_{\mathfrak{m}}(N_s)$, $s = 0, 1, 2$. From (1) and (2) we have

$$h_{T_0}^{(1,1)}(i,j) + h_{T_2}^{(1,1)}(i,j) - h_{T_1}^{(1,1)}(i,j) = h_V^{(1,1)}(i,j) - h_W^{(1,1)}(i,j). \tag{4}$$

Because of (3) we have

$$\begin{aligned} h_V^{(1,1)}(i,j) - h_W^{(1,1)}(i,j) &= h_L^{(1,1)}(i+1,j) - h_L^{(1,1)}(i,j) \\ &= h_L^{(0,1)}(i+1,j). \end{aligned} \tag{5}$$

Hence by (4) and (5)

$$h_{T_0}^{(1,1)}(i,j) + h_{T_2}^{(1,1)}(i,j) - h_{T_1}^{(1,1)}(i,j) = h_L^{(0,1)}(i+1,j)$$

is a polynomial of degree at most $d-1$, which concludes the proof. ◻

The above general definition of a c^D-multiplicity sequence for a graded modulo allows us to define next the main multiplicity sequence associated to an ideal:

Definition 2.4. Let $(A, \mathfrak{m})$ be a local ring, $I \subseteq A$ an ideal of A, M a finitely generated A-module and $N := G_I(M))$. Let $D \geq \dim M$. We set $c_k^D(I, M) := c_k^D(N)$ for all $k = 0, \ldots, D$; in the case where $D = \dim M$ we call the sequence

$$c_k(I, M) := c_k^{\dim M}(I, M), \text{ for all } k = 0, \ldots, \dim M$$

the *Achilles-Manaresi multiplicity sequence* associated to the ideal I with respect to M.

To be more explicit, consider the bigraded ring $R := G_{\mathfrak{m}}(G_I(A))$ and the bigraded R-module $T := G_{\mathfrak{m}}(G_I(M))$. Then $R = \oplus_{i,j=0}^{\infty} R_{i,j}$ and $T = \oplus_{i,j=0}^{\infty} T_{i,j}$ with

$$R_{i,j} = \left(\mathfrak{m}^i I^j + I^{j+1}\right) / \left(\mathfrak{m}^{i+1} I^j + I^{j+1}\right)$$

and

$$T_{i,j} = \left(\mathfrak{m}^i I^j M + I^{j+1} M\right) / \left(\mathfrak{m}^{i+1} I^j M + I^{j+1} M\right),$$

respectively.

Observe that $R_{0,0} = A/\mathfrak{m}$ is a field and T has dimension $d = \dim M$. We denote the Hilbert-Samuel function $\ell_{R_{0,0}}(T_{i,j})$ of $T = G_{\mathfrak{m}}(G_I(M))$ by $h_{(I,M)}(i,j)$ and its Hilbert sum by $h^{(1,1)}(i,j)$. That is,

$$h^{(1,1)}(i,j) = \sum_{v=0}^{j} \sum_{u=0}^{i} h(i,j),$$

which for $i, j \gg 0$ can be written as

$$h_{(I,M)}^{(1,1)}(i,j) = \sum_{k=0}^{d} \frac{c_k(I,M)}{k!(d-k)!} i^k j^{d-k} + \cdots,$$

where $\cdots$ means lower degree terms.

Remark 2.5. The coefficient just defined are a generalization of the classical Hilbert coefficients. Indeed, when the ideal I is $\mathfrak{m}$-primary then $c_0(I,M) = e(I,M)$ and $c_1(I,M) = \cdots = c_{\dim M}(I,M) = 0$. In fact, if I is $\mathfrak{m}$-primary, there exists a t such that $\mathfrak{m}^t \subseteq I$ and then, for i, j large enough,

$$h^{(1,1)}_{(I,M)}(i,j) = \ell(M/I^{j+1}M).$$

Therefore, the Achilles-Manaresi multiplicity sequence generalizes the Hilbert-Samuel multiplicity for $\mathfrak{m}$-primary ideals.

3. Integral closure and reduction. For completeness, we recall next some properties of reduction of ideals with respect to an A-module M in a Noetherian ring A.

An ideal $I \subseteq J$ is said to be a *reduction* of (J,M) if $IJ^nM = J^{n+1}M$ for at least one positive integer n.

Next we define a notion for an element of the ring A to be integral over a pair (J,M). As for the authors' knowledge, this definition is new. Since it generalize the usual definition of integral dependence and it is also equivalent to the notion of reduction of a pair (J,M), as shown in Proposition 3.3, we think it is interesting to include it here, even though we will only use it at one place in this work, namely Theorem 4.6.

Definition 3.1. Let M be an A-module, and let I be an ideal of A. An element $z \in A$ is said to be *integral* over (I,M) if it satisfies the following relation

$$z^n + a_1 z^{n-1} + \cdots + a_n \in (0:M)$$

with $n \in \mathbf{N}$ and $a_i \in I^i$ for all $i = 1, \ldots, n$. The set of all elements in A which are integral over (I,M) is denoted by $\overline{(I,M)}$, and it is called the *integral closure* of (I,M).

Remark 3.2. (i) Notice that if M is a faithful A-module, then $\overline{(I,M)} = \overline{I}$.

(ii) Notice that $z \in \overline{(I,M)}$ if and only if $\overline{z} \in \overline{I + (0:M)/(0:M)}$ as an element in $A/(0:M)$. Hence, $\overline{(I,M)}$ is an ideal of A which satisfies the relation $I + (0:M) \subseteq \overline{(I,M)} \subseteq \sqrt{I + (0:M)}$.

Proposition 3.3. *Let $I \subseteq J$ be ideals of A, and let M be an A-module. Then the following conditions are equivalent*:

(i) *J is integral over (I, M);*

(ii) *I is is a reduction of (J, M);*

(iii) *the Rees module $\mathcal{R}(J, M)$ is finite over the Rees ring $\mathcal{R}(I, A)$.*

Proof. We prove first the equivalence (i) $\Leftrightarrow$ (ii). Notice that by Remark 3.2 we have that J is integral over (I, M) if and only if $J + (0 : M)/(0 : M)$ is integral over $I + (0 : M)/(0 : M)$ as ideals in $A/(0 : M)$, which is equivalent to $I + (0 : M)/(0 : M)$ being a reduction of $J + (0 : M)/(0 : M)$ as ideals in $A/(0 : M)$. But we know that this happens if and only if $(I + (0 : M))(J + (0 : M))^n/(0 : M) = (J + (0 : M))^{n+1}/(0 : M)$ for some $n \in N$ or equivalently $IJ^n + (0 : M)/(0 : M) = J^{n+1} + (0 : M)/(0 : M)$ for some $n \in N$. But this last equality is equivalent to $IJ^nM = J^{n+1}M$. Hence, the result follows. The equivalence of (ii) and (iii) follows by [**5**, Lemma 2.3 (a)]. □

Proposition 3.4. *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of local rings such that $\mathfrak{m}B = \mathfrak{n}$, and let I be an ideal of A. Then $c_k(I, A) = c_k(IB, B)$ for all $k = 0, \dots, d$.*

Proof. It is well known that $\dim B = \dim A + \dim (B/\mathfrak{m}B)$, see, e.g., [**11**, (13.B)]. Hence, since $\mathfrak{m}B = \mathfrak{n}$, we have that $\dim B = \dim A$. Moreover, if M is an A-module and the length $\ell_B(B/\mathfrak{m}B)$ is finite, then $\ell_B(M \otimes_A B) = \ell_B(B/\mathfrak{m}B) \cdot \ell_A(M)$ (see, for example, [**10**, Lemma (1.28), page 13]). Note that $\ell_B(B/\mathfrak{m}B) = 1$ since $\mathfrak{m}B = \mathfrak{n}$. Thus, $\ell_B(M \otimes_A B) = \ell_A(M)$.

Putting $M = (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1})$ with nonnegative integers i and j and $J := IB$, we obtain

$$\begin{aligned} M \otimes_A B &\cong \left(\mathfrak{m}^i I^j + I^{j+1}\right) B / \left(\mathfrak{m}^{i+1} I^j + I^{j+1}\right) B \\ &\cong \left(\mathfrak{n}^i J^j + J^{j+1}\right) / \left(\mathfrak{n}^{i+1} J^j + J^{j+1}\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\ell_A \left(\left(\mathfrak{m}^i I^j + I^{j+1}\right) / \left(\mathfrak{m}^{i+1} I^j + I^{j+1}\right)\right) \\ &\qquad = \ell_B \left(\left(\mathfrak{n}^i J^j + J^{j+1}\right) / \left(\mathfrak{n}^{i+1} J^j + J^{j+1}\right)\right). \end{aligned}$$

The result follows now by definition of the Achilles-Manaresi multiplicity sequence. □

4. Main results. In this section we prove our main results. We prove first that the Achilles-Manaresi multiplicity sequence, like the classical Hilbert-Samuel multiplicity, is additive with respect to exact sequence of modules. As for the case of Hilbert-Samuel multiplicity, we prove the associativity formula for the Achilles-Manaresi multiplicity sequence. The properties for the coefficient c_0 are also known results. They are proved for example in [**6**, 6.1.7, 6.1.8]. As a consequence, we give new proofs for two results already known. First, the Achilles-Manaresi multiplicity sequence is an invariant up to reduction, a result first proved by Ciupercă in [**4**]. Second, $I \subseteq J$ is a reduction of (J, M) if and only if $c_0(I_{\mathfrak{p}}, M_{\mathfrak{p}}) = c_0(J_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$, a result first proved by Flenner and Manaresi in [**5**]. In the literature, the coefficient $c_0(I, M)$ is also referred to as the j-multiplicity of (I, M).

Theorem 4.1 (**Additivity**). *Let $(A, \mathfrak{m})$ be a local ring, and let I be an ideal of A. Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence of finitely generated A-modules and D an integer with $D \geq d := \dim M_1$. Then*

$$h^{(1,1)}_{(I,M_0)}(r,n) + h^{(1,1)}_{(I,M_2)}(r,n) - h^{(1,1)}_{(I,M_1)}(r,n)$$

is, for $r, n \gg 0$, a polynomial of degree at most $d-1$. In particular, we have that

$$c_k^D(I, M_1) = c_k^D(I, M_0) + c_k^D(I, M_2)$$

for all $k = 0, \dots, D$.

Proof. Let $N_i := \mathcal{R}_e(I, M_i)$ be the extended Rees module associated to M_i. Set $N_0' := \ker(N_1 \to N_2) = \oplus_{n \in \mathbf{Z}} M_0 \cap I^n M_1 t^n$. We consider the natural diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_0'(1) & \longrightarrow & N_1(1) & \longrightarrow & N_2(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t^{-1}} & & \downarrow{\scriptstyle t^{-1}} & & \downarrow{\scriptstyle t^{-1}} & & \\
0 & \longrightarrow & N_0' & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & 0
\end{array}$$

which gives an exact sequence of cokernels

$$0 \longrightarrow G' := \frac{N'_0}{t^{-1}N'_0} \longrightarrow G_I(M_1) \longrightarrow G_I(M_2) \longrightarrow 0. \tag{6}$$

Next consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_0(1) & & N'_0(1) & \longrightarrow & L(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t^{-1}} & & \downarrow{\scriptstyle t^{-1}} & & \downarrow{\scriptstyle t^{-1}} & & \\
0 & \longrightarrow & N_0 & \longrightarrow & N'_0 & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

where $L := \operatorname{coker}(N_0 \to N'_0)$. The snake-lemma yields an exact sequence

$$0 \longrightarrow U \longrightarrow G_I(M_0) \longrightarrow G' \longrightarrow V \longrightarrow 0. \tag{7}$$

where $U := \ker(L(1) \to L)$ and $V := \operatorname{coker}(L(1) \to L)$. We also have the exact sequence

$$0 \longrightarrow U \longrightarrow L(1) \longrightarrow L \longrightarrow V \longrightarrow 0. \tag{8}$$

For $n \leq 1$ the coefficient modules of t^n in $\mathcal{R}_e(I, M_0)$ and in N'_0 coincide, hence the action of t^{-1} on L is nilpotent. Therefore, the dimension of L is at most that of G', which is bounded by d. Thus, all modules occurring in the exact sequence (8) have dimension at most d.

For any bigraded algebra $E = \oplus_{r,n\in\mathbf{N}} E_{r,n}$, consider the Hilbert-Samuel functions $h_E(r,n) := \ell(E_{r,n})$ and their Hilbert sums

$$h_E^{(1,1)}(r,n) := \sum_{v=0}^{n} \sum_{u=0}^{r} h_E(u,v).$$

Now (6), (7) and (8) are exact sequences of graded $G_I(A)$-modules of dimension at most d. Hence, we may apply the same arguments as in the proof of Proposition 2.3 along these sequences and obtain

$$\begin{aligned}
h_{(I,M_0)}^{(1,1)}(r,n) + h_{(I,M_2)}^{(1,1)}(r,n) - h_{(I,M_1)}^{(1,1)}(r,n) \\
&= h_L^{(1,1)}(r+1,n) - h_L^{(1,1)}(r,n) \\
&= h_L^{(0,1)}(r+1,n).
\end{aligned}$$

which, for $r, n \gg 0$, is a polynomial of degree at most $d-1$ because L has dimension at most d. □

A standard application of the additivity formula is the following so-called associativity formula.

Corollary 4.2 (**Associativity**). *Let $(A, \mathfrak{m})$ be a local ring, let I be an ideal of A, and let M be a d-dimensional finitely generated A-module. Then,*

$$c_k(I, M) = \sum_{\mathfrak{p}} c_k(IA/\mathfrak{p}, A/\mathfrak{p})\, \ell(M_{\mathfrak{p}}),$$

where the sum is taken over all prime ideals $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}) = d$.

Proof. We write $\sigma := \sum_{\mathfrak{p}} \ell(M_{\mathfrak{p}})$ where the sum is taken over all prime ideals $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}) = d$, and proceed by induction on σ. If $\sigma = 0$ then $M = 0$ and so the formula is obvious. If $\sigma > 0$, choose a prime ideal $\mathfrak{p}_0$ with $\dim(A/\mathfrak{p}_0) = d$ and $\ell(M_{\mathfrak{p}_0}) > 0$; then $\mathfrak{p}_0 \in \operatorname{Supp}(M)$. Thus M contains a submodule N isomorphic to $A/\mathfrak{p}_0$. If $\dim M/N < d$, then $\sigma = \ell(M_{\mathfrak{p}_0}) = 1$ and by Theorem 4.1 $c_k(I, M) = c_k(I, N)$. If $\dim M/N = d$, then $\sigma(M/N) < \sigma(M) = \sigma$ and so by induction hypothesis

$$c_k(I, M/N) = \sum_{\mathfrak{p}} c_k(IA/\mathfrak{p}, A/\mathfrak{p})\, \ell((M/N)_{\mathfrak{p}}).$$

Now $\ell(M_{\mathfrak{p}}) = \ell((M/N)_{\mathfrak{p}})$ for $\mathfrak{p} \neq \mathfrak{p}_0$ and $\ell(M_{\mathfrak{p}_0}) = \ell((M/N)_{\mathfrak{p}_0}) + 1$. Since by Theorem 4.1 $c_k(I, M) = c_k(I, M/N) + c_k(I, A/\mathfrak{p}_0)$, the assertion follows. □

The next theorem is one of the main results of this work, which says that the Achilles-Manaresi multiplicity sequence is an invariant of I up to reduction.

Theorem 4.3. *Let $(A, \mathfrak{m})$ be a local Noetherian ring, and let M be a finitely generated A-module of dimension d. Let $I \subseteq J$ be proper arbitrary ideals of A. If I is a reduction of (J, M), then $c_k(I, M) = c_k(J, M)$ for all $k = 0, \ldots, d$.*

Proof. Since I is a reduction of (J, M), we have that $I(J^iM) = J^{i+1}M = J(J^iM))$ for some $i \geq 0$. Hence, we have that $G_I(J^iM) = G_J(J^iM)$, and thus $c_k^d(I, J^iM) = c_k^d(J, J^iM)$ for all $k = 0, \dots, d$. On the other hand, set $M_j := J^jM/J^{j+1}M$ for $j \geq 0$. Notice that $JM_j = 0 = IM_j$; hence, $G_I(M_j) = M_j = G_J(M_j)$ and then $c_k^d(I, M_j) = c_k^d(J, M_j)$ for all $k = 0, \dots, d$.

Using the additivity of the c^d-multiplicity sequence as proven in Theorem 4.1, we now conclude that

$$\begin{aligned} c_k(I, M) &= c_k^d(I, J^iM) + \sum_{j=0}^{i-1} c_k^d(I, M_j) \\ &= c_k^d(J, J^iM) + \sum_{j=0}^{i-1} c_k^d(J, M_j) \\ &= c_k(J, M). \qquad \square \end{aligned}$$

The above theorem was also proved by Ciupercă by different methods [**4**, Proposition 2.7]. Ciupercă proved first that the Achilles-Manaresi multiplicity sequence behaves well with respect to general hyperplane sections and then, applying [**1**, Proposition 2.3], he used induction on the dimension of the local ring. Our approach is much simpler.

Example 4.4. Let $A = k[x, y, z]$ be the ring of polynomials in three variables over the field k, and let $\mathfrak{m} = (x, y, z)$ be the maximal homogeneous ideal. As in the local case, one can define the Achilles-Manaresi multiplicity sequence. For any ideal I of any d-dimensional ring A, we denote by $c(I, A)$ the Achilles-Manaresi multiplicity sequence of I, $c(I, A) = (c_0(I, A), \dots, c_d(I, A))$.

(1) Let $I = (x^4z^2, y^2z^4)$, and let $J = (x^4z^2, y^2z^4, x^2yz^3)$. Notice that $JI = J^2$, i.e., I is a reduction of J. Using the intersection algorithm [**1**, Theorem 4.1] for computing the Achilles-Manaresi multiplicity sequence, one easily shows that in this particular case $c(I, A) = c(J, A) = (0, 24, 2, 0)$.

(2) Let $A = k[x, y]$ be the ring of polynomials in two variables over the field k, and let $\mathfrak{m} = (x, y)$ be the maximal homogeneous ideal. Let $I = (x^5y^3, x^2y^7)$ and $J = (x^5y^3, x^2y^7, x^6y)$ be ideals of

A. Then a similar computation shows that $c(J,A) = (29,5,0)$ and $c(I,A) = (40,3,0)$. Hence, those ideals do not have the same integral closure.

4.1. Local criterion for reduction. In this section we give a criterion for reduction of ideals involving the c_0-multiplicity in all localizations in prime ideals.

In order to be able to prove the main result in a simple way, we need some notations. Let $(A,\mathfrak{m})$ be a Noetherian local ring. Given a finitely generated A-module N_j, the extended Rees module

$$\mathcal{R}_e(\mathfrak{m}, N_j) := \bigoplus_{i\in\mathbf{Z}} \mathfrak{m}^i N_j u^i$$

will be denoted by P_j. It gives rise to the associated graded module

$$G_j := \frac{P_j}{u^{-1}P_j} = G_{\mathfrak{m}}(N_j) := \bigoplus_{i\in\mathbf{N}} [G_j]_i$$

where

$$[G_j]_i := \frac{\mathfrak{m}^i N_j}{\mathfrak{m}^{i+1} N_j}.$$

In general, given any graded module F, its ith homogeneous component will be denoted by F_i, i.e., $F = \oplus_{i\in\mathbf{N}} F_i$.

Lemma 4.5. *Let N_j, $j = 0,1,2$, be finitely generated A-modules and*

$$0 \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow 0$$

an exact sequence. Set $P_0' := ker(P_1 \to P_2)$. Let L be the cokernel of the natural embedding $P_0 \hookrightarrow P_0'$. Let U and V denote the kernel and cokernel of the map $u^{-1} : L(1) \to L$, respectively. Then we have that

(i) $\ell([G_1]_i) = \ell([G_0]_i) + \ell([G_2]_i) - [\ell(U_i) - \ell(V_i)]$;

(ii) $0 \to U \to L(1) \to L \to V \to 0$ *is an exact sequence and*

(iii) *all modules occurring in* (ii) *have dimension at most* $\dim(N_1)$.

Proof. The proof is analogous to that of [**6**, Theorem 1.2.6.]. □

The next theorem provides a crucial step in the proof of our main result of this section.

Theorem 4.6. *Let $(A, \mathfrak{m})$ be a local Noetherian ring, $I \subseteq J$ ideals of A, and let M be a finitely generated d-dimensional A-module. Assume that $I_{\mathfrak{q}}$ is a reduction of $(J_{\mathfrak{q}}, M_{\mathfrak{q}})$ for every prime $\mathfrak{q}$ of A with $\mathfrak{q} \neq \mathfrak{m}$. Then*

(i) *$c_k(I, M) = c_k(J, M)$ for all $k = 1, \dots, d$ and $c_0(I, M) \geq c_0(J, M)$.*

(ii) *Suppose that M is a quasi-unmixed A-module. If $c_0(I, M) \leq c_0(J, M)$, then I is a reduction of (J, M).*

Proof. We may factor out the annihilator of M to assume that M is a faithful A-module. Passing to the completion, we might assume that A, and hence M, is complete (see Proposition 3.4 and [**9**, Corollary (4.12)]). In particular, A is equidimensional of dimension d in the setting of (ii). Theorem 4.3 shows that $c_k(I, M)$ does not change when we replace I by the ideal generated by the elements in J that are integral over I on M. Notice that, by Remark 3.2 (i), the notions of being integral over (I, M) and over I coincide. Thus, by our assumption on I and J we may suppose that JM/IM has finite length over A.

For any $j \in \mathbf{N}$, consider the following exact sequences of A-modules

$$0 \longrightarrow \frac{J^{j+1}M}{I^{j+1}M} \longrightarrow \frac{J^jM}{I^{j+1}M} \longrightarrow \frac{J^jM}{J^{j+1}M} \longrightarrow 0 \tag{9}$$

and

$$0 \longrightarrow \frac{I^jM}{I^{j+1}M} \longrightarrow \frac{J^jM}{I^{j+1}M} \longrightarrow \frac{J^jM}{I^jM} \longrightarrow 0. \tag{10}$$

Applying Lemma 4.5 (i) to the exact sequence (9), we have that

$$\ell\left(\frac{J^jM}{\mathfrak{m}^iJ^jM + I^{j+1}M}\right) = \ell\left(\frac{J^jM}{\mathfrak{m}^iJ^jM + J^{j+1}M}\right) + \ell\left(\frac{J^{j+1}M}{I^{j+1}M}\right) - [h_L^{(1,0)}(i+1, j) - h_L^{(1,0)}(i, j)] \tag{11}$$

where $L := \oplus_{i,j \in \mathbf{N}} L_{i,j}$ with

$$L_{i,j} := \frac{\mathfrak{m}^iJ^jM \cap J^{j+1}M + I^{j+1}M}{\mathfrak{m}^iJ^{j+1}M + I^{j+1}M},$$

and where we use the fact that $\ell(J^{j+1}M)/(I^{j+1}M) < \infty$ and hence for $i \gg 0$,

$$\ell\left(\frac{J^{j+1}M}{\mathfrak{m}^i J^{j+1}M + I^{j+1}M}\right) = \ell\left(\frac{J^{j+1}M}{I^{j+1}M}\right).$$

Applying Lemma 4.5 (i) to the exact sequence (10) we have that

$$\ell\left(\frac{J^jM}{\mathfrak{m}^i J^jM + I^{j+1}M}\right) = \ell\left(\frac{I^jM}{\mathfrak{m}^i I^jM + I^{j+1}M}\right) + \ell\left(\frac{J^jM}{I^jM}\right) - [h^{(1,0)}_{L'}(i+1,j) - h^{(1,0)}_{L'}(i,j)] \tag{12}$$

where $L' := \oplus_{i,j\in\mathbf{N}} L'_{i,j}$ with

$$L'_{i,j} := \frac{\mathfrak{m}^i J^jM \cap I^jM + I^{j+1}M}{\mathfrak{m}^i I^jM + I^{j+1}M},$$

and where we use the fact that $\ell(J^jM)/(I^jM) < \infty$, and hence for $i \gg 0$,

$$\ell\left(\frac{J^jM}{\mathfrak{m}^i J^jM + I^jM}\right) = \ell\left(\frac{J^jM}{I^jM}\right).$$

By Lemma 4.5 (iii), L and L' have dimension at most d, hence $[h^{(1,0)}_{L}(i+1,j) - h^{(1,0)}_{L}(i,j)]$ and $[h^{(1,0)}_{L'}(i+1,j) - h^{(1,0)}_{L'}(i,j)]$ are eventually polynomials of degree at most $d-2$. On the other hand, by Amaos's theorem [**3**], $\ell(J^jM)/(I^jM)$ is eventually a polynomial in j of degree at most d. Write its leading coefficient by $a^*(I/J;M)$.

Therefore, by equalities (11) and (12), we have that

$$\sum_{u=0}^{j} \ell\left(\frac{J^uM}{\mathfrak{m}^i J^uM + J^{u+1}M}\right) + \sum_{u=0}^{j}\left[\ell\left(\frac{J^{u+1}M}{I^{u+1}M}\right) - \ell\left(\frac{J^uM}{I^uM}\right)\right]$$

and

$$\sum_{u=0}^{j} \ell\left(\frac{I^uM}{\mathfrak{m}^i I^uM + I^{u+1}M}\right)$$

are eventually polynomials of degree at most d with the same leading coefficients. Hence, $c_k(I,M) = c_k(J,M)$ for all $k = 1,\dots,d$ and $c_0(I,M) = c_0(J,M) + a^*(I/J;M)$, which proves (i).

Under the assumptions of (ii) we have that $a^*(I/J; M) = 0$ which, by Rees-Amaos's theorem [**12**, Theorem 2.1], implies that I is a reduction of (J, M). □

We are now ready to assemble the proof of the main theorem of this subsection.

Theorem 4.7. *Let $(A, \mathfrak{m})$ be a local Noetherian ring and M a quasi-unmixed finitely generated d-dimensional A-module. Let $I \subseteq J \subseteq \mathfrak{m}$ ideals of A. Then the following are equivalent.*

(i) *I is a reduction of (J, M).*

(ii) $c_0(I_\mathfrak{p}, M_\mathfrak{p}) = c_0(J_\mathfrak{p}, M_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec}(A)$.

(iii) $c_0(I_\mathfrak{p}, M_\mathfrak{p}) \leq c_0(J_\mathfrak{p}, M_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 4.3 and (ii) $\Rightarrow$ (iii) is trivial. To show that (i) follows by (iii), we let $\mathfrak{q}$ be any prime ideal of A. Assuming that (iii) holds we prove by induction on $e := \dim A_\mathfrak{q}$ that $I_\mathfrak{q}$ is a reduction of $(J_\mathfrak{q}, M_\mathfrak{q})$. If $e = 0$, then the result follows because the ring $A_\mathfrak{q}$ is a local Artinian ring, and hence the maximal ideal is nilpotent. In this case any two comparable ideals form a reduction. Hence, we may suppose that $e > 0$. Let $\mathfrak{p}$ be any prime ideal of A such that $\dim A_\mathfrak{p} = e - 1$ and $\mathfrak{p} \subset \mathfrak{q}$. Hence, by induction hypotheses we have that $I_\mathfrak{p}$ is a reduction of $(J_\mathfrak{p}, M_\mathfrak{p})$. The quasi-unmixedness assumption on M is preserved by localization. Thus, in the local ring $(A_\mathfrak{q}, \mathfrak{q}_\mathfrak{q})$, all the assumptions of Theorem 4.6 hold, hence $I_\mathfrak{q}$ is a reduction of $(J_\mathfrak{q}, M_\mathfrak{q})$. □

Notice that, by [**1**, Proposition 2.3, Corollary 3.4], the above theorem coincides with Flenner-Manaresi's theorem [**5**, Theorem 3.3]. Their proof relies heavily on the behavior of the j-multiplicity under hyperplane sections and ours on the additivity property for the Achilles-Manaresi multiplicity sequence.

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REFERENCES

1. R. Achilles and M. Manaresi, *Multiplicities of a bigraded ring and intersection theory*, Math. Ann. **309** (1997), 573–591.

2. R. Achilles and S. Rams, *Intersection numbers, Segre numbers and generalized Samuel multiplicities*, Arch. Math. **77** (2001), 391–398.

3. J.O. Amao, *On a certain Hilbert polynomial*, J. London Math. Soc. **14** (1976), 13–20.

4. C. Ciupercă, *A numerical characterization of the S_2-ification of a Rees algebra*, J. Pure Appl. Algebra **178** (2003), 25–48.

5. H. Flenner and M. Manaresi, *A numerical characterization of reduction ideals*, Math. Z. **238** (2001), 205–214.

6. H. Flenner, L. O'Carroll and W. Vogel, *Joins and intersections*, Springer Mono. Math,. Springer-Verlag, Berlin, 1999.

7. H. Flenner and W. Vogel, *On multiplicities of local rings*, Manuscripta Math. **78** (1993), 85–97.

8. T. Gaffney and R. Gassler, *Segre numbers and hypersurface singularities*, J. Algebraic Geom. **8** (1999), 695–736.

9. M. Herrmann, S. Ikeda and U. Orbanz, *Equimultiplicity and blowing up. An algebraic study*, Springer-Verlag, Berlin, 1988.

10. M. Herrmann, R. Schmidt and W. Vogel, *Theorie der normalen Flachheit.* Teubner-Texte zur Mathematik. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1977.

11. H. Matsumura, *Commutative algebra*, Nagoya University, Nagoya, Japan, 1970.

12. D. Rees, *a-transform of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17.

13. B. Teissier, *Cycles évanescents, sections planes et condition de Whitney*, Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci. Cargèse, 1972), Astŕisque **7, 8**, Soc. Math. France, 1973.

14. B.L. Van der Waerden, *On Hilbert function series of composition of ideals and a generalisation of the theorem of Bezout*, Proc. Roy. Acad. Amsterdam. **31** (1929), 749–770.

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