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QUASI-CONVEX MAPPINGS OF ORDER α ON THE UNIT POLYDISK IN $\mathbf{C}^n$

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ABSTRACT. In this paper, a sufficient condition is first obtained for the quasi-convex mapping of type B and order α on the unit ball in a complex Banach space. Sharp estimations of all homogeneous expansions are then provided for some quasi-convex mappings $f(z)$ of type B and order α on the open unit polydisk in $\mathbf{C}^n$, where $f(z) = (f_1(z), f_2(z), \dots, f_n(z))'$ is a k-fold symmetric normalized mapping, or $z = 0$ is a zero of order $k+1$ of $f(z) - z$. These results generalize some results in the literature.

1. Introduction. The analytic functions of one complex variable which map the unit disk onto starlike or convex domains have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions. In moving to higher dimensions, several difficulties arise. Some are predictable, some are somewhat surprising. In the case of one complex variable, the following well known theorems have been established.

Theorem A [**3, 11**]. *Suppose that $0 \le \alpha < 1$, and let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a holomorphic function on the unit disk U in $\mathbf{C}$. If $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha$, then f is a univalent starlike function of order α, $0 \le \alpha < 1$, on the unit disk U in $\mathbf{C}$. If $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le 1-\alpha$, then f is a univalent convex function of order α, $0 \le \alpha < 1$, on the unit disk U in $\mathbf{C}$.*

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Theorem B [**12**]. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a univalent convex function of order α, $0 \leq \alpha < 1$, on the unit disk U in $\mathbf{C}$, then*

$$|a_n| \leq \frac{1}{n!} \prod_{k=2}^{n} (k - 2\alpha), \quad n = 2, 3, \dots .$$

In the case of several complex variables, Roper and Suffridge [**13**] provided a sufficient condition for a normalized biholomorphic convex mapping on the open Euclidean unit ball in $\mathbf{C}^n$ in 1999. After that, Zhu [**16**] provided a brief proof of the theorem of Roper and Suffridge. Liu and Zhu [**7, 8**] gave some sufficient conditions for a normalized biholomorphic convex mapping on some Reinhardt domains in $\mathbf{C}^n$.

With respect to the estimation of homogeneous expansions for subclasses of biholomorphic mappings in the case of several complex variables, only a few results have been given until now (see [**1, 5, 9, 10, 15**]). In spite of that, investigating these estimations is still interesting and significant. An important reason is that the estimations of all homogeneous expansions for starlike mappings on the open unit polydisk U^n in $\mathbf{C}^n$ are similar to the famous Bieberbach conjecture in the case of one complex variable.

Conjecture 1.1 [**1**]. *If $f : U^n \to \mathbf{C}^n$ is a normalized biholomorphic starlike mapping, where U^n is the open unit polydisk in $\mathbf{C}^n$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leq m\|z\|^m, \quad z \in U^n, \quad m = 2, 3, \dots .$$

As for the estimation of the homogeneous expansion for quasi-convex mappings of type B on the Euclidean unit ball in $\mathbf{C}^n$, since Roper and Suffridge [**13**] provided a counterexample to show that the following similar conjecture to the above does not hold for $m = 2$, only the estimation of the homogeneous expansion shall be studied here for quasi-convex mappings of type B and order α on U^n in $\mathbf{C}^n$.

Conjecture 1.2. *If $f : B_2^n \to \mathbf{C}^n$ is a normalized biholomorphic quasi-convex mapping of type B, where B_2^n is the Euclidean unit ball* *in* $\mathbf{C}^n$, *then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leq \|z\|^m, \quad z \in B_2^n,\ m = 2, 3, \dots .$$

Throughout this paper, let X be a complex Banach space with the norm $\|.\|$, let X^* denote the dual space of X, E be the open unit ball in X, and U represent the Euclidean open unit disk in $\mathbf{C}$. Also, let U^n be the open unit polydisk in $\mathbf{C}^n$, and let $\mathbf{N}$ denote the set of all positive integers. Let ∂U^n be the boundary of U^n, $\partial_0 U^n$ be the distinguished boundary of U^n. Let the symbol $'$ mean transpose. For each $x \in X\backslash\{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1,\ T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(E)$ be the set of all holomorphic mappings from E into X. It is well known that, if $f \in H(E)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in E$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \dots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^n X$ into X.

A holomorphic mapping $f : E \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(E)$. A mapping $f \in H(E)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in E$. If $f : E \to X$ is a holomorphic mapping, then f is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I stands for the identity operator from X into X.

A normalized biholomorphic mapping $f : E \to X$ is said to be starlike if $f(E)$ is a starlike domain with respect to the origin. Also, a normalized biholomorphic mapping $f : E \to X$ is said to be convex if $f(E)$ is a convex domain.

Definition 1.1. Suppose $\alpha \in [0,1)$ and $f : E \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\,\{T_x[(Df(x))^{-1}(D^2f(x)(x^2) + Df(x)x)]\} \geq \alpha\|x\|, \quad x \in E,$$

then f is said to be quasi-convex of type B and order α.

When $\alpha = 0$, Definition 1.1 is the definition of the quasi-convex mapping of type B, which was introduced by Roper, Suffridge [**13**] and Gong [**2**]; when $X = \mathbf{C}$, Definition 1.1 is the same as the definition of the normalized univalent convex function of order α in one complex variable.

Definition 1.2 [**2, 4**]. Let $f \in H(E)$. It is said that f is k-fold symmetric if $\exp(-(2\pi i)/k))f(e^{(2\pi i)/k}x) = f(x)$ for all $x \in E$, where $k \in \mathbf{N}$ and $i = \sqrt{-1}$.

Definition 1.3 [**6**]. Suppose Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \ldots, D^{k-1}f(0) = 0$, but $D^kf(0) \neq 0$, where $k \in \mathbf{N}$.

The definition is the same as in the case $X = \mathbf{C}$. According to Definitions 1.2 and 1.3, it is obvious that $x = 0$ is a zero of order $k+1(k \in \mathbf{N})$ of $f(x) - x$ if f is a k-fold symmetric normalized holomorphic mapping $f(x)(f(x) \not\equiv x)$ defined on E. However, the converse does not hold.

Let $S(E)$ be the set of all normalized biholomorphic mappings on E, and let $K(E)$ be the set of all normalized biholomorphic convex mappings on E. Let $QK_B^\alpha(E)(0 \leq \alpha < 1)$ be the set of all quasi-convex mappings of type B and order α on E, and let $K_\alpha(U)(0 \leq \alpha < 1)$ be the set of all normalized univalent convex functions of order α on the unit disk U.

In one complex variable, it is well known that $K_\alpha(U)$, $0 \leq \alpha < 1$, is an important subclass of univalent functions; there are a few beautiful and significant results for them on the unit disk U (for instance, Theorems A and B). A natural question arises as to how to attain the generalizations of these results in several complex variables. The object of this paper is to answer this question.

The paper is organized as follows. In Section 2, a sufficient condition for quasi-convex mappings of type B and order α on the unit ball E in a complex Banach space is provided. In Section 3, sharp estimations of all homogeneous expansions are established for quasi-convex mappings of type B and order α defined on the open unit polydisk U^n in $\mathbf{C}^n$, which satisfy some conditions.

2. The construction of quasi-convex mappings of order α.

The following lemmas are given to obtain the desired theorem in this section.

Lemma 2.1. *Suppose that $\alpha \in [0,1)$, and $f \in S(U)$. Then*

$$F(x) = \frac{f(T_u(x))}{T_u(x)} x \in QK_B^{\alpha}(E)$$

if and only if $f \in K_\alpha(U)$, where $\|u\| = 1$ and

$$\left.\frac{f(T_u(x))}{T_u(x)}\right|_{T_u(x)=0} = 1.$$

Proof. Denote

$$g(x) = \frac{f(T_u(x))}{T_u(x)},$$

since $F(x) = g(x)x$, we obtain

$$DF(x)\xi = (Dg(x)\xi)x + g(x)\xi, \quad \xi \in X.$$

Also, since $f \in S(U)$, we have

$$\frac{f'(T_u(x))T_u(x)}{f(T_u(x))} \neq 0, \quad x \in E.$$

Straightforward calculation yields

$$\frac{Dg(x)x}{g(x)} = \frac{f'(T_u(x))T_u(x)}{f(T_u(x))} - 1.$$

Hence, $(Dg(x)x)/(g(x)) + 1 \neq 0$. It is not difficult to verify that

$$DF(x)\left[\frac{1}{g(x)}\left(\xi - \frac{(Dg(x)\xi)x}{g(x)+Dg(x)x}\right)\right] = \xi, \quad \xi \in X.$$

Thus,

$$(DF(x))^{-1}\xi = \frac{1}{g(x)}\left(\xi - \frac{(Dg(x)\xi)x}{g(x)+Dg(x)x}\right), \quad \xi \in X.$$

Hence $F(x)$ is a normalized locally biholomorphic mapping on E. Note that $DF(x)x = f'(T_u(x))x$, we have

$$D^2F(x)(x^2)+DF(x)x = D(DF(x)x)x = f''(T_u(x))T_u(x)x + f'(T_u(x))x,$$

which implies that $D^2F(x)(x^2) = f''(T_u(x))T_u(x)x$. It follows that

$$\begin{aligned}
&(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)x)\\
&= x + (DF(x))^{-1}(D^2F(x)(x^2))\\
&= x + \frac{1}{g(x)}\left(f''(T_u(x))T_u(x)x - \frac{Dg(x)x}{g(x)+Dg(x)x} f''(T_u(x))T_u(x)x\right),\\
&\qquad x \in E.
\end{aligned}$$

Also,

$$\frac{g(x)}{g(x)+Dg(x)x} = \frac{f(T_u(x))}{f'(T_u(x))T_u(x)},$$

it yields that

$$(2.1)\quad \begin{aligned}&(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)x)\\ &= (1 + \frac{f''(T_u(x))T_u(x)}{f'(T_u(x))})x, \quad x \in E.\end{aligned}$$

Thus, by (2.1), we have

$$\begin{aligned}
&\Re e\,\{T_x[(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)x)]\} \geq \alpha\|x\|\\
&\Longleftrightarrow \Re e\left(1 + \frac{f''(T_u(x))T_u(x)}{f'(T_u(x))}\right) \geq \alpha.
\end{aligned}$$

Hence, by Definition 1.1, we obtain that $F \in QK_B^{\alpha}(E)$ if and only if $f \in K_{\alpha}(U)$. This completes the proof. □

The following lemma can be easily proved.

Lemma 2.2. *Suppose that* $\alpha \in [0,1)$, *and* f *is a normalized locally biholomorphic mapping on* U^n. *Then* $f \in QK_B^{\alpha}(U^n)$ *if and only if*

$$\Re e\, \frac{g_j(z)}{z_j} \geq \alpha, \quad z = (z_1, \dots, z_n)' \in U^n,$$

where $g(z) = (g_1(z), \dots, g_n(z))' = (Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)z)$, $z \in U^n$ *is a column vector in* $\mathbf{C}^n$ *and* j *satisfies* $|z_j| = \|z\| = \max_{1 \leq k \leq n}\{|z_k|\}$.

Theorem 2.1. *Suppose that* $\alpha \in [0,1)$. *If* $f \in H(E)$, $f(0) = 0$, $Df(0) = I$, *and*

$$\sum_{m=2}^{\infty} \frac{m(m-\alpha)\|D^m f(0)\|}{m!} \leq 1 - \alpha,$$

where $\|D^m f(0)\| = \sup_{\|x^{(k)}\|=1, 1 \leq k \leq m} \|D^m f(0)(x^{(1)}, x^{(2)}, \dots, x^{(m)})\|$. *Then* $f \in QK_B^{\alpha}(E)$.

In particular, when $\alpha = 0$, then f is a quasi-convex mapping of type B on E.

Proof. First, since

$$f(x) = x + \sum_{m=2}^{\infty} \frac{D^m f(0)(x^m)}{m!} \in H(E),$$

we have

$$Df(x) = I + \sum_{m=2}^{\infty} \frac{m D^m f(0)(x^{m-1}, .)}{m!}. \tag{2.2}$$

Also, since

$$\sum_{m=2}^{\infty} \frac{m(m-\alpha)\|D^m f(0)\|}{m!} \le 1-\alpha,$$

from (2.2), we obtain

$$\begin{aligned}\|Df(x)-I\| &\le \sum_{m=2}^{\infty} \frac{m\|D^m f(0)\|\|x\|^{m-1}}{m!}\\ &\le \frac{1}{2-\alpha}\sum_{m=2}^{\infty} \frac{m(m-\alpha)\|D^m f(0)\|}{m!}\\ &\le \frac{1-\alpha}{2-\alpha} < 1.\end{aligned}$$

Hence we obtain that $Df(x) = I-(I-Df(x))$ is an invertible linear operator, see [**14**, page 192], and

$$\begin{aligned}\|(Df(x))^{-1}\| &\le \frac{1}{1-\|I-Df(x)\|}\\ &\le \frac{1}{1-\sum_{m=2}^{\infty}(m\|D^m f(0)\|\|x\|^{m-1})/m!}.\end{aligned} \tag{2.3}$$

Next,

$$\begin{aligned}\|D^2 f(x)(x^2)\| &= \|\sum_{m=2}^{\infty} \frac{m(m-1)}{m!} D^m f(0)(x^m)\|\\ &\le \sum_{m=2}^{\infty} \frac{m(m-\alpha+\alpha-1)}{m!}\|D^m f(0)\|\|x\|^m\\ &\le (1-\alpha)\|x\|^2\Big(1-\sum_{m=2}^{\infty}\|D^m f(0)\|\|x\|^{m-2}\Big).\end{aligned} \tag{2.4}$$

Hence, from (2.3) and (2.4), we obtain

$$\begin{aligned}&\Re e\,\{T_x[(Df(x))^{-1}(D^2 f(x)(x^2)+Df(x)x)]\} - \alpha\|x\|\\ &\ge (1-\alpha)\|x\| - |T_x[(Df(x))^{-1}D^2 f(x)(x^2)]|\\ &\ge (1-\alpha)\|x\| - \|(Df(x))^{-1}D^2 f(x)(x^2)\|\\ &\ge (1-\alpha)\|x\| - \frac{(1-\alpha)\|x\|^2(1-\sum_{m=2}^{\infty}(m\|D^m f(0)\|\|x\|^{m-2})/m!)}{1-\sum_{m=2}^{\infty}(m\|D^m f(0)\|\|x\|^{m-1})/m!}\\ &\ge (1-\alpha)\|x\|(1-\|x\|) \ge 0.\end{aligned}$$

Thus, by Definition 1.1, we deduce that $f \in QK_B^\alpha(E)$. This completes the proof. □

Remark 2.1. (1) Theorem 2.1 tells us that

$$f(x) = x + \sum_{m=2}^{\infty} (D^m f(0)(x^m))/m! \in QK_B^\alpha(E)$$

when the coefficients $(\|D^m f(0)\|)/m!$, $m = 2, 3, \dots$, are sufficiently small.

(2) Setting $X = \mathbf{C}$, $E = U$ in Theorem 2.1; we get the corresponding result for the normalized univalent convex function of order α on the unit disk in $\mathbf{C}$ of [**3**]. Furthermore, by setting $X = \mathbf{C}^n$, $E = U^n$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Suppose that* $\alpha \in [0, 1)$*. If* $f \in H(U^n)$*,* $f(0) = 0$*,* $Df(0) = I$*, and*

$$\sum_{m=2}^{\infty} \frac{m(m-\alpha)\|D^m f(0)\|}{m!} \le 1 - \alpha,$$

where $\|D^m f(0)\| = \sup_{\|z^{(k)}\|=1, 1 \le k \le m} \|D^m f(0)(z^{(1)}, z^{(2)}, \dots, z^{(m)})\|$*. Then* $f \in QK_B^\alpha(U^n)$*.*

Example 2.1. Suppose that $\alpha \in [0, 1)$. If

$$\sum_{k=2}^{n} |a_{km}| \le (1-\alpha)/(m(m-1)) \text{ for } m = 2, 3, \dots ,$$

then

$$f(z) = \Big(z_1 + \sum_{k=2}^{n} a_{km} z_k^m, z_2, \dots , z_n\Big)' \in QK_B^\alpha(U^n) \text{ for } m = 2, 3, \dots .$$

Proof. It is obvious that f is a normalized biholomorphic mapping on U^n. Simple computation yields

$$(Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)z) = \Big(z_1 + (m^2 - m) \sum_{k=2}^{n} a_{km} z_k^m, z_2, \dots , z_n\Big)'.$$

If there exists a j, $2 \le j \le n$, such that $|z_j| \ge |z_1|$, then we have

$$\Re e \frac{g_j(z)}{z_j} - \alpha = \Re e(1-\alpha) > 0, \tag{2.5}$$

where $g(z) = (g_1(z), \dots, g_j(z), \dots, g_n(z))' = (Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)z)$.

If $|z_k| < |z_1|$ for all $k = 2, \dots, n$, then we obtain

$$\begin{aligned} \Re e \frac{g_1(z)}{z_1} - \alpha &= \Re e \left[1 + (m^2 - m) \sum_{k=2}^{n} a_{km} \frac{z_k^m}{z_1}\right] - \alpha \\ &\ge 1 - \alpha - (m^2 - m) \sum_{k=2}^{n} |a_{km}| \ge 0. \end{aligned} \tag{2.6}$$

Hence by Lemma 2.2, (2.5) and (2.6), we conclude that

$$f(z) = \left(z_1 + \sum_{k=2}^{n} a_{km} z_k^m, z_2, \dots, z_n\right)' \in QK_B^{\alpha}(U^n) \text{ for } m = 2, 3, \dots,$$

and the proof is complete. □

In view of Example 2.1, it is easy to know that $QK_B^{\alpha}(U^n) \subset K(U^n)$ is invalid.

Example 2.2. Suppose that $\alpha \in [0,1)$. If

$$\begin{aligned} f(z) = \bigg(\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}, z_2 \int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}} / z_1, \dots, \\ z_n \int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}} / z_1 \bigg)', \end{aligned}$$

then $f(z) \in QK_B^{\alpha}(U^n)$, where the power function is chosen such that

$$(1-t^k)^{(2(1-\alpha))/k} = 1 - \frac{2(1-\alpha)}{k} t^k + \cdots. \tag{2.7}$$

Proof. First, it is obvious that

$$\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}} \in K_\alpha(U).$$

By Lemma 2.1, we deduce that

$$z\int_0^{T_u(z)} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}} /T_u(z) \in QK_B^\alpha(U^n),$$

where $\|u\|=1$.

By taking $u=(1,0,\dots,0)'$, notice that $T_u=(1,0,\dots,0)$, we have $T_u(z)=z_1$; thus,

$$f(z)=\Big(\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}, z_2\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1,\dots, \\ z_n\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1\Big)' \in QK_B^\alpha(U^n),$$

and the proof is complete. □

3. Sharp estimations of all homogeneous expansions for quasi-convex mappings of order α. In order to obtain the desired theorems in this section, we need to establish the following lemmas. Using a method similar to that in [**10**], we can derive the inequalities asserted by Lemma 3.1 below.

Lemma 3.1. *Suppose that $\alpha\in[0,1)$, and*

$$g(z)=(g_1(z),g_2(z),\dots,g_n(z))' \in H(U^n),\ g(0)=0, Dg(0)=I.$$

If $\Re e\,(g_j(z))/z_j \ge \alpha$, $z\in U^n$, where $|z_j|=\|z\|=\max_{1\le k\le n}\{|z_k|\}$, then

$$\frac{\|D^m g(0)(z^m)\|}{m!} \le 2(1-\alpha)\|z\|^m, \quad z\in U^n,\ m=2,3,\dots.$$

Lemma 3.2. *If $f(z)$ is a k-fold symmetric normalized locally biholomorphic mapping on U^n, and $g(z)=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z) \in H(U^n)$, then*

$$(k+1)k\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!} = \frac{D^{k+1}g(0)(z^{k+1})}{(k+1)!},$$

and

$$\begin{aligned}
&(qk+1)qk\frac{D^{qk+1}f(0)(z^{qk+1})}{(qk+1)!}\\
&=\frac{D^{qk+1}g(0)(z^{qk+1})}{(qk+1)!}+(k+1)\\
&\quad\times\frac{D^{k+1}f(0)(z^k,(D^{(q-1)k+1}g(0)(z^{(q-1)k+1}))/(((q-1)k+1)!))}{(k+1)!}\\
&\quad+\cdots+((q-1)k+1)\\
&\quad\times\frac{D^{(q-1)k+1}f(0)(z^{(q-1)k},(D^{k+1}g(0)(z^{k+1}))/((k+1)!))}{((q-1)k+1)!},
\end{aligned}$$

for $z\in U^n$, $q=2,3,\dots$.

Proof. Since $g(z)=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z)\in H(U^n)$, we have

$$D^2f(z)(z^2)+Df(z)z=Df(z)g(z).$$

Notice that

$$f(z)=z+\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!}+\cdots+\frac{D^{qk+1}f(0)(z^{qk+1})}{(qk+1)!}+\cdots,$$

we obtain

$$\begin{aligned}
&z+(k+1)^2\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!}+\cdots+(qk+1)^2\frac{D^{qk+1}f(0)(z^{qk+1})}{(qk+1)!}+\cdots\\
&=\Big(I+(k+1)\frac{D^{k+1}f(0)(z^k,.)}{(k+1)!}+\cdots+(qk+1)\frac{D^{qk+1}f(0)(z^{qk},.)}{(qk+1)!}+\cdots\Big)\\
&\quad\cdot\Big(Dg(0)z+\frac{D^2g(0)(z^2)}{2!}+\cdots+\frac{D^{qk+1}g(0)(z^{qk+1})}{(qk+1)!}\\
&\qquad+\cdots+\frac{D^mg(0)(z^m)}{m!}+\cdots\Big).
\end{aligned}$$

Comparing with the homogeneous expansion of both sides of the above equality, we obtain

$$\begin{aligned}
&Dg(0)z=z,\ D^{qk+l}g(0)(z^{qk+l})=0,\\
&l=2,\dots,k(k\geq 2),\ q=0,1,2,\dots,
\end{aligned}$$

and

$$(k+1)^2\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!} = \frac{D^{k+1}g(0)(z^{k+1})}{(k+1)!} + (k+1)\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!},$$

and

$$\begin{aligned}&(qk+1)^2\frac{D^{qk+1}f(0)(z^{qk+1})}{(qk+1)!}\\ &\quad= \frac{D^{qk+1}g(0)(z^{qk+1})}{(qk+1)!} + (k+1)\\ &\qquad\times \frac{D^{k+1}f(0)(z^k, (D^{(q-1)k+1}g(0)(z^{(q-1)k+1}))/(((q-1)k+1)!))}{(k+1)!}\\ &\qquad+\cdots+((q-1)k+1)\\ &\qquad\times \frac{D^{(q-1)k+1}f(0)(z^{(q-1)k}, (D^{k+1}g(0)(z^{k+1}))/((k+1)!))}{((q-1)k+1)!}\\ &\qquad+(qk+1)\frac{D^{qk+1}f(0)(z^{qk+1})}{(qk+1)!}, \quad z\in U^n,\ q=2,3,\dots .\end{aligned}$$

Hence, the desired results hold true, and the proof is complete. □

Lemma 3.3. *Suppose that $f(z)$ is a normalized locally biholomorphic mapping on U^n. If $z=0$ is a zero of order $k+1 (k\in \mathbf{N})$ of $f(z)-z$, and $g(z)=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z)\in H(U^n)$, then for each $z\in U^n$, we have*

$$\begin{aligned}&m(m-1)\frac{D^m f(0)(z^m)}{m!}\\ &= \begin{cases} (D^m g(0)(z^m))/m! & m=k+1,\dots,2k;\\ (D^m g(0)(z^m))/m! \\ \quad +(k+1)[(D^{k+1}f(0)(z^k, (D^{m-k}g(0)(z^{m-k}))/((m-k)!)))/(k+1)!] & m=2k+1.\end{cases}\end{aligned}$$

Proof. Since $g(z)=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z)\in H(U^n)$, we have

$$D^2f(z)(z^2)+Df(z)z = Df(z)g(z).$$

Notice that

$$f(z) = z + \frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!} + \cdots + \frac{D^m f(0)(z^m)}{m!} + \cdots,$$

and we obtain

$$\begin{aligned}
&z + (k+1)^2 \frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!} + \cdots + m^2 \frac{D^m f(0)(z^m)}{m!} + \cdots \\
&= \Big(I + (k+1)\frac{D^{k+1}f(0)(z^k, .)}{(k+1)!} + \cdots + m\frac{D^m f(0)(z^{m-1}, .)}{m!} + \cdots\Big) \\
&\quad \cdot \Big(Dg(0)z + \frac{D^2 g(0)(z^2)}{2!} + \cdots + \frac{D^{k+1}g(0)(z^{k+1})}{(k+1)!} \\
&\qquad\qquad + \cdots + \frac{D^m g(0)(z^m)}{m!} + \cdots\Big).
\end{aligned}$$

Comparing with the homogeneous expansion of both sides of the above equality, we obtain

$$Dg(0)z = z, \qquad D^l g(0)(z^l) = 0, \quad l = 2, 3, \ldots, k, \ k \geq 2,$$

and

$$m^2 \frac{D^m f(0)(z^m)}{m!} = \begin{cases} (D^m g(0)(z^m))/(m!) + m(D^m f(0)(z^m))/m! \\ \qquad\qquad m = k+1, \ldots, 2k; \\ (D^m g(0)(z^m))/(m!) \\ \quad +(k+1)[(D^{k+1}f(0)(z^k, (D^{m-k}g(0)(z^{m-k}))/((m-k)!)))/(k+1)!] \\ \quad +m(D^m f(0)(z^m))/m!, \qquad m = 2k+1. \end{cases}$$

Hence, the desired results hold true, and the proof is complete. □

Now the main results in this section can be proved as follows.

Theorem 3.1. *Suppose that* $\alpha \in [0, 1)$. *If* f *is a* k*-fold symmetric quasi-convex mapping of type* B *and order* α, *and*

$$D^{sk+1}f_p(0)(z^{sk+1}) = z_p\bigg(\sum_{l=1}^{n} a_{pl(sk+1)} z_l^{sk}\bigg),$$
$$z \in U^n, \quad p = 1, 2, \ldots, n, \ s = 1, 2, \ldots,$$

where

$$a_{pl(sk+1)} = \frac{\partial^{sk+1} f_p(0)}{\partial z_p \partial z_l^{sk}}, \quad p, l = 1, 2, \dots, n, \; s = 1, 2, \dots,$$

then

$$\frac{\|D^{sk+1} f(0)(z^{sk+1})\|}{(sk+1)!} \le \frac{\prod_{m=1}^{s}((m-1)k + 2 - 2\alpha)}{(sk+1) \cdot s! k^s} \|z\|^{sk+1}, \quad z \in U^n, \quad s = 1, 2, \dots. \tag{3.1}$$

All of the above estimations are sharp.

Proof. Fix $z \in U^n \setminus \{0\}$, set $z_0 = z/\|z\|$. Since f is a k-fold symmetric quasi-convex mapping of type B and order α, by Lemma 2.2, Lemma 3.1 (when $m = k + 1$) and Lemma 3.2 (when $q = 1$), we have

$$\frac{\|D^{k+1} f(0)(z^{k+1})\|}{(k+1)!} \le \frac{2 - 2\alpha}{(k+1)k} \|z\|^{k+1}, \quad z \in U^n,$$

which shows that (3.1) holds true for $s = 1$. Suppose now that (3.1) holds true for $1 \le s \le q$ for some integer $q \ge 2$. Then

$$\frac{\|D^{sk+1} f(0)(z^{sk+1})\|}{(sk+1)!} \le \frac{\prod_{m=1}^{s}((m-1)k + 2 - 2\alpha)}{(sk+1) \cdot s! k^s} \|z\|^{sk+1}, \quad z \in U^n, \quad s = 1, 2, \dots, q. \tag{3.2}$$

Thus, by (3.2), we have

$$\|D^{sk+1} f(0)(z_0^{sk+1})\| \le \frac{\prod_{m=1}^{s}((m-1)k + 2 - 2\alpha)}{(sk+1) \cdot s! k^s} \cdot (sk+1)!. \tag{3.3}$$

Also

$$D^{sk+1} f_p(0)(z^{sk+1}) = z_p \left(\sum_{l=1}^{n} a_{pl(sk+1)} z_l^{sk} \right), \quad z \in U^n, \quad p = 1, 2, \dots, n, \; s = 1, 2, \dots, q,$$

where $a_{pl(sk+1)} = (\partial^{sk+1} f_p(0))/(\partial z_p \partial z_l^{sk})$, $p,l = 1,2,\dots,n$, $s = 1,2,\dots,q$. Hence, from (3.3), we have

$$\left|\frac{z_j}{\|z\|}\left(\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk}\right)\right|$$
$$= \left|\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk}\right| \le \frac{\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!,$$

where $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$. Especially, when $z_l = e^{-(i\alpha_l)/(sk)} \times \|z\|$ with $a_l = \arg a_{jl(sk+1)}$, $l = 1,2,\dots,n$, we obtain

$$\sum_{l=1}^{n} |a_{jl(sk+1)}| \le \frac{\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!. \tag{3.4}$$

Denote

$$w = \frac{D^{(q-s+1)k+1} g(0)(z^{(q-s+1)k+1})}{((q-s+1)k+1)!}, \quad s = 1,2,\dots,q.$$

Notice that

$$D^{sk+1} f_p(0)(z^{sk+1}) = z_p\left(\sum_{l=1}^{n} a_{pl(sk+1)} z_l^{sk}\right),$$
$$z \in U^n,\ p = 1,2,\dots,n,\ s = 1,2,\dots,q.$$

For any λ with $|\lambda| < 1/(2-2\alpha)$, we have

$$\frac{sk+1}{2^{sk+1}} D^{sk+1} f_j(0)(z^{sk}, w)$$
$$= \frac{d}{d\lambda}\left[D^{sk+1} f_j(0)\left(\underbrace{\frac{z+\lambda w}{2}, \frac{z+\lambda w}{2}, \cdots, \frac{z+\lambda w}{2}}_{sk+1}\right)\right]|_{\lambda=0}$$
$$= \frac{d}{d\lambda}\left[\frac{z_j+\lambda w_j}{2}\left(\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l+\lambda w_l}{2}\right)^{sk}\right)\right]|_{\lambda=0}$$
$$= \frac{sk\cdot z_j(\sum_{l=1}^{n} a_{jl(sk+1)} z_l^{sk-1} w_l) + w_j(\sum_{l=1}^{n} a_{jl(sk+1)} z_l^{sk})}{2^{sk+1}},$$

where j satisfies $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$, that is,

$$\begin{aligned}(3.5)\quad & D^{sk+1}f_j(0)(z^{sk}, w)\\ &= \frac{1}{sk+1}\left[sk\cdot z_j\left(\sum_{l=1}^{n} a_{jl(sk+1)} z_l^{sk-1} w_l\right) + w_j\left(\sum_{l=1}^{n} a_{jl(sk+1)} z_l^{sk}\right)\right].\end{aligned}$$

Thus, by Lemma 3.1, (3.4) and (3.5), we obtain

$$\begin{aligned}
&\left|D^{sk+1}f_j(0)\left(z_0^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right| = \frac{1}{sk+1}\\
&\times\left|sk\cdot\frac{z_j}{\|z\|}\left(\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk-1}\frac{D^{(q-s+1)k+1}g_l(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right.\\
&\qquad\left.+\ \frac{D^{(q-s+1)k+1}g_j(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\left(\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk}\right)\right|\\
&\le \frac{1}{sk+1}\left[sk\left|\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk-1}\frac{D^{(q-s+1)k+1}g_l(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right|\right.\\
&\qquad\left.+\ \frac{|D^{(q-s+1)k+1}g_j(0)(z_0^{(q-s+1)k+1})|}{((q-s+1)k+1)!}\left|\sum_{l=1}^{n} a_{jl(sk+1)}\left(\frac{z_l}{\|z\|}\right)^{sk}\right|\right]\\
&\le \frac{1}{sk+1}\left[sk\sum_{l=1}^{n} |a_{jl(sk+1)}|\left(\frac{|z_l|}{\|z\|}\right)^{sk-1}\right.\\
&\qquad\times\ \frac{|D^{(q-s+1)k+1}g_l(0)(z_0^{(q-s+1)k+1})|}{((q-s+1)k+1)!}\\
&\qquad\left.+\ \frac{|D^{(q-s+1)k+1}g_j(0)(z_0^{(q-s+1)k+1})|}{((q-s+1)k+1)!}\sum_{l=1}^{n} |a_{jl(sk+1)}|\left(\frac{|z_l|}{\|z\|}\right)^{sk}\right]\\
&\le \frac{1}{sk+1}\left[sk\cdot\frac{\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!(2-2\alpha)\right.\\
&\qquad\left.+\ (2-2\alpha)\cdot\frac{\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!\right]\\
&= \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!.
\end{aligned}$$

This implies that

$$\begin{aligned}(3.6)\quad &\left|D^{sk+1}f_j(0)\left(z_0^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right| \\ &\leq \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!, \quad z_0\in\partial U^n.\end{aligned}$$

Especially, when $z_0 \in \partial_0 U^n$, from (3.6), we have

$$\begin{aligned}(3.7)\quad &\left|D^{sk+1}f_p(0)\left(z_0^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right| \\ &\leq \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!, \quad p=1,2,\ldots,n.\end{aligned}$$

Because

$$D^{sk+1}f_p(0)\left(z^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\in H(\overline{U^n}), \quad p=1,2,\ldots,n,$$

by the maximum modulus principle of holomorphic functions on the unit polydisk and (3.7), we have

$$\begin{aligned}&\left|D^{sk+1}f_p(0)\left(z_0^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right| \\ &\leq \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!, \\ &\qquad z_0\in\partial U^n, \quad p=1,2,\ldots,n.\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}&\left\|D^{sk+1}f(0)\left(z_0^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z_0^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right\| \\ &\leq \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!,\end{aligned}$$

which implies that

$$\left\| D^{sk+1}f(0)\left(z^{sk}, \frac{D^{(q-s+1)k+1}g(0)(z^{(q-s+1)k+1})}{((q-s+1)k+1)!}\right)\right\| \le \frac{(2-2\alpha)\prod_{m=1}^{s}((m-1)k+2-2\alpha)}{(sk+1)\cdot s!k^s}\cdot(sk+1)!\|z\|^{(q+1)k+1}. \tag{3.8}$$

According to Lemma 3.2 and (3.8), we obtain

$$\begin{aligned}
&\frac{((q+1)k+1)(q+1)k\|D^{(q+1)k+1}f(0)(z^{(q+1)k+1})\|}{((q+1)k+1)!}\\
&\le \frac{\|D^{(q+1)k+1}g(0)(z^{(q+1)k+1})\|}{((q+1)k+1)!}\\
&\quad + (k+1)\frac{\|D^{k+1}f(0)(z^k,[D^{qk+1}g(0)(z^{qk+1})/(qk+1)!])\|}{(k+1)!}\\
&\quad + \cdots + ((q-1)k+1)\\
&\quad \times \frac{\|D^{(q-1)k+1}f(0)(z^{(q-1)k},[D^{2k+1}g(0)(z^{2k+1})/(2k+1)!])\|}{((q-1)k+1)!}\\
&\quad + (qk+1)\frac{\|D^{qk+1}f(0)(z^{qk},[D^{k+1}g(0)(z^{k+1})/(k+1)!])\|}{(qk+1)!}\\
&\le (2-2\alpha)\|z\|^{(q+1)k+1} + (k+1)\frac{(2-2\alpha)(2-2\alpha)}{(k+1)k}\|z\|^{(q+1)k+1}\\
&\quad + \cdots + ((q-1)k+1)\\
&\quad \times \frac{(2-2\alpha)\prod_{m=1}^{q-1}((m-1)k+2-2\alpha)}{((q-1)k+1)\cdot(q-1)!k^{q-1}}\|z\|^{(q+1)k+1}\\
&\quad + (qk+1)\frac{(2-2\alpha)\prod_{m=1}^{q}((m-1)k+2-2\alpha)}{(qk+1)\cdot q!k^q}\|z\|^{(q+1)k+1}\\
&= \Bigg[2-2\alpha+\frac{(2-2\alpha)(2-2\alpha)}{k}+\cdots\\
&\quad + \frac{(2-2\alpha)\prod_{m=1}^{q-1}((m-1)k+2-2\alpha)}{(q-1)!k^{q-1}}\\
&\qquad\qquad + \frac{(2-2\alpha)\prod_{m=1}^{q}((m-1)k+2-2\alpha)}{q!k^q}\Bigg]\|z\|^{(q+1)k+1}\\
&= \frac{\prod_{m=1}^{q+1}((m-1)k+2-2\alpha)}{q!k^q}\|z\|^{(q+1)k+1}.
\end{aligned}$$

This implies that

$$\frac{\|D^{(q+1)k+1}f(0)(z^{(q+1)k+1})\|}{((q+1)k+1)!} \le \frac{\prod_{m=1}^{q+1}((m-1)k+2-2\alpha)}{((q+1)k+1)\cdot(q+1)!k^{q+1}}\|z\|^{(q+1)k+1}, \quad z \in U^n.$$

Therefore (3.1) holds true for $s = q+1$.

Finally, we verify that the inequalities (3.1) are sharp. By Example 2.2, it is easy to verify that

$$f(z) = \left(\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}},\ z_2\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1, \dots, z_n\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1\right)', \quad z \in U^n$$

satisfies the condition of Theorem 3.1, where the power function is chosen such that (2.7) holds.

Taking $z = (r, 0, \dots, 0)'$, $0 \le r < 1$, then

$$\frac{\|D^{qk+1}f(0)(z^{qk+1})\|}{(qk+1)!} = \frac{\prod_{m=1}^{q}((m-1)k+2-2\alpha)}{(qk+1)\cdot q!k^q} r^{qk+1}, q = 1, 2, \dots .$$

Hence, the estimations (3.1) of Theorem 3.1 are sharp, and the proof is complete. □

Remark 3.1. Setting $k = 1$, $\alpha = 0$ in Theorem 3.1, we get Theorem 3.1 in [**9**]; setting $X = \mathbf{C}$, $E = U$, $k = 1$ in Theorem 3.1, we obtain the corresponding results of normalized univalent convex function of order α, $0 \le \alpha < 1$, in [**12**].

By Theorem 3.1, making use of a method similar to that in [**9**], the following corollaries can be proved (the details of the proof are omitted here).

Corollary 3.1. *Suppose that $\alpha \in [0, 1)$. If f is a k-fold symmetric quasi-convex mapping of type B and order α, and*

$$D^{qk+1}f_p(0)(z^{qk+1}) = z_p\left(\sum_{l=1}^{n} a_{pl(qk+1)}z_l^{qk}\right), \quad z \in U^n,$$
$$p = 1, 2, \dots, n,\ q = 1, 2, \dots ,$$

where

$$a_{pl(qk+1)} = \frac{\partial^{qk+1} f_p(0)}{\partial z_p \partial z_l^{qk}}, \quad p, l = 1, 2, \dots, n, \ q = 1, 2, \dots,$$

then

$$\|f(z)\| \le \int_0^{\|z\|} \frac{dt}{(1 - t^k)^{(2(1-\alpha))/k}}, \quad z \in U^n.$$

The above estimation is sharp.

Corollary 3.2. *Suppose that $\alpha \in [0, 1)$. If f is a k-fold symmetric quasi-convex mapping of type B and order α, and*

$$D^{qk+1} f_p(0)(z^{qk+1}) = z_p \left(\sum_{l=1}^{n} a_{pl(qk+1)} z_l^{qk} \right), \quad z \in U^n,$$
$$p = 1, 2, \dots, n, \ q = 1, 2, \dots,$$

where

$$a_{pl(qk+1)} = \frac{\partial^{qk+1} f_p(0)}{\partial z_p \partial z_l^{qk}}, \quad p, l = 1, 2, \dots, n, \ q = 1, 2, \dots,$$

then

$$\|Df(z)z\| \le \frac{\|z\|}{(1 - \|z\|^k)^{(2(1-\alpha))/k}}, \quad z \in U^n.$$

The above estimation is sharp.

By Example 2.2, it is easy to verify

$$f(z) = \left(\int_0^{z_1} \frac{dt}{(1 - t^k)^{(2(1-\alpha))/k}}, \frac{z_2}{z_1} \int_0^{z_1} \frac{dt}{(1 - t^k)^{(2(1-\alpha))/k}}, \dots, \right.$$
$$\left. \frac{z_n}{z_1} \int_0^{z_1} \frac{dt}{(1 - t^k)^{(2(1-\alpha))/k}} \right)', \quad z \in U^n$$

satisfies the condition of Corollaries 3.1 and 3.2, where the power function is chosen such that (2.7) holds.

Taking $z=(r,0,\dots,0)'$, $0\le r<1$, we have

$$\|f(z)\|=\int_0^r \frac{dt}{(1-t^k)^{(2(1-\alpha)/k}}, \qquad \|Df(z)z\|=\frac{r}{(1-r^k)^{(2(1-\alpha))/k}}.$$

Hence, the estimations of Corollaries 3.1 and 3.2 are sharp.

Remark 3.2. Setting $k=1$, $\alpha=0$ in Corollaries 3.1 and 3.2, we get Corollaries 3.1 and 3.2 of [**9**]; setting $X=\mathbf{C}$, $E=U$, $k=1$ in Corollaries 3.1 and 3.2, we obtain the corresponding results of normalized univalent convex function of order α, $0\le\alpha<1$, in [**3**].

Theorem 3.2. *Suppose that $\alpha\in[0,1)$, $k\in\mathbf{N}$ and $f\in QK_B^\alpha(U^n)$. If $z=0$ is a zero of order $k+1$ of $f(z)-z$, and*

$$D^m f_p(0)(z^m)=z_p\bigg(\sum_{l=1}^n a_{plm}z_l^{m-1}\bigg),$$
$$z\in U^n,\ p=1,2,\dots,n,\ m=k+1,k+2,\dots,2k+1,$$

where

$$a_{plm}=\frac{\partial^m f_p(0)}{\partial z_p\partial z_l^{m-1}},$$
$$p,l=1,2,\dots,n,\ m=k+1,k+2,\dots,2k+1,$$

then for all $z\in U^n$,

$$\frac{\|D^m f(0)(z^m)\|}{m!} \le \begin{cases} (2-2\alpha)/(m(m-1))\|z\|^m & m=k+1,k+2,\dots,2k;\\ [(2-2\alpha)(k+2-2\alpha)/m(m-1)k]\|z\|^m & m=2k+1.\end{cases} \tag{3.9}$$

When $m=k+1$ or $m=2k+1$, the above estimations are sharp.

Proof. When $m=k+1$, $k+2,\dots,2k$. By Lemmas 3.1 and 3.3, we have

$$\frac{\|D^m f(0)(z^m)\|}{m!}=\frac{1}{m(m-1)}\frac{\|D^m g(0)(z^m)\|}{m!}\le\frac{2-2\alpha}{m(m-1)}\|z\|^m,\ z\in U^n.$$

When $m = 2k+1$. By applying a method similar to the proof of Theorem 3.1, we can deduce that

$$\frac{\|D^{k+1}f(0)(z^k, [D^{m-k}g(0)(z^{m-k})/(m-k)!])\|}{(k+1)!} \le \frac{(2-2\alpha)^2}{(k+1)k}\|z\|^m, \quad z \in U^n,\ m = 2k+1.$$

Hence, it follows from Lemma 3.3 that

$$\begin{aligned}
&m(m-1)\frac{\|D^m f(0)(z^m)\|}{m!}\\
&= \left\|\frac{D^m g(0)(z^m)}{m!} + (k+1)\frac{D^{k+1}f(0)(z^k, [D^{m-k}g(0)(z^{m-k})/(m-k)!])}{(k+1)!}\right\|\\
&\le \frac{\|D^m g(0)(z^m)\|}{m!}\\
&\quad + (k+1)\frac{\|D^{k+1}f(0)(z^k, [D^{m-k}g(0)(z^{m-k})/(m-k)!])\|}{(k+1)!}\\
&\le (2-2\alpha)\|z\|^m + (k+1)\frac{(2-2\alpha)^2}{(k+1)k}\|z\|^m\\
&= \frac{(2-2\alpha)(k+2-2\alpha)}{k}\|z\|^m, \quad z \in U^n,\ m = 2k+1,
\end{aligned}$$

which implies the inequalities (3.9) hold true.

Finally, we verify the inequalities (3.9) are sharp for $m = k+1$ or $m = 2k+1$. By Example 2.2, it is easy to verify that

$$f(z) = \left(\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}},\ z_2\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1, \dots,\ z_n\int_0^{z_1} \frac{dt}{(1-t^k)^{(2(1-\alpha))/k}}/z_1\right)', \quad z \in U^n$$

satisfies the condition of Theorem 3.2, where the power function is chosen such that (2.7) holds.

Taking $z = (r, 0, \dots, 0)'$, $0 \le r < 1$, we have

$$\frac{\|D^m f(0)(z^m)\|}{m!} = \begin{cases} (2-2\alpha)/(m(m-1))r^m & m = k+1;\\ [(2-2\alpha)(k+2-2\alpha)]/(m(m-1)k)r^m & m = 2k+1.\end{cases}$$

Hence, the inequalities (3.9) of Theorem 3.2 are sharp for $m = k+1$ or $m = 2k+1$, and the proof is complete. $\square$

Remark 3.3. Setting $X = \mathbf{C}$, $E = U$ in Theorem 3.2, we get the corresponding result of a normalized univalent convex function of order α, $0 \le \alpha < 1$.

By Lemmas 3.1 and 3.3, the following corollary can be easily proved.

Corollary 3.3. *Suppose that $\alpha \in [0,1)$, $k \in \mathbf{N}$ and $f \in QK_B^\alpha(U^n)$. If $z = 0$ is a zero of order $k+1$ of $f(z) - z$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \le \frac{2-2\alpha}{m(m-1)}\|z\|^m,$$
$$z \in U^n, \quad m = k+1, k+2, \dots, 2k.$$

When $m = k+1$, the above estimation is sharp.

Remark 3.4. Setting $\alpha = 0$ in Corollary 3.3, we obtain the corresponding result of [**10**].

Corollary 3.4. *Suppose that $\alpha \in [0,1)$. If $f \in QK_B^\alpha(U^n)$, then*

$$\|(Df(z))^{-1}D^2 f(z)(z^2)\| \le \frac{(2-2\alpha)\|z\|^2}{1-\|z\|}, \quad z \in U^n.$$

The above estimation is sharp.

Proof. Set

$$g(z) = (Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)z).$$

From Lemma 3.1 and the hypothesis of Corollary 3.4, we obtain

$$\begin{aligned}\|(Df(z))^{-1}D^2f(z)(z^2)\| &= \left\| \sum_{m=2}^{\infty} \frac{D^m g(0)(z^m)}{m!} \right\| \\ &\le \sum_{m=2}^{\infty} \frac{\|D^m g(0)(z^m)\|}{m!} \\ &\le \sum_{m=2}^{\infty} (2-2\alpha)\|z\|^m \\ &= \frac{(2-2\alpha)\|z\|^2}{1-\|z\|}, \quad z \in U^n.\end{aligned}$$

Hence, the desired result holds true.

Finally, it is easy to verify that

$$\begin{aligned}f(z) = \bigg(\int_0^{z_1} \frac{dt}{(1-t)^{2(1-\alpha)}}, z_2 \frac{dt}{(1-t)^{2(1-\alpha)}}/z_1, \dots , \\ z_n \frac{dt}{(1-t)^{2(1-\alpha)}}/z_1 \bigg)', \quad z \in U^n\end{aligned}$$

satisfies the condition of Corollary 3.4, where the power function is chosen such that (2.7) holds.

Taking $z = (r, 0, \dots , 0)'$, $0 \le r < 1$, we have

$$\|(Df(z))^{-1}D^2f(z)(z^2)\| = \frac{(2-2\alpha)r^2}{1-r}.$$

Hence, the estimation of Corollary 3.4 is sharp. This completes the proof. □

Remark 3.5. Setting $X = \mathbf{C}$, $E = U$ in Corollary 3.4, we get the corresponding result of normalized univalent convex function of order α, $0 \le \alpha < 1$.

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