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FOUR-DIMENSIONAL LIE ALGEBRAS WITH A PARA-HYPERCOMPLEX STRUCTURE

NOVICA BLAŽIĆ AND SRDJAN VUKMIROVIĆ

Novica Blažić passed away on Monday, 10 October 2005
and this paper is dedicated to his memory

ABSTRACT. In this paper we classify four-dimensional real Lie algebras $\mathfrak{g}$ admitting an integrable, left invariant, para-hypercomplex structure. The equivalence classes of compatible structures are classified. The metric of split signature (2; 2), canonically determined by the para-hypercomplex structure, is very convenient in understanding the structure of $\mathfrak{g}$. Moreover, these structures provide many examples of left invariant metrics of anti-self-dual metric of split signature. Conformal geometry and the curvature of the canonical metric on the corresponding Lie groups are also discussed. For example, the holonomy algebras of this canonical metrics are determined.

1. Introduction. Invariant structures of complex or quaternionic type on Lie groups are important from a geometric view point as well as from algebraic view point. For example, Snow [**11**] and Ovando [**10**] classified the invariant complex structures on four-dimensional, solvable, simply-connected real Lie groups. Invariant hypercomplex structures on four-dimensional real Lie groups are classified by Barberis [**5**] (see Section 2 for details). There is the unique (up to a homothety) positive definite Hermitan metric associated with such a structure. Andrada and Salamon [**4**] have shown that any para-hypercomplex structure on a real Lie algebra $\mathfrak{g}$ rise to a hypercomplex structure on its complexification $\mathfrak{g}^{\mathbf{C}}$ (considered as a real Lie algebra). They referred to para-hypercomplex structure as complex product structure.

An additional interest in integrable hypercomplex and para-hypercomplex structure is provided by the fact that each of these structures

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implies the anti self-duality of a canonical metric. Lie groups with a positive definite, left-invariant, anti-self dual metrics, which are not conformally flat, were classified by de Smedt and Salamon [**7**].

We have three goals. The first one is to classify four-dimensional real Lie algebras $\mathfrak{g}$ which admit an integrable para-hypercomplex structure in order to describe the corresponding left invariant structures on Lie groups. Let us state the main theorem (proved in subsection 3.4).

Theorem 1.1. *The only* 4*-dimensional Lie algebras* $\mathfrak{g}$ *admitting an integrable para-hypercomplex structure are*:

$$\mathbf{R}^4, \mathbf{R} \oplus \mathfrak{s}l_2(\mathbf{R}), \mathbf{R} \oplus \mathfrak{r}_{3,1}, \mathbf{R} \oplus \mathfrak{h}_3,$$
$$\mathbf{R}^2 \oplus \mathfrak{aff}(\mathbf{R}), \mathfrak{d}_4, \mathfrak{d}_{4,\lambda}, \mathfrak{aff}(\mathbf{C}), \mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R}),$$

$\mathfrak{t}_{4,1,c}$, $\mathfrak{r}_{4,1}$ *and* $\mathfrak{h}_4$.

In this case the corresponding Hermitian pseudo-Riemannian metric determined by the para-hypercomplex structure is also unique up to a constant, but has to be of signature $(2,2)$. As noticed by Salamon this metric is anti self-dual (see also [**8**]).

Although the metric is not involved in the statement of the main theorem we use it to naturally define subclasses of the structures in terms of the signature of the induced metric on the commutator subalgebra $\mathfrak{g}'$ and center $Z(\mathfrak{g})$. In the proof we study these classes separately in details. Since metrics induced by compatible structures are isometric, we classified equivalence classes of structures up to a compatibility, which is our second goal. Moreover, in a few cases the equivalence of the structure in a given compatible class is also established. Explicit examples of all such classes are given in Section 5. The third goal is to study the geometry and the curvature of the corresponding left invariant metrics. The holonomy algebras of the constructed left-invariant metrics are computed.

Here is a brief outline of the paper. In Section 2 we first give necessary definitions and prove some basic properties of para-hypercomplex structures that we use later. Some of these properties are of general interest in study of these structures. In Section 3 we step-by-step prove Theorem 1.1. First, in subsection 3.1 we classify four-dimensional Lie algebras with a non-trivial center and admitting a para-hypercomplex structure. Further on, we suppose that algebra $\mathfrak{g}$ has a trivial center. In subsections 3.2 and 3.3 we classify solvable four-dimensional Lie algebras $\mathfrak{g}$ admitting a para-hypercomplex structure (Theorems 3.2, 3.3 and 3.4 depending on the dimension of the commutator subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$). In subsection 3.4 we prove Theorem 1.1 using previous classifications, and we provide examples of para-hypercomplex structures on the corresponding algebras. Our results are compared with the results of Barberis [**5**] in Section 4. In Section 5 we classify the *para-hypercomplex* structures up to equivalence. Finally, in Section 6 we study conformal geometry, curvature and holonomy of the metrics induced by the *para-hypercomplex* structures.

2. Preliminaries. Let us recall some standard notation of four-dimensional Lie algebras (see [**2**]):

- $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R}) : \ [e_1, e_2] = e_4, \ [e_2, e_4] = -e_1, \ [e_4, e_1] = e_2,$
- $\mathbf{R} \oplus \mathfrak{r}_{3,1} : \ [e_1, e_2] = e_2, \ [e_1, e_4] = e_4,$
- $\mathbf{R} \oplus \mathfrak{h}_3 : \ [e_1, e_2] = e_3,$
- $\mathbf{R}^2 \oplus \mathfrak{aff}(\mathbf{R}) : \ [e_1, e_2] = e_1,$
- $\mathfrak{d}_4 : \ [e_1, e_4] = e_3, \ [e_1, e_2] = e_1, \ [e_2, e_4] = e_4,$
- $\mathfrak{d}_{4,\lambda} : \ [e_4, e_3] = e_3, \ [e_1, e_2] = e_3, \ [e_4, e_1] = \lambda e_1, \ [e_4, e_2] = (1-\lambda) e_2,$
- $\mathfrak{aff}(\mathbf{C}) : \ [e_4, e_2] = e_2, \ [e_4, e_3] = e_3, \ [e_1, e_2] = e_3, \ [e_1, e_3] = -e_2,$
- $\mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R}) : \ [e_1, e_3] = e_1, \ [e_2, e_4] = e_2,$
- $\mathfrak{t}_{4,1,\lambda} : \ [e_4, e_1] = e_1, \ [e_4, e_2] = e_2, \ [e_4, e_3] = \lambda e_3,$
- $\mathfrak{h}_4 : \ [e_4, e_3] = e_3, \ [e_1, e_2] = e_3, \ [e_4, e_2] = e_2/2, \ [e_4, e_1] = e_2 + e_1/2,$
- $\mathfrak{r}_{4,\lambda} : \ [e_4, e_1] = e_1, \ [e_4, e_2] = \lambda e_2, \ [e_4, e_3] = e_2 + \lambda e_3.$

In order to provide more uniform view we also use the following notation for 4-dimensional Lie algebras:

(PHC1) γ is abelian,

(PHC2) $[X, Y] = W, \ [Y, W] = -X, \ [W, X] = Y,$

(PHC3) $[X, Y] = Y, \ [X, W] = W,$

(PHC4) $[X, Y] = Z,$

(PHC5) $[X, Y] = X,$

(PHC6) $[X,W]=Z$, $[X,Y]=X$, $[Y,W]=W$,

(PHC7) $[X,Z]=X$, $[X,W]=Y$, $[Y,Z]=Y$, $[Y,W]=aX+bY$, $a,b\in\mathbf{R}$,

(PHC8) $[X,Z]=X$, $[Y,W]=Y$,

(PHC9) $[Z,W]=Z$, $[Y,W]=Y$, $[X,W]=cX+aY+bZ$, $c\neq 0$, $a\in\mathbf{R}$, $b\in\{0,1\}$,

(PHC10) $[Y,X]=\lambda Z$, $[W,Z]=Z$, $[W,X]=\lambda X+bY+aZ$, $[W,Y]=(1-\lambda)Y$, $\lambda\neq 0,1$.

In the previous list the additive basis of algebra $\mathfrak{g}$ is (X,Y,Z,W), and only the non-zero commutators are given.

The relations between these lists of algebras are described in the following lemma.

Lemma 2.1. *Lie algebras* (PHC3), (PHC4), (PHC5), (PHC6) *and* (PHC8) *in our list are respectively Lie algebras* $\mathbf{R}\oplus\mathfrak{r}_{3,1}$, $\mathbf{R}\oplus\mathfrak{h}_3$, $\mathbf{R}\oplus\mathfrak{aff}(\mathbf{R})$, $\mathfrak{d}_4$ *and* $\mathfrak{aff}(\mathbf{R})\oplus\mathfrak{aff}(\mathbf{R})$.

Lie algebra (PHC7) *is* $\mathfrak{d}_{4;1}$ *for* $4a+b^2=0$; $\mathfrak{aff}(\mathbf{C})$ *for* $4a+b^2<0$; $\mathfrak{aff}(\mathbf{R})\oplus\mathfrak{aff}(\mathbf{R})$ *for* $4a+b^2>0$.

Also, (PHC9) *corresponds to* $\mathfrak{t}_{4,1,c}$ *for* $c\neq 1$; $\mathfrak{t}_{4,1,1}$ *for* $c=1$ *and* $a=b=0$; *and* $\mathfrak{r}_{4,1}$ *for* $c=1$ *and* $a^2+b^2\neq 0$.

In the (PHC10) *case, we have* $\mathfrak{d}_{4,\lambda}$ *for* $\lambda\neq 1/2$; $\mathfrak{d}_{4,1/2}$ *for* $\lambda=1/2$, $b=0$; *and* $\mathfrak{h}_4$ *for* $\lambda=1/2$, $b\neq 0$.

Let V be a real vector space. A *complex structure* on V is an endomorphism J_1 of V satisfying the condition

$$J_1^2=-1.$$

Existence of a complex structure implies that V has to be of an even dimension. A *product structure* on V is an endomorphism J_2 of V satisfying the conditions

$$J_2^2=1,\qquad J_2\neq\pm 1.$$

A *para-hypercomplex structure* on V is a pair (J_1,J_2) of anti-commuting complex structure J_1 and product structure J_2, i.e., satisfying the relations

$$J_1^2 = -1, \qquad J_2^2 = 1, \qquad J_1 J_2 = -J_2 J_1. \tag{2.1}$$

This structure is also known as Cliff $(1,1)$-structure. If both structures J_1 and J_2 are complex, then the pair (J_1, J_2) is called a *hypercomplex structure* on V. In the sequel we concentrate on the case of para-hypercomplex structure.

It is customary to denote $J_3 = J_1 J_2$. Note that the structure J_3 is a product structure. The Lie subalgebra of End(V) spanned by J_1, J_2 and J_3 is isomorphic to $\mathfrak{s}l_2(\mathbf{R})$. Any $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ defines a structure by the formula

$$J_x := x_1 J_1 + x_2 J_2 + x_3 J_3.$$

Denote by

$$\langle x, y \rangle = x_1 y_1 - x_2 y_2 - x_3 y_3,$$

$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, the inner product in $\mathbf{R}^3 = \mathbf{R}^{1,2}$ and by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1),$$

the usual cross product. The structure J_x is a complex structure provided that

$$\langle x, x \rangle = x_1^2 - x_2^2 - x_3^2 = 1,$$

and a product structure provided that

$$\langle x, x \rangle = x_1^2 - x_2^2 - x_3^2 = -1.$$

Hence, a para-hypercomplex structure (J_1, J_2) defines a 2-sheeted hyperboloid $\mathcal{S}^-$ of complex structures and a 1-sheeted hyperboloid $\mathcal{S}^+$ of product structures.

Proposition 2.1. *If* (J_1, J_2) *is a para-hypercomplex structure on a vector space* V, *then*:

i) $J_x J_y = -\langle x, y \rangle 1 + J_{x \times y}$.

ii) *The pair* $(J_x, J_y) \in \mathcal{S}^- \times \mathcal{S}^+$ *is a para-hypercomplex structure if and only if* $x \perp y$.

Proof. From the relations

$$J_1J_2 = J_3 = -J_2J_1, \quad J_1J_3 = -J_2 = -J_3J_1, \quad J_2J_3 = -J_1 = -J_3J_2,$$

statement i) follows by a direct calculation. Since J_x is a complex structure and J_y is a product structure, the pair (J_x, J_y) is a para-hypercomplex structure if and only if J_x and J_y anti-commute. Using the relation i) and the anti-commutativity of the cross product we have

$$0 = J_xJ_y + J_yJ_x = -2\langle x, y\rangle 1.$$

Hence, statement ii) is proved. □

The para-hypercomplex structures (J_1, J_2) and (J_x, J_y) are called *compatible*. A consequence of Proposition 2.1 is that all compatible structures are parameterized by the group $SO(1,2)_o$ which acts on them.

An *almost para-hypercomplex structure* on a manifold M is a pair (J_1, J_2) of sections of $\mathrm{End}\,(TM)$ satisfying the relations (2.1). It is a para-hypercomplex structure if both structures are *integrable*, that is, if the corresponding Nijenhuis tensors

$$(2.2) \quad \mathcal{N}_\alpha(X,Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] \pm [X,Y],$$

$\alpha = 1, 2$, vanish on all vector fields X, Y. In this formula sign $-$ occurs in the case of a complex structure and sign $+$ occurs in the case of a product structure.

If $M = G$ is a Lie group we additionally assume that the para-hypercomplex structure is left invariant. This allows us to also describe a para-hypercomplex structure on its Lie algebra $\mathfrak{g}$. Hence, a para-hypercomplex structure (J_1, J_2) on $\mathfrak{g}$ satisfies both relations (2.1) and (2.2).

Proposition 2.2. *Let (J_1, J_2) be an integrable para-hypercomplex structure on a Lie algebra $\mathfrak{g}$.*

i) *The product structure $J_3 = J_1J_2$ is integrable.*

ii) *Any compatible para-hypercomplex structure (J_x, J_y) is integrable.*

Proof. Statement i) follows from the relation

$$\begin{aligned}2\mathcal{N}_3(X,Y) = &\mathcal{N}_1(J_2X,J_2Y)+\mathcal{N}_2(J_1X,J_1Y)\\ &-J_1\mathcal{N}_2(J_1X,Y)-J_1\mathcal{N}_2(X,J_1Y)\\ &+\mathcal{N}_2(X,Y)-J_2\mathcal{N}_1(J_2X,Y)\\ &-J_2\mathcal{N}_1(X,J_2Y)-\mathcal{N}_1(X,Y),\end{aligned}$$

where $\mathcal{N}_3$ is the Nijenhuis tensor of the product structure J_3.

To prove ii) denote by $\mathcal{N}_x$ the Nijenhuis tensor corresponding to the structure J_x, $x=(x_1,x_2,x_3)$. One can check that

$$\begin{aligned}\mathcal{N}_x = &\, x_1^2\mathcal{N}_1+x_2^2\mathcal{N}_2+x_3^2\mathcal{N}_3+x_1x_2(J_3\mathcal{N}_1+J_3\mathcal{N}_2+J_3\mathcal{N}_3J_1)\\ &+x_2x_3(J_1\mathcal{N}_2-J_1\mathcal{N}_3-J_1\mathcal{N}_1J_2)\\ &+x_1x_3(-J_2\mathcal{N}_1-J_2\mathcal{N}_3+J_2\mathcal{N}_2J_3)\end{aligned}$$

holds, where we have used the notation, for instance,

$$J_2\mathcal{N}_2J_3(X,Y)=J_2\mathcal{N}_2(J_3X,J_3Y).$$

Now, statement ii) follows using statement i). □

Let g be an inner product on the vector space V. A para-hypercomplex structure (J_1,J_2) on V is called *Hermitian* with respect to g if

$$g(J_\alpha X,Y)=-g(X,J_\alpha Y),\quad X,Y\in V \tag{2.3}$$

holds, i.e., if both structures J_1 and J_2 are Hermitian. It is easy to prove that a Hermitian complex structure is an isometry and a Hermitian product structure is an anti-isometry, i.e.,

$$g(J_1X,J_1Y)=g(X,Y),\qquad g(J_2X,J_2Y)=-g(X,Y).$$

Existence of an anti-isometry implies that the inner product g must be of neutral, (n,n) signature.

Proposition 2.3. *Let (J_1,J_2) be a para-hypercomplex structure Hermitian with respect to the scalar product g on the vector space V.*

i) *The product structure* $J_3 = J_1 J_2$ *is Hermitian.*

ii) *Any compatible para-hypercomplex structure* (J_x, J_y) *is Hermitian.*

Proof. i) If J_1 and J_2 are Hermitian, then J_3 is Hermitian since we have

$$\langle J_3 X, Y\rangle = \langle J_1 J_2 X, Y\rangle = -\langle J_2 X, J_1 Y\rangle = \langle X, J_2 J_1 Y\rangle = -\langle X, J_3 Y\rangle.$$

ii) Since the condition of any J_x to be Hermitian is linear with respect to x, statement ii) follows from statement i). □

Now, we specialize to the four-dimensional case and prove some lemmas which will be useful in the sequel.

Lemma 2.2. *If* (J_1, J_2) *is a para-hypercomplex structure on a real four-dimensional vector space* V*, then*:

i) *There is an inner product* g *on* V*, unique up to a non-zero constant, such that the structure* (J_1, J_2) *is Hermitian with respect to* g*.*

ii) *Any compatible para-hypercomplex structure* (J_x, J_y) *determines the same inner product* g *on* V*.*

Proof. First, we prove the existence of such an inner product. If $(\cdot,\cdot)$ is an arbitrary positive definite inner product on V, then the inner product

$$g(X, Y) := (X, Y) + (J_1 X, J_1 Y) - (J_2 X, J_2 Y) - (J_3 X, J_3 Y) \tag{2.4}$$

satisfies the properties (2.3).

To see the uniqueness let $g'(\cdot,\cdot)$ be another inner product on V satisfying (2.3). As remarked before both products are of signature $(2, 2)$. There exists a vector X which is not null with respect to the both inner products, for instance

$$g(X, X) = 1, \quad g'(X, X) = \lambda \neq 0.$$

The relations (2.1) and (2.3) imply that the vectors X, J_1X, J_2X, J_3X are mutually orthogonal with respect to both inner products. Moreover,

$$g(X,X) = g(J_1X, J_1X) = -g(J_2X, J_2X) = -g(J_3X, J_3X) = 1$$
$$g'(X,X) = g'(J_1X, J_1X) = -g'(J_2X, J_2X) = -g'(J_3X, J_3X) = \lambda.$$

Hence, $g'(\cdot,\cdot) = \lambda g(\cdot,\cdot)$, $\lambda \neq 0$.

ii) According to Proposition 2.3 the structure (J_x, J_y) is Hermitian with respect to g. The statement follows from the uniqueness of g (up to a non-zero scalar). □

Remark 2.1. In light of Lemma 2.2 we see that the notion of null vector N (such that $g(N,N) = 0$) depends only on the compatibility class of Hermitian structure (J_1, J_2) and not on a particular inner product.

From the proof of Lemma 2.2 we also obtain the following.

Lemma 2.3. *If (J_1, J_2) is a is a para-hypercomplex structure on a real four-dimensional vector space V, then (X, J_1X, J_2X, J_3X) is a basis of V if and only if X is not null.*

Lemma 2.4. *If J_α is an endomorphism of a 4-dimensional Lie algebra $\mathfrak{g}$ such that $J_\alpha^2 = \pm 1$ and $(X, J_\alpha X, Y, J_\alpha Y)$ is a basis of $\mathfrak{g}$ then the corresponding Nijenhuis tensor N_α vanishes if and only if $N_\alpha(X,Y) = 0$.*

Proof. One can easily show that $\mathcal{N}_\alpha(J_\alpha X, Y) = -J_\alpha \mathcal{N}_\alpha(X,Y)$. The lemma follows from the fact that $\mathcal{N}_\alpha$ is antisymmetric and bilinear. □

Lemma 2.5. *Let (J_1, J_2) be a para-hypercomplex structure on a real four-dimensional vector space V, and let $W \subset V$ be a two-dimensional subspace. Then there exists a compatible para-hypercomplex structure (J_1', J_2') such that:*

i) *If W is definite (contains no null directions) then $J_1'W = W$.*

ii) *If W is Lorentz (contains exactly two null directions), then $J_2'W = W$.*

iii) *If* W *is totally null* (*every vector in* W *is a null vector*) *then either*

(a) $J_2'|_W = 1$, $V = W \oplus J_1' W$, *or*

(b) *there exists a non-null vector* X *such that*

$$W = \mathbf{R}\langle J_1'X + J_2'X, X - J_3'X\rangle, \quad J(W) = W \text{ for all } J \in \mathcal{S}^{\pm}.$$

iv) *If the induced metric on* W *is of rank* 1 (W *contains exactly one null direction* N) *then* $N = J_1'X - J_2'X$ (*for any given vector* $X \in W$, $|X|^2 \neq 0$).

Proof of i) *and* ii). Let (X, Y) be a pseudo-orthonormal basis of W ($|X|^2 = -|Y|^2 = 1$ and $\langle X, Y\rangle = 0$ with respect to the induced inner product on W). Then, according to Lemma 23. vectors X, J_1X, J_2X and J_3X form a pseudo-orthonormal basis of V and we have $Y = x_1J_1X + x_2J_2X + x_3J_3X$ with $x_1^2 - x_2^2 - x_3^2 = \pm 1$, where $-$ occurs if W is Lorentz and $+$ if W is positive or negative definite. The structure

$$J_x = x_1J_1 + x_2J_2 + x_3J_3$$

preserves W. It is a product structure if W is Lorentz (and we set $J_2' = J_x$) or a complex structure if W is definite (and we set $J_1' = J_x$). The second structure can be chosen such that (J_1', J_2') forms a compatible para-hypercomplex structure. Note that there cannot exist a product structure preserving a definite W since a product structure is an anti-isometry. Similarly, a complex structure preserving a Lorentz W cannot exist.

Proof of iii). Let $N_1 \in W$ be a null vector. There exists a non-null vector $X \in V$ perpendicular to N_1. Hence

$$N_1 = \alpha J_1X + \beta J_2X + \gamma J_3X \quad \text{and} \quad \alpha^2 - \beta^2 - \gamma^2 = 0,$$

so $\alpha \neq 0$ and we may assume that $\alpha = 1$. Then $J_2' = \beta J_2 + \gamma J_3$ is a product structure, the structure (J_1', J_2'), $J_1' = J_1$ is a compatible para-hypercomplex structure, and we have

$$N_1 = J_1'X + J_2'X.$$

Any null vector $aX + bJ_1'X + cJ_2'X + dJ_3'X$ which is orthogonal to the vector N_1 is of the form

$$N^{\pm} = aX + bJ_1'X + bJ_2'X \pm aJ_3'X.$$

Notice that the vector N_1 is also of the form $N^{\pm}$ and that there exist exactly two null planes $W^{\pm}$ containing the vector N_1. They can be written in the form

$$W^{\pm} = \mathbf{R}\langle N_1, N_2^{\pm} = X \pm J_3'X\rangle.$$

The plane W^- is the $+1$-eigenspace of the product structure J_3' and the vectors N_1, N_2^-, $J_1'N_1$, $J_1'N_2^-$ are independent, so $V = W^- \oplus J_1'W^-$ and iii) (a) holds.

In the case of the plane W^+ one easily checks that $J_1'W^+ = W^+ = J_2'W^+$ and hence statement iii) (b) follows.

Proof of iv). The proof is similar to the first part of the previous proof (with $N_1 = N$). □

Lemma 2.6. *Let (J_1, J_2) be a para-hypercomplex structure on a real four-dimensional vector space V, and let $W \subset V$ be a three-dimensional subspace such that the induced metric on W is degenerate (that is $W^{\perp} \subset W$). For $N \in W^{\perp}$ and $X \in W$, $|X|^2 \neq 0$, there exists a compatible para-hypercomplex structure (J_1', J_2') on V such that $N = J_1'X - J_2'X$ and the arbitrary null vector in W belongs to the union of two-dimensional planes $\pi_1 = \mathbf{R}\langle N, J_1'N\rangle$ and $\pi_- = \{V \mid J_3'V = -V\}$, i.e.,*

$$\begin{aligned}\operatorname{null}(W) &:= \{U \in W \mid |U|^2 = 0\} = \pi_1 \cup \pi_- \\ &= \mathbf{R}\langle N, J_1'N\rangle \cup \{V \mid J_3'V = -V\}.\end{aligned}$$

Proof. Since we have $|N|^2 = 0$, $|X|^2 \neq 0$, $\langle N, X\rangle = 0$ the existence of a compatible structure (J_1', J_2') such that $N = J_1'X - J_2'X$ follows from Lemma 2.5 iv). Moreover, $(N, J_1'N, X)$ is a basis of W and $(N, J_1'N, X, J_1'X)$ is a basis of V. Thus, for $U \in \operatorname{null}(W)$ of the form $U = \alpha N + \beta J_1'N + \gamma X$ we get

$$0 = |U|^2 = \gamma(\gamma - 2\beta)|X|^2.$$

The case $\gamma = 0$ gives the plane $\pi_1 = \mathbf{R}\langle N, J_1 N\rangle$. For $\gamma = 2\beta$ one can check that $J_3'(U) = -U$, so U belongs to the -1 eigenspace of J_3'. □

3. Lie algebras admitting a para-hypercomplex structure.

3.1. Case when $\mathfrak{g}$ has a non-trivial center.

Theorem 3.1. *A four-dimensional Lie algebra $\mathfrak{g}$ admitting a para-hypercomplex structure and with a non-trivial center $Z(\mathfrak{g})$ is one of algebras $\mathbf{R}^4$, $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R})$, $\mathbf{R} \oplus \mathfrak{r}_{3,1}$, $\mathbf{R} \oplus \mathfrak{h}_3$, $\mathbf{R}^2 \oplus \mathfrak{aff}(\mathbf{R})$, $\mathfrak{d}_4$ (PHC1-PHC6).*

As a consequence of the Levi decomposition theorem and the classification of real semisimple Lie algebras the only non-solvable Lie algebras which are four-dimensional are $\mathbf{R} \oplus so(3)$ and $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R})$. Since they both have a non-trivial center, as a consequence of Theorem 3.1 we have the following corollary.

Corollary 3.1. *The only non-solvable, real four-dimensional Lie algebra admitting a para-hypercomplex structure is $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R})$.*

Proof of Theorem 3.1. In order to prove that these are the only Lie algebras with non-trivial center which admit a para-hypercomplex structure we consider two cases.

Case 1. There exists a non-null central element Z. Let (J_1, J_2) be a para-hypercomplex structure on $\mathfrak{g}$ and denote

$$X = J_1 Z, \quad Y = J_2 Z, \quad W = J_3 Z.$$

Then

$$[X, Y] = aZ + bX + cY + dW. \tag{3.1}$$

According to Lemma 2.4 integrability of J_1 is equivalent to

$$0 = \mathcal{N}_1(Z, Y) = [X, W] - J_1[X, Y]. \tag{3.2}$$

Similarly, the integrability of J_2 is equivalent to

$$0 = \mathcal{N}_2(X, Z) = [Y, W] - J_2[X, Y]. \tag{3.3}$$

From relations (3.1), (3.2) and (3.3) we get

$$[X, W] = -bZ + aX - dY + cW, \quad [Y, W] = cZ - dX + aY - bW.$$

The Jacobi identity is equivalent to

$$\begin{aligned} 0 &= [[X, Y], W] + [[Y, W], X] + [[W, X], Y] \\ &= 2(-a^2 - b^2 + c^2)Z - 2cdX - 2dbY - 2adW. \end{aligned}$$

If $a = b = c = d = 0$ then the algebra γ is abelian $\mathbf{R}^4$ (PHC1). If $a = b = c = 0$ and $d \neq 0$ then after scaling $\mathfrak{g} \cong R \oplus \mathfrak{s}l_2(\mathbf{R})$ (PHC2).

If $d = 0$ and $0 \neq c^2 = a^2 + b^2$, then the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is two-dimensional since

$$c[Y, W] = a[X, Y] + b[W, X].$$

It is generated by the vectors $W_1 = [X, Y]$, $Y_1 = [W, X]$. The vectors $Z_1 = Z$, $X_1 = X/c$, Y_1 and W_1 are linearly independent and we get algebra $\mathbf{R} \oplus \mathfrak{r}_{3,1}$ (PHC3).

Case 2. All central vectors are null vectors. Denote one of them by N. According to Lemma 2.5 iv), we can assume that $N = J_1X - J_2X$ for a non-null vector $X \in \mathfrak{g}'$. Then the vectors N, J_1N, X and J_1X form a basis of $\mathfrak{g}$ and the structure J_2 expressed in the terms of that basis reads

$$J_2X = J_1X - N, \quad J_2J_1N = N, \quad J_2J_1X = J_1N + X, \quad J_2N = J_1N. \tag{3.4}$$

The integrability of the structure J_1 gives the following conditions

$$0 = \mathcal{N} =_1 (X, N) = [J_1X, J_1N] - J_1[X, J_1N]. \tag{3.5}$$

Since the vectors N, J_2N, X and J_2X form a basis of $\mathfrak{g}$, the integrability of the product structure J_2 is equivalent to

$$0 = \mathcal{N}_2(X, N) = [J_1X, J_1N] - J_2[X, J_1N]. \tag{3.6}$$

The vector $[X, J_1N]$ is of the form $[X, J_1N] = aN+bJ_1N+cX+dJ_1X$. Using the relations (3.5) and (3.6) we get
(3.7)
$$[X, J_1N] = aN + bJ_1N + 2bX, \quad [J_1X, J_1N] = -bN + aJ_1N + 2bJ_1X.$$

If we write $[X, J_1X] = \alpha N + \beta J_1N + \gamma X + \delta J_1X$ and impose the Jacobi identity on the vectors J_1N, X and J_1X we get the following system of equations:

$$\begin{aligned} -4\alpha b - b^2 - \delta b + \gamma a - a^2 &= 0,\\ -4b\beta + a\delta + b\gamma &= 0,\\ b(a+\gamma) &= 0,\\ b(b-\delta) &= 0. \end{aligned} \tag{3.8}$$

The system has three classes of solutions.

Case 2a). $a = 0 = b$. In this case the only non-zero commutator is
$$[X, J_1X] = \alpha N + \beta J_1N + \gamma X + \delta J_1X.$$

If $\gamma = 0 = \delta$, the change of the basis $Y = J_1X$, $N_1 = \alpha N + \beta J_1N$, $N_2 \in \mathbf{R}\langle N, J_1N\rangle$ gives $\mathbf{R} \oplus \mathfrak{h}_3$ (PHC4). If $\delta \neq 0$, then the change $Y[X, J_1X]/\delta$, $N_1 = N, N_2 = J_1N$ gives $\mathbf{R}^2 \oplus \mathfrak{aff}(\mathbf{R})$ (PHC5). The case $\delta = 0, \gamma \neq 0$, similarly reduces to $\mathbf{R} \oplus \mathfrak{r}_{2,0}$ (PHC5).

Case 2b). $b = \delta \neq 0$, $a = -\gamma$, $\beta = 0$ and $\alpha = -(a^2+b^2)/(2b)$. This case reduces to $\mathbf{R} \oplus \mathfrak{r}_{3,1}$ (PHC3).

Case 2c). $a = \gamma \neq 0$. Then, $b = \delta = 0$. Moreover, we may assume that $a = 1$ to obtain $\mathfrak{d}_4$ (PHC6). □

3.2. Case of solvable Lie algebra $\mathfrak{g}$ and $\dim \mathfrak{g}' \leq 2$.

Theorem 3.2. *Let $\mathfrak{g}$ be a four-dimensional real Lie algebra admitting a para-hypercomplex structure and* $\dim \mathfrak{g}' = 1$. *Then $\mathfrak{g}$ is $\mathbf{R} \oplus \mathfrak{h}_3$ or $\mathbf{R} \oplus \mathfrak{aff}(\mathbf{R})$.*

Proof. If $\mathfrak{g}$ has a non-trivial center ξ, then from Theorem 3.1 it is one of the algebras $\mathbf{R} \oplus \mathfrak{h}_3$, $\mathbf{R} \oplus \mathfrak{aff}(\mathbf{R})$. Now, as in [**5**, Proposition 3.2], suppose that the center ξ of $\mathfrak{g}$ is trivial, and let X be a non-zero element of $\mathfrak{g}'$. There exists a Y such that $[Y,X]=X$. Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g}=\ker(\mathrm{ad}_X)\cap\ker(\mathrm{ad}_Y)\oplus\mathbf{R}X\oplus\mathbf{R}Y.$$

From the Jacobi identity we get that $\xi=\ker(\mathrm{ad}_X)\cap\ker(\mathrm{ad}_Y)$, a contradiction. Hence, solvable $\mathfrak{g}$ without center and with $\dim\mathfrak{g}'=1$ does not exist (this does not depend on the existence of para-hypercomplex structure). □

Theorem 3.3. *Let $\mathfrak{g}$ be a four-dimensional solvable Lie algebra admitting a para-hypercomplex structure and with* $\dim\mathfrak{g}'=2$. *If $\mathfrak{g}$ has a non-trivial center, then it is algebra $\mathbf{R}\oplus\mathfrak{r}_{3,1}$. If $\mathfrak{g}$ has a trivial center, then $\mathfrak{g}$ is one of algebras $\mathfrak{d}_{4;1}$, $\mathfrak{aff}(\mathbf{C})$, $\mathfrak{aff}(\mathbf{R})\oplus\mathfrak{aff}(\mathbf{R})$.*

Remark 3.1. Using the notation introduced by Snow [**11**], these Lie algebras are $S11$, $S8$ and $S10$, respectively. The class $S11$ contains as a special case the Lie algebra $\mathfrak{aff}(\mathbf{C})$ which is the unique solvable Lie algebra with 2-dimensional derived algebra which admits hypercomplex structure [**5**].

Proof. Suppose that the center of $\mathfrak{g}$ is trivial and that (J_1,J_2) is a para-hypercomplex structure on $\mathfrak{g}$. According to Lemma 2.2 and Remark 2.1 the structure (J_1,J_2) determines the inner product on $\mathfrak{g}=V$ and the notion of a null vector. As in Lemma 2.5 we have to consider the cases concerning the rank and the signature of the induced inner product on $\mathfrak{g}'=W$.

Case i) Induced metric on $\mathfrak{g}'$ is definite. Because of Lemma 2.5 i) we may assume that $\mathfrak{g}'$ is invariant with respect to the complex structure J_1, $J_1\mathfrak{g}'=\mathfrak{g}'$, and $\mathfrak{g}=\mathfrak{g}'\oplus J_2\mathfrak{g}'$. Let $\{X,J_1X=Y\}$ be a basis of $\mathfrak{g}'$ and $\{X,Y,J_2X,J_2Y\}$ be a basis of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}'$ is abelian since $\mathfrak{g}$ is solvable and by the integrability of the product structure J_2 we have $\mathcal{N}_2(X,J_1X)=0$ and

$$[J_2X,J_2Y]=0,\quad [J_2X,Y]=[J_2Y,X]. \tag{3.9}$$

Because of the integrability of the complex structure J_1, $\mathcal{N}_1(X,J_2X)=0$ and

$$[X,J_2X]=-[Y,J_2Y]. \tag{3.10}$$

For arbitrary vectors V and W in $\mathfrak{g}$,

$$[V,W]=\alpha(V,W)X+\beta(V,W)Y,$$

where α and β are skew-symmetric bilinear forms on $\mathfrak{g}$. From the Jacobi identity we have

$$\alpha(X,J_2X)=\beta(X,J_2Y),\quad \alpha(J_2Y,X)=\beta(X,J_2X),$$

and the bracket in $\mathfrak{g}$ is determined by $c=\alpha(X,J_2X)$ and $d=\beta(X,J_2X)$ as follows:

$$(3.11)\quad [X,J_2X]=-[Y,J_2Y]=cX+dY,\ [X,J_2Y]=[Y,J_2X]=-dX+cY.$$

Since $\dim\mathfrak{g}'=2$, $c^2+d^2\neq 0$ and we may choose

$$\widetilde{x}=(c^2+d^2)^{-1}(cX+dY),\qquad \widetilde{y}=(c^2+d^2)^{-1}(-dX+cY),$$
$$\widetilde{Z}=(c^2+d^2)^{-1}(cJ_2X-dJ_2Y),\qquad \widetilde{W}=(c^2+d^2)^{-1}(dJ_2X+cJ_2Y),$$

and hence

$$[\widetilde{X},\widetilde{Z}]=\widetilde{X},\qquad [\widetilde{X},\widetilde{W}]=\widetilde{Y},$$
$$[\widetilde{Y},\widetilde{Z}]=\widetilde{Y},\qquad [\widetilde{Y},\widetilde{X}]=-\widetilde{X},$$

so we get the algebra PHC7 for $a=-1$, $b=0$ ($\mathfrak{g}\equiv\mathfrak{a}ff(\mathbf{C})$).

Case ii) Induced metric on γ' is indefinite, of Lorentz type $(-+)$. Because of Lemma 2.5 ii) we may assume that $\mathfrak{g}'$ is invariant with respect to the product structure J_2, $J_2\mathfrak{g}'=\mathfrak{g}'$, and $\mathfrak{g}=\mathfrak{g}'\oplus J_1\mathfrak{g}'$. Let $\{X,J_2X=Y\}$ be a basis of $\mathfrak{g}'$ and $\{X,Y,J_1X,J_1Y\}$ a basis of $\mathfrak{g}$. By the integrability of the complex structure J_1, $\mathcal{N}_1(X,Y)=0$ and

$$(3.12)\qquad [J_1X,J_1Y]=0,\qquad [J_1X,Y]=[J_1Y,X].$$

Because of the integrability of the product structure J_2, $\mathcal{N}_2(X,J_1X)=0$ and

$$(3.13)\qquad [X,J_1X]=[Y,J_1Y].$$

From the Jacobi identity we have

$$\alpha(X,J_1X)=\beta(X,J_1Y),\qquad \alpha(J_1Y,X)=-\beta(X,J_1X),$$

and the bracket in $\mathfrak{g}$ is determined by $c = \alpha(X, J_1X)$ and $d = \beta(X, J_1X)$ as follows:

$$[X, J_1X] = [Y, J_1Y] = cX + dY, \qquad [X, J_1Y] = [Y, J_1X] = dX + cY.$$

Since $\dim \mathfrak{g}' = 2$, $c^2 - d^2 \neq 0$ and we may choose

$$\widetilde{X} = (c^2 - d^2)^{-1}(cX + dY), \qquad \widetilde{Y} = (c^2 - d^2)^{-1}(dX + cY),$$
$$\widetilde{Z} = (c^2 - d^2)^{-1}(cJ_1X - dJ_1Y), \quad \widetilde{W} = (c^2 - d^2)^{-1}(-dJ_1X + cJ_1Y),$$

and hence

$$[\widetilde{X}, \widetilde{Z}] = \widetilde{X}, \qquad [\widetilde{X}, \widetilde{W}] = \widetilde{Y},$$
$$[\widetilde{Y}, \widetilde{Z}] = \widetilde{Y}, \qquad [\widetilde{Y}, \widetilde{W}] = \widetilde{X},$$

and we get algebra $\mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R})$ (PHC7 for $a = 1, b = 0$).

Case iii) $\mathfrak{g}'$ is a totally null plane. According to Lemma 2.5 iii) we have to consider two geometrically different cases.

In the first case we can assume that $J_2|_{\mathfrak{g}'} = 1$ and $\mathfrak{g} = \mathfrak{g}' + J_1\mathfrak{g}'$ holds. If (X, Y) is a basis of $\mathfrak{g}'$ we have

$$J_2X = X, \quad J_2Y = Y, \quad J_2J_1X = -J_1X, \quad J_2J_1Y = -J_1Y.$$

One easily checks that the integrability of the complex structure J_1 is equivalent to the relations

$$[J_1X, J_1Y] = 0, \quad [X, J_1Y] = [Y, J_1X].$$

It is interesting that the product structure J_2 is automatically integrable. Hence, the possible non-null commutators are

$$T' = [X, J_1X] = aX + bY,$$
$$Y' = [Y, J_1Y] = cX + dY,$$
$$X' = [X, J_1Y] = eX + fY.$$

The Jacobi identity is equivalent to the equations

$$(3.14) \qquad (e - d)X' + fY' - cT' = 0, \quad (a - f)X' + bY' - eT' = 0,$$

or equivalently

$$e(e - d) + c(f - a) = 0, \quad ef = bc, \quad af - f^2 + bd - be = 0.$$

If X' is a zero vector, then we get the algebra $\mathfrak{a}ff(\mathbf{R})\oplus\mathfrak{a}ff(\mathbf{R})$. Suppose that X' is a non-zero vector. If Y' or T' is a zero vector then we get an algebra PHC7 for $a = 0 = b$. Suppose that none of the vectors X', Y', Z' is the zero vector. We can suppose that one of the pairs X', Y' and X', T' is independent, say X', T'. If the vectors X' and Y' are collinear then we get the algebra PHC7 for $a = 0$, $b = 1$. Finally, if both the pairs X', T' and X', Y' are independent then introduce a new basis (X', Y', Z', W') satisfying

$$Z' = \frac{1}{D}(fJ_1X - bJ_1Y), \quad W' = \frac{1}{D}(-eJ_1X + aJ_1Y),$$

where $D = af - be \neq 0$. In the new basis the commutator relations take the very simple form

$$[X', Z'] = X', \quad [X', W'] = Y', \quad [Y', Z'] = Y',$$
$$[Y', W'] = \frac{fc - de}{D}X' + \frac{ad - bc}{D}Y'.$$

Since X' and Y' are independent then $cf - de \neq 0$, that is, $a \neq 0$ in the algebra PHC7.

In the second case we can assume that (N_1, N_2) is a basis of $\mathfrak{g}'$ and $\mathfrak{g}'$ is invariant with respect to J_1, J_2, J_3. Then a possible basis of $\mathfrak{g}$ is

$$N_1 = J_1X + J_2X, \quad N_2 = X - J_3X, \quad N_3 = J_1X - J_2X, \quad N_2 = X + J_3X.$$

We calculate the structures in terms of that basis:

$$J_1N_1 = -N_2, \quad J_1N_3 = -N_4,$$
$$J_2N_1 = N_2, \quad J_2N_3 = -N_4,$$
$$J_3N_1 = N_1, \quad J_3N_2 = -N_2, \quad J_3N_3 = -N_3, \quad J_3N_4 = N_4.$$

By the integrability of J_3,

$$J_3[N_1, N_4] = [N_1, N_4], \quad J_3[N_2, N_3] = -[N_2, N_3].$$

Thus,

$$[N_1, N_4] = \mu N_1, \quad [N_2, N_3] = \lambda N_2.$$

The integrability of J_1 and J_2 is equivalent to

$$0 = -[N_2, N_4] - \lambda N_1 + \mu N_2 + [N_1, N_3]$$

After imposing the Jacobi identity this reduces to the algebra PHC3.

Case iv): the induced metric on $\mathfrak{g}'$ is of rank 1. Denote by N the null vector belonging to $\mathfrak{g}'$ (which is unique up to a scaling constant).

According to Lemma 2.5 iv) we can choose a product structure J_2 such that for the basis (X, N) of $\mathfrak{g}'$ one has

$$N = J_1X - J_2X, \quad N \text{ is null.} \tag{3.15}$$

Then (X, N, J_1X, J_1N) is a basis of $\mathfrak{g}$. One easily calculates the following relations

$$J_2X = J_1X - N, \quad J_2N = J_1N.$$

The integrability of J_1 is equivalent to $\mathcal{N}_1[X, N] = 0$, i.e., to the relations

$$[J_1X, J_1N] = 0, \quad [X, J_1N] = [N, J_1X].$$

Since (X, N, J_2X, J_2N) is a basis of $\mathfrak{g}$ the integrability of the product structure J_2 is equivalent to $\mathcal{N}_2(X, N) = 0$ which gives the condition

$$[N, J_1N] = 0.$$

The commutator relations now read

$$[X, J_1X] = aX + bN, \quad [X, J_1N] = cX + dN,$$

where a, b, c, d are unknown coefficients. The Jacobi identity is now equivalent to the following relations

$$c = 0, \quad d(a - d) = 0. \tag{3.16}$$

The case $d = 0$ gives the algebra with $\dim \mathfrak{g}' = 1$ which we have already discussed. The remaining case $a = d \neq 0$, after the change

$$\widetilde{Y} = N, \quad \widetilde{Z} = J_1N, \quad \widetilde{X} = \frac{1}{a}X, \quad \widetilde{W} = \frac{1}{a}J_1X - \frac{b}{a^2}J_1N, \tag{3.17}$$

takes the form

$$[\widetilde{Y}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{W}] = \widetilde{Y}, \quad [\widetilde{X}, \widetilde{Z}] = \widetilde{Y}, \quad [\widetilde{X}, \widetilde{W}] = \widetilde{X}$$

of the algebra PHC7 for $a = 0 = b$. □

3.3. Case of solvable Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}' = 3$.

Theorem 3.4. *Let $\mathfrak{g}$ be a four-dimensional solvable Lie algebra admitting a para-hypercomplex structure and with $\dim \mathfrak{g}' = 3$. If $\mathfrak{g}$ has a nontrivial center it is algebra $\mathfrak{d}_4$, otherwise it is algebra of the type* PHC9 *or* PHC10.

Proof. If the algebra $\mathfrak{g}$ is solvable then its derived algebra $\mathfrak{g}'$ is nilpotent. Up to isomorphism the only 3-dimensional nilpotent Lie algebras are abelian algebra and Heizenberg algebra generated by X, Y and Z with nonzero commutator

$$[X, Y] = Z.$$

Let $\mathfrak{g}$ be with trivial center, admitting a para-hypecomplex structure (J_1, J_2), and let $\langle \cdot, \cdot \rangle$ be a compatible inner product on $\mathfrak{g}$. First, we discuss the case of $\mathfrak{g}'$ being abelian.

Suppose that $\mathfrak{g}'$ is a non-degenerate subspace and X is normal vector of $\mathfrak{g}'$. Then $|X|^2 \neq 0$ and $\mathfrak{g}' = \mathbf{R}\langle J_1X, J_2X, J_3X\rangle$. From the integrability of J_1 and J_2, we have

$$[X, J_\alpha J_\beta X] = J_\alpha [X, J_\beta X],$$

for $\alpha, \beta \in 1, 2, 3$, $\alpha \neq \beta$. Hence, $[X, J_\alpha X] = \lambda J_\alpha X$, and we get the algebra PHC9 for $a = 0 = b$ (the Lie algebra corresponding to the real hyperbolic spaces).

Assume now that $\mathfrak{g}'$ is a degenerate subspace and N is normal vector of $\mathfrak{g}'$. Then $|N|^2 = 0$ and $N \in \mathfrak{g}'$. According to Lemma 2.5 iv) we can choose a compatible structure (J_1, J_2) such that $N = J_1X - J_2X$ for any $X \in \mathfrak{g}'$, $|X|^2 \neq 0$. Since J_1N is orthogonal to N we also have $J_1N \in \mathfrak{g}'$. Hence we may suppose that $\mathfrak{g}' = \mathbf{R}\langle N, J_1N, X\rangle$. Moreover the (N, J_1N, X, J_1X) is a basis of $\mathfrak{g}$. The integrability of J_1 and J_2 implies

$$[J_1N, J_1X] = J_1[N, J_1X] = J_2[N, J_1X], \tag{3.18}$$

i.e., $[N, J_1X] = dN$ and $[J_1N, J_1X] = dJ_1N$, $d \neq 0$. Moreover, we may choose X such that $d = 1$ to get algebra PHC9.

Now we turn to the case when $\mathfrak{g}'$ is a Heisenberg algebra. Let $\mathfrak{g}' = \mathbf{R}\langle X, Y, Z\rangle$ and $\mathfrak{g} = \mathbf{R}\langle X, Y, Z, W\rangle$. One can easily check that the center $\mathbf{R}\langle Z\rangle$ is an ideal of $\mathfrak{g}$, and hence

$$[W, Z] = \lambda Z, \quad \lambda \neq 0,$$

no matter how the vector W that does not belong to $\mathfrak{g}'$ is chosen. At the other side, independently of the choice of non-central vectors $X, Y \in \mathfrak{g}'$ their commutator is always in the center. Moreover, non-commutativity of $\mathfrak{g}'$ implies that $[X, Y] \neq 0$, and by scaling of Z we can achieve

$$[X, Y] = Z.$$

Also, $\lambda \neq 0$ since otherwise Z would be a non-zero central element of $\mathfrak{g}$. Hence, it remains to calculate the commutators $[W, X]$ and $[W, Y]$. This approach we use to prove the remaining part of the theorem.

We consider the cases depending on degeneracy of $\mathfrak{g}'$ with respect to the induced compatible metric. Also there are different subcases depending on the norm of a central element of $\mathfrak{g}'$.

i) Suppose that $\mathfrak{g}'$ is not degenerated, and let W be its normal vector. Denote by $Z = \xi(\mathfrak{g}')$ a non-zero central element of $\mathfrak{g}'$. As an element of $\mathfrak{g}'$, Z is orthogonal to W. Now we have the following cases.

W and Z have the same sign: Using Lemma 2.5 i) we may choose a compatible structure (J_1, J_2) such that $Z = J_1W$. Then the (J_1W, J_2W, J_3W) is a basis of $\mathfrak{g}'$. After a simple calculation (and scaling) we get the commutator relation:

$$[W, J_1W] = 2J_1W,\ [W, J_2W] = J_2W,\ [W, J_3W]J_3W,\ [J_2W, J_3W] = J_1W.$$

That is a special form of algebra PHC10.

W and Z have the opposite sign: Using Lemma 2.5 ii) we may choose a compatible structure (J_1, J_2) such that $Z = J_2W$. Then the (J_1W, J_2W, J_3W) is a basis of $\mathfrak{g}'$. After a simple calculation (and scaling) we get the commutator relation:

$$[W, J_1W] = J_1W,\ [W, J_2W] = 2J_2W,\ [W, J_3W]J_3W,\ [J_1W, J_3W] = J_2W.$$

That is again a special form of algebra PHC10.

The center Z of $\mathfrak{g}'$ is a null vector: We have: $|W|^2 \neq 0$, $|Z|^2 = 0$, $Z \perp X$, so using Lemma 2.5 iv) we may choose a structure (J_1, J_2) such that

$$N = Z = J_1W - J_2W.$$

Moreover there is a decomposition

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathbf{R}W = \mathbf{R}\langle N, J_1W, J_3W\rangle \oplus \mathbf{R}W.$$

Now we have

$$[J_1W, J_3W] = \lambda N, \qquad [W, N] = N, \quad \lambda \neq 0.$$

After imposing the integrability condition for the structure (J_1, J_2) we get a contradiction. Hence, this case does not give a solution.

ii) Suppose that $\mathfrak{g}'$ is degenerated, and let $N \in \mathfrak{g}'$ be its normal vector and $Z \in \mathfrak{g}'$, a non-zero central element of $\mathfrak{g}'$. We now discuss cases depending on the type of vector Z.

Z is a non null vector, $|Z|^2 \neq 0$: Let $X = Z$. Consider the basis:

$$\mathfrak{g} = \mathbf{R}\langle N, J_1N, X, J_1X\rangle, \qquad \mathfrak{g}' = \mathbf{R}\langle N, J_1N, X\rangle.$$

Let $[N, J_1N] = X$ and $[J_1X, X] = \lambda X$. Then

$$(J_1 - J_2)[N, J_1X] = -X,$$

whaich is again a contradiction.

Z is a null vector, $|Z|^2 = 0$: According to Lemma 2.6 all null vectors of $\mathfrak{g}'$ are contained in two two-dimensional planes:

$$\operatorname{null}(\mathfrak{g}') = \pi_1 \cup \pi_- = \mathbf{R}\langle N, J_1N\rangle \cup \{V | J_3V = -V\}.$$

We now study three possible cases $Z = N$, $Z \in \pi_-$ and $Z \in \pi_1$.

$\mathbf{Z} = \mathbf{N}$ (the normal to $\mathfrak{g}'$ is a center of $\mathfrak{g}'$): Then we have a decomposition:

$$\mathfrak{g} = \mathbf{R}\langle N, J_1N, X, J_1X\rangle, \qquad \mathfrak{g}' = \mathbf{R}\langle N, J_1N, X\rangle.$$

Because of the integrability of para-hypecomplex structure (J_1, J_2), we have
(3.19)

$$[J_1N, X] = \lambda N,\ [J_1X, N] = N,\ [J_1X, X] = aN + bJ_1N + cX,\ \lambda \neq 0.$$

The Jacobi identity is equivalent to $c = \lambda$. After some scaling we get the algebras PHC10.

$Z \in \pi_-, Z \neq N$, (Z is -1 eigenvector of J_3). Then $Z = aN + b(J_1N + 2X)$ and we have the decomposition:

$$\mathfrak{g} = \mathbf{R}\langle N, J_1N, Z, J_1Z\rangle, \qquad \mathfrak{g}' = \mathbf{R}\langle N, J_1N, Z\rangle.$$

Due to the Heisenberg algebra structure of $\mathfrak{g}'$ we may assume

$$[Z, J_1Z] = Z, \quad [N, J_1N] = \lambda Z, \quad \lambda \neq 0.$$

Because of the integrability of J_1 and J_2 we have

$$[J_1N, J_1Z] = J_1[N, J_1Z] = J_2[N, J_1Z],$$

and then

$$[N, J_1Z] = \alpha N, \text{ and } [J_1N, J_1Z] = \alpha J_1N, \quad \alpha \neq 0.$$

Now, by the Jacobi identity,

$$\begin{aligned} [N, J_1Z] &= \alpha N, & [Z, J_1Z] &= 2\alpha Z, \\ [J_1N, J_1Z] &= \alpha J_1N, & [N, J_1N] &= \lambda Z, \end{aligned}$$

$\alpha, \lambda \neq 0$. After scaling it is a special case of relations PHC10.

$\mathbf{Z} \in \boldsymbol{\pi}_1$, $Z = aN + J_1N$, $a \in \mathbf{R}$. Consider the decomposition

$$\mathfrak{g} = \mathbf{R}\langle N, Z, X, J_1X\rangle, \qquad \mathfrak{g}' = \mathbf{R}\langle N, Z, X\rangle.$$

Let $[N, X] = Z$ and $[J_1X, Z] = \lambda Z$. By the integrability,

$$(J_1 - J_2)[N, J_1X] = 2\lambda Z - 2aN,$$

which implies $\lambda = 0$, i.e., Z is in the center of $\mathfrak{g}$. That is a contradiction.

3.4. The proof of Theorem 1.1. According to the Levi decomposition theorem every Lie algebra $\mathfrak{g}$ decomposes into direct sum

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s},$$

where $\mathfrak{r}$ is the maximal solvable ideal (radical) and $\mathfrak{s}$ is the semisimple part. Since $\mathfrak{so}(3)$ and $\mathfrak{sl}_2(\mathbf{R})$ are the only semisimple Lie algebras of dimension less or equal to 4, the only non-solvable Lie algebras of dimension four are

$$\mathbf{R} \oplus \mathfrak{so}(3) \quad \text{and} \quad \mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R}).$$

They both have a non-trivial center $\mathbf{R}$, so from Theorem 3.1 we conclude that the unique non-solvable Lie algebra admitting a para-hypercomplex structure is $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R})$, i.e., PHC2. Solvable four-dimensional Lie algebras with nontrivial center and admitting a para-hypercomplex structure are PHC1 and PHC3-PHC6 (Theorem 3.1). Solvable four-dimensional Lie algebras with trivial center and admitting a para-hypercomplex structure are PHC7-PHC10 (Theorems 3.2, 3.3 and 3.4).

The examples of para-hypercomplex structures on the algebras are given in Section 5. □

4. Comparisons with the work of Barberis. In this section we compare our results with the classification of hypercomplex structures in the paper of Barberis [**5**]. We see that there are many more four-dimensional Lie algebras with para-hypercomplex structure than Lie algebras with hypercomplex structure.

Namely, we have the following.

Theorem 4.1 ([**5**]). *The only four-dimensional Lie algebras admitting an integrable hypercomplex structure are*:

(HC1) $\mathfrak{g}$ *is abelian,*

(HC2) $[X,Y]=W$, $[Y,W]=X$, $[W,X]=Y$,

(HC3) $[X,Z]=X$, $[X,W]=Y$, $[Y,Z]=Y$, $[Y,W]=-Y$,

(HC4) $[W,X]=X$, $[W,Y]=Y$, $[W,Z]=Z$,

(HC5) $[W,X]=X$, $[W,Y]=Y/2$, $[W,Z]=Z/2$, $[Z,Y]=X$.

The Lie algebra HC2 is isomorphic to $\mathbf{R}\oplus\mathfrak{so}(3)$ and it does not admit a para-hypercomplex structure. Its counterpart admitting a para-hypercomplex (but not hypercomplex) structure is algebra $\mathbf{R}\oplus\mathfrak{sl}(2)$.

No algebra $\mathfrak{g}$ with $\dim\mathfrak{g}'=1$ admits a hypercomplex structure, while algebras $\mathbf{R}\oplus\mathfrak{h}_3$ and $\mathbf{R}\oplus\mathfrak{aff}(\mathbf{R})$ admit a para-hypercomplex structure and satisfy $\dim\mathfrak{g}'=1$.

The Lie algebra HC3 is isomorphic to $\mathfrak{aff}(\mathbf{C})$ and it is the only Lie algebra with dim $\mathfrak{g}'=2$ admitting a hyper-complex structure. It also admits a para-hypercomplex structure.

The Lie algebra HC4 corresponds to real hyperbolic space $\mathbf{R}H^4$. It admits both hypercomplex and para-hypercomplex structure $(\mathfrak{t}_{4,1,1})$.

Finally, the Lie algebra HC5 corresponds to complex hyperbolic space $\mathbf{C}H^2$. It admits both hypercomplex and para-hypercomplex structure $(\mathfrak{d}_{4,(1/2)})$.

5. Equivalence of structures. In Lemma 2.2 we have proved that any compatible structure (J_x,J_y) on the Lie algebra $\mathfrak{g}$ gives rise to the same geometry of $\mathfrak{g}$, i.e., the induced metrics are isometric. Hence, we do not distinguish compatible structures from a geometrical point of view. Thus, we will also use the weaker version of the standard notion of equivalent structures.

Definition 5.1. The structures (J_1,J_2) and (J_1',J_2') on Lie algebra $\mathfrak{g}$ are *compatibly equivalent* if there exist an automorphism ϕ of Lie algebra $\mathfrak{g}$ and structures (J_x,J_y) and $(J_{x'},J_{y'})$ compatible with (J_1,J_2) and (J_1',J_2'), respectively, which commute with ϕ:

$$\phi\circ J_x=J_{x'}\circ\phi,\qquad \phi\circ J_y=J_{y'}\circ\phi.$$

The equivalence ϕ in that case is conformal with respect to the induced metrics on $\mathfrak{g}$ (it is a homothety). The other way around, non-equivalent structures may induce metrics on $\mathfrak{g}$ which are not conformal to each other.

In this section we find equivalence classes of *para-hypercomplex* structures on four-dimensional algebras $\mathfrak{g}$.

In the case of abelian four-dimensional Lie algebra $\mathfrak{g} = \mathbf{R}^4$ the automorphism ϕ is any linear map, since there are no obstructions coming from the commutator relations. Therefore, any two structures on the abelian algebra are equivalent.

5.1. Algebra PHC2.

Theorem 5.1. *Integrable* para-hypercomplex *structure on* $\mathbf{R} \oplus \mathfrak{sl}_2(\mathbf{R})$ *is unique up to the equivalence.*

Let $\mathfrak{g}$ be a Lie algebra $\mathbf{R}\oplus\mathfrak{sl}_2(\mathbf{R})$ with an integrable *para-hypercomplex* structure (J_1, J_2). By the proof of Theorem 3.1, case 1, for non-null central element Z and the basis $(Z, X = J_1Z, Y = J_2Z, W = J_3Z)$, we have

$$[X, Y] = W, \quad [X, W] = -Y, \quad [Y, W] = -X,$$

and similar relations for arbitrary *para-hypercomplex* structure $\{\widetilde{J}_1, \widetilde{J}_2\}$ on $\mathfrak{g}$ for some $\widetilde{Z}$. The automorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$, $\Phi Z = \widetilde{Z}$, $\Phi J_\alpha Z = \widetilde{J}_\alpha \widetilde{Z}$, $\alpha \in \{1, 2, 3\}$ is the equivalence between these two *para-hypercomplex* structures.

In the basis (Z, J_1Z, J_2Z, J_3Z) this structure reads:

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

We will refer to this structure as $J2$.

Remark 5.1. Note that the center Z of $\mathfrak{g}$ is non-null vector and that Z is orthogonal to $\mathfrak{g}'$.

Remark 5.2. The simply connected Lie group with Lie algebra $\mathbf{R} \oplus \mathfrak{sl}(2, \mathbf{R})$ is the multiplicative group $\widetilde{\mathbf{H}}^*$ of para-quaternionic numbers of norm one.

5.2. Algebra PHC3. The equivalence of structures is Lie algebra automorphism and hence preserves the center of algebra $\mathfrak{g}$. On the other hand the equivalence is conformal and hence preserves the metric type of the center. From the proof of Theorem 3.1, we see that $\mathbf{R} \oplus \mathfrak{r}_{3,1}$ (PHC3) appears twice and we have to consider the following non-equivalent classes: the central element is non-null and the central element is null.

Theorem 5.2. *On the algebra* $\mathbf{R} \oplus \mathfrak{r}_{3,1}$ *there is* S^1*-family of non-equivalent integrable* para-hypercomplex *structures such that the center of the algebra is non-null. There is a unique integrable* para-hypercomplex *structure* (J_1, J_2) *with the null center* (*up to a compatibilty equivalence*)*.*

Proof. Let $\mathfrak{g}$ be a Lie algebra $\mathbf{R} \oplus \mathfrak{r}_{3,1}$, and (J_1, J_2) a *para-hypercomplex* structure on $\mathfrak{g}$. Suppose that the center of $\mathfrak{g}$ is not null with respect to the metric induced by (J_1, J_2). By the proof of Theorem 3.1, case 1, for non-null central element Z and the basis $(Z, X = J_1 Z, Y = J_2 Z, W = J_3 Z)$, we have

$$[X, Y] = aZ + bX + cY, \tag{5.2}$$

$$[X, W] = -bZ + aX + cW, \tag{5.3}$$

$$[Y, W] = cZ + aY - bW, \tag{5.4}$$

for some $a, b, c \in \mathbf{R}$, $a^2 + b^2 = c^2$, $c \neq 0$. Similarly, for arbitrary *para-hypercomplex* structure $(\widetilde{J}_1, \widetilde{J}_2)$ on $\mathfrak{g}$ we obtain the corresponding relations for some $\widetilde{a}, \tilde{b}, \widetilde{c} \in \mathbf{R}$, $\widetilde{a}^2 + \tilde{b}^2 = \widetilde{c}^2$, $\widetilde{c} \neq 0$. Suppose that $\Phi : \gamma \to \gamma$ is an equivalence of these two *para-hypercomplex* structures. Since the center is one-dimensional, $\Phi Z = \alpha Z$, $\alpha \neq 0$ and $\Phi X = \alpha \widetilde{X}$, $\Phi Y = \alpha \widetilde{Y}$, $\Phi W = \alpha \widetilde{W}$, where $\widetilde{J}_1 Z = \widetilde{X}$, $\widetilde{J}_2 Z = \widetilde{Y}$,

$\widetilde{J}_3 Z = \widetilde{W}$. Then, using the relations $\Phi[X,Y] = [\Phi X, \Phi Y]$, from (5.2) we get $(a,b,c) = \alpha(\widetilde{a}, \tilde{b}, \widetilde{c})$. That is, the equivalence classes of *para-hypercomplex* structures are parameterized by the points of $\mathbf{R}P^1 = S^1$.

The one-parameter family of *para-hypercomplex* structures $J3A(\phi)$ is given in the following way

(5.5)

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 1 & -\cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & -\cos\phi & 0 \\ 0 & \sin\phi & 1 & \cos\phi \end{pmatrix}.$$

Now we consider *para-hypercomplex* structures with null center. Let $\mathfrak{g}$ be a Lie algebra $\mathbf{R} \oplus \mathfrak{r}_{3,1}$, and (J_1, J_2) and $(\widetilde{J}_1, \widetilde{J}_2)$ two equivalent *para-hypercomplex* structures on $\mathfrak{g}$. According to Lemma 2.5, up to a compatibility, we can assume $N = J_1 X - J_2 X$ and $N = \widetilde{J}_1(\widetilde{J}_1 - \widetilde{J}_2)\widetilde{X}$ for some unit vectors $X, \widetilde{X} \in \mathfrak{g}$. Moreover, from the proof of Theorem 3.1, case $2b$, in the basis $(N, J_1 N, X, J_1 X)$ the commutators are

(5.6)

$$[X, J_1 N] = aN + bJ_1 N + 2bX, \quad [J_1 X, J_1 N] = -bN + aJ_1 N + 2bJ_1 X,$$

$$[X, J_1 X] = -\frac{a^2 + b^2}{2b} N - aX + bJ_1 X, \tag{5.7}$$

where $b \neq 0$. For any other *para-hypercomplex* structure $(\widetilde{J}_1, \widetilde{J}_2)$, with a null center, the commutators in the basis $(N, \widetilde{J}_1 N, \widetilde{X}, \widetilde{J}_1 \widetilde{X})$ have the similar form for some $\widetilde{a}, \tilde{b}, \widetilde{c} \in \mathbf{R}$, $\tilde{b} \neq 0$. Let $\Phi : \mathfrak{g} \to \mathfrak{g}$ be the equivalence between these two structures. Let $\Phi X = \alpha N + \beta \widetilde{J}_1 N + \mathbf{C}X + \delta \widetilde{J}_1 X$. Since the algebra $\mathbf{R} \oplus \mathfrak{r}_{3,1}$ is with one-dimensional center $\Phi N = pN$, $p \neq 0$. By applying Φ to the relation $NJ_1 X - J_2 X$ we get $\gamma = 2\beta + p$, $\delta = 0$ and hence

$$\Phi X = \alpha N + \beta \widetilde{J}_1 N + (2\beta + p)X.$$

Since Φ is an equivalence,

$$\Phi[X, J_1 N] = [\Phi X, \widetilde{J}_1 \Phi N], \qquad \Phi[J_1 X, J_1 N] = [\widetilde{J}_1 \Phi X, \widetilde{J}_1 \Phi N]$$

and

$$p = \frac{b}{\tilde{b}}, \quad \alpha = \frac{\widetilde{a} - a}{2\tilde{b}}, \quad \beta = \frac{\tilde{b} - b}{2\tilde{b}}.$$

The compatibility condition for the third commutator is fulfilled automatically since the third commutator is dependent.

Thus, Φ is an equivalence for all $a, b; \widetilde{a}, \tilde{b}$, with $b, \tilde{b} \neq 0$. The conclusion is that all structures with null center are equivalent. The structure $J3B$ for $a = 0$, $b = 1$ is

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \qquad \square \tag{5.8}$$

5.3. Algebra PHC4.

Theorem 5.3. *If the algebra* $\mathbf{R} \oplus \mathfrak{h}_3$ *admits an integrable* para-hypercomplex *structure, then its two-dimensional center is necessarily totally null. There is a* S^1 *family of non-equivalent integrable* para-hypercomplex *structures on the algebra* $\mathbf{R} \oplus \mathfrak{h}_3$*. Any such structure is compatibly equivalent to some structure from the given* S^1*-family.*

Proof. In the proof of Theorem 3.1, this algebra appears only in the case $2a$) where all central elements are null. There exist an isotropic central element N, and $X, \widetilde{X} \in \mathfrak{g}$ such that

$$N = J_1X - J_2X = \widetilde{J}_1\widetilde{X} - \widetilde{J}_2\widetilde{X}.$$

Also, from the proof of Theorem 3.1 we know that in the corresponding basses (N, J_1N, X, X, J_1X) and $(N, \widetilde{J}_1N, \widetilde{X}, \widetilde{X}_1\widetilde{X})$ the nonzero commutators are

$$[X, J_1X] = mN + nJ_1N, \quad [\widetilde{X}, \widetilde{J}_1\widetilde{X}] = \widetilde{m}N + \widetilde{n}\widetilde{J}_1N. \tag{5.9}$$

The equivalence $\Phi : \mathfrak{g} \to \mathfrak{g}$ of the *para-hypercomplex* structures (J_1, J_2) and $(\widetilde{J}_1, \widetilde{J}_2)$ is of the form

$$\Phi X = \alpha N + \beta\widetilde{J}_1N + \gamma\widetilde{X} + \delta\widetilde{J}_1\widetilde{X}, \qquad \Phi N = pN + q\widetilde{J}_1N, \tag{5.10}$$

where $\alpha, \beta, \gamma, \delta, p, q \in \mathbf{R}$. If we replace the relation $N = J_1X - J_2X$ into (5.10), we get $p = \gamma - 2\beta$, $q = 0 = \delta$, i.e.,

$$\Phi N = (\gamma - 2\beta)N, \qquad \Phi X = \alpha N + \beta\widetilde{J}_1N + \gamma\widetilde{X},$$

with $\gamma \neq 0$, $\gamma - 2\beta \neq 0$. From the commutators (5.9), we see that Φ is an equivalence if and only if the system

$$m(\gamma - 2\beta) = \gamma^2 \widetilde{m}, \qquad n(\gamma - 2\beta) = \gamma^2 \widetilde{n}\widetilde{n}, \tag{5.11}$$

has a non-trivial solution for $(\alpha, \beta, \gamma, \delta)$, that is, if and only if the nonzero vectors (m, n) and $(\widetilde{m}, \widetilde{n})$ are proportional and we have an S^1 family of non-equivalent structures. To find them explicitly, denote $Y = J_1X$, $Z = mN + nJ_1N$, $W = -J_1Z$. Then in the basis (X, Y, Z, W) the commutator of $\mathbf{R} \oplus \mathfrak{h}_3$ reads $[X, Y] = Z$. One can check easily that the S^1 family $J4(\phi)$ of non-equivalent structures is:
(5.12)

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_2^{\phi} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\cos\phi & \sin\phi & \sin 2\phi & \cos 2\phi \\ \sin\phi & \cos\phi & \cos 2\phi & -\sin 2\phi \end{pmatrix}. \square$$

5.4. Algebra PHC5.

Theorem 5.4. *If the algebra $\mathbf{R} \oplus \mathfrak{aff}(\mathbf{R})$ admits an integrable* para-hypercomplex *structure, then its two-dimensional center is necessarily totally null. Moreover, every such structure is compatibly equivalent to some structure from the two S^1-families of non-equivalent* para-hypercomplex *structures on the algebra $\mathbf{R} \oplus \mathfrak{aff}(\mathbf{R})$. For the first family the commutator g' is null space and for the second g' is not null space.*

Proof. This algebra appears only in the case 2a) of the proof of Theorem 3.1, where all central elements are null. To find non-equivalent structures we start similarly as in the proof of Theorem 5.3. That is, for the bases (N, J_1N, X, J_1X), $N = J_1X - J_2X$ and $(N, \widetilde{J}_1N, \widetilde{X}, \widetilde{J}_1\widetilde{X})$, $N = \widetilde{J}_1\widetilde{X} - \widetilde{J}_2\widetilde{X}$, the endomorphism Φ compatible with the structure (J_1, J_2) is of the form:

$$\Phi N = (\gamma - 2\beta)N, \qquad \Phi X = \alpha N + \beta \widetilde{J}_1 N + \gamma \widetilde{X},$$

where $\gamma(\gamma - 2\beta) \neq 0$. The corresponding non-zero commutators are of the form:

$$[X, J_1X] = mN + nJ_1N + kX + fJ_1X,$$
$$[\widetilde{X}, \widetilde{J}_1\widetilde{X}] = \widetilde{m}N + \widetilde{n}\widetilde{J}_1N + \tilde{k}\widetilde{X} + \tilde{f}\widetilde{J}_1\widetilde{X},$$

with k^2+f^2, $\tilde{k}^2+\tilde{f}^2 \neq 0$. If we assume that $|X|^2=1$, then the square norm of the commutator is

$$|[X, J_1X]|^2 = f^2+k^2+2(fm-kn).$$

It is clear that if the structures (J_1, J_2) and $(\widetilde{J}_1, \widetilde{J}_2)$ are equivalent then the commutators are simultaneously null or non-null, i.e.,

$$f^2+k^2+2(fm-kn) = \lambda(\tilde{f}^2+\tilde{k}^2+2(\tilde{f}\widetilde{m}-\tilde{k}\widetilde{n})), \quad \lambda \neq 0.$$

The condition that Φ is Lie algebra endomorphism is equivalent to the following:

$$k=\gamma\tilde{k}, \quad f=\gamma\tilde{f} \tag{5.13}$$

$$\alpha(-k)+\beta(f+2m) = \gamma(m-\gamma m'),$$
$$\alpha(-f)+\beta(-k+2n) = \gamma(n-\gamma n'). \tag{5.14}$$

From (5.13) it follows that the structures are equivalent only if $(k,f) = \gamma(\tilde{k},\tilde{f})$ what we assume in the sequel. Equations (5.14) are then linear equations over α and β for some fixed γ. The determinant of that system $D=f^2+k^2+2(fm-kn)$ is exactly the square norm of the commutator $[X, J_1X]$.

If $D \neq 0$ then there exist the unique solution α, β. The condition $\gamma(\gamma-2\beta)\neq 0$ (non-degeneracy of Φ) is equivalent to $\tilde{f}^2+\tilde{k}^2+2(\tilde{f}\widetilde{m}-\tilde{k}\widetilde{n}) = |[\widetilde{X}, \widetilde{J}_1\widetilde{X}]|^2 \neq 0$. Hence, in the case of non-null commutator g' there exist S^1 family of non-equivalent structures.

If $D=0$ then one can show, by using (5.13) that equations (5.14) are dependent and the solution is not unique. The condition $\gamma(\gamma-2\beta)\neq 0$ is easily achieved and we again have S^1 family of non-equivalent structures.

Let us write these structures explicitly. In the case of non-null commutator g' we may choose $m=n=0$ and introduce a new basis

$$Y' = \cos\phi X + \sin\phi J_1 X, \quad X' = -J_1Y', \quad Z' = N, \quad W' = J_1N.$$

In that basis the only nonzero commutator is $[X', Y'] = Y'$ and the structures $J5A(\phi)$ are given by:

(5.15)

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_2^{\phi} = \begin{pmatrix} -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ -\cos 2\phi & \sin 2\phi & 0 & 0 \\ -\sin\phi & -\cos\phi & 0 & 1 \\ -\cos\phi & \sin\phi & 1 & 0 \end{pmatrix}.$$

In the case of null commutator g' we may choose

$$(m,n,k,f) = (-(1/2)\sin\phi, (1/2)\cos\phi, \cos\phi, \sin\phi),$$

and introduce the new basis

$$Y' = -(1/2)\sin\phi\, N + (1/2)\cos\phi\, J_1 N + \cos\phi\, X + \sin\phi\, J_1 X,$$
$$X' = -J_1 Y', \quad Z' = N, \quad W' = J_1 N.$$

In that basis the only nonzero commutator is $[X', Y'] = Y'$ and the structures $J5B(\phi)$ are given by:
(5.16)

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2^{\phi} = \begin{pmatrix} -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ -\cos 2\phi & \sin 2\phi & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad \square$$

5.5. Algebra PHC6.

Theorem 5.6. *If the algebra $\mathfrak{d}_4$ admits an integrable* para-hypercomplex *structure, then its one-dimensional center is necessarily null and the induced metric on three-dimensional $\mathfrak{g}'$ is degenerate (of rank 2). Any such structure is compatibly equivalent to the some of the following non-equivalent structures: $(J_1^{0,0}, J_2^{0,0})$ $(J6C)$, $(J_1^{0,1}, J_2^{0,1})$ $(J6B)$ and the one-parameter family $(J_1^{1,n}, J_2^{1,n})$, $n \in \mathbf{R}$, $(J6A)$ defined below.*

Proof. This algebra appears only in case 2c) of the proof of Theorem 3.1 where all central elements are null. As before we suppose that

$$[X, J_1N] = N, \quad [J_1X, J_1N] = J_1N, \quad [X, J_1X] = mN + nJ_1N + X,$$

and

$$[\widetilde{X}, \widetilde{J}_1N] = N, \quad [\widetilde{J}_1\widetilde{X}, \widetilde{J}_1N] = \widetilde{J}_1N, \quad [\widetilde{X}, \widetilde{J}_1\widetilde{X}] = \widetilde{m}N + \widetilde{n}\widetilde{J}_1N + \widetilde{X}.$$

An equivalence $\Phi : \mathfrak{g} \to \mathfrak{g}$ is determined by $\Phi X = \alpha N + \beta\widetilde{J}_1N + \gamma\widetilde{X}$, $\Phi N = pN$, $p = \gamma - 2\beta$ and it exists if and only if the system of equations

$$mp = \widetilde{m}, \quad (n-1)p = \widetilde{n} - 1, \quad \gamma = 1,$$

has a solution such that $p \neq 0$. The result follows by studding of the equations. To get the structures explicitly choose the base

$$X' = X + \frac{n}{2} J_1 N, \quad Y' = J_1 X - m J_1 N, \quad Z' = N, \quad W' = J_1 N.$$

We get the commutators for $\mathfrak{d}_4$ and the corresponding *para-hypercomplex* structures $(J_1^{m,n}, J_2^{m,n})$ are defined by

$$J_1^{m,n} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -n/2 & m & 0 & -1 \\ m & n/2 & 1 & 0 \end{pmatrix},$$

$$J_2^{m,n} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ (n-2)/2 & -m & 0 & 1 \\ m & (2-n)/2 & 1 & 0 \end{pmatrix}. \qquad \square$$

5.6. Algebras PHC7 and PHC8. Depending on the sign of $4a+b^2$ the algebra PHC7 is $\mathfrak{d}_{4;1}$, $\mathfrak{aff}(\mathbf{C})$, or $\mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R})$. The algebras PHC8 and $\mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R})$ by definition coincide.

Each of these algebras appears twice in the proof of Theorem 3.3 depending on the metric type induced on its two-dimensional derived algebra $\mathfrak{g}'$, and we have to analyze these situations. It is interesting that the case of totally null $\mathfrak{g}'$ happens for each algebra.

Theorem 5.6. *On the algebra $\mathfrak{g} = \mathfrak{aff}(\mathbf{C})$ there are two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to the one of this two. With respect to one of them the derived algebra $\mathfrak{g}'$ is definite and with respect to the other it is totally null.*

Proof. As first, suppose that two-dimensional commutator subalgebra $\mathfrak{g}'$ is definite with respect to the metric induced by structure (J_1, J_2). Using a compatible structure if necessary we can suppose that J_1 preserves $\mathfrak{g}'$. Following the proof of Theorem 3.3, case i) and formulas (3.11), for $X \in \mathfrak{g}'$, in the basis $(X, Y = J_1 X, Z = J_2 X, W = -J_3 X)$ the commutator relations are:

$$[X, Z] = cX + dY = [Y, W], \quad [X, W] = dX - cY = -[Y, Z], \ c^2 + d^2 \neq 0.$$

The same construction for some $\widetilde{X} \in \mathfrak{g}'$ and structure $(\widetilde{J}_1, \widetilde{J}_2)$ yields similar relations over some $\widetilde{c}, \tilde{d}$, $\widetilde{c}^2 + \tilde{d}^2 \neq 0$. One can easily check that the automorphism given by

$$\Phi(X) = \frac{c\widetilde{c} + d\tilde{d}}{\widetilde{c}^2 + \tilde{d}^2} X + \frac{\widetilde{c}d - c\tilde{d}}{\widetilde{c}^2 + \tilde{d}^2} Y$$

is equivalence of (J_1, J_2) and $(\widetilde{J}_1, \widetilde{J}_2)$. Hence, all such structures are equivalent to $J71A$

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, suppose that the subalgebra $\mathfrak{g}'$ is totaly null with respect to the metric induced by structure (J_1, J_2). We use a different approach, namely, we choose the basis (X, Y, Z, W) in which the algebra $\mathfrak{g} = \mathfrak{a}ff(\mathbf{C})$ has the simplest commutators

$$[X, Z] = X = -[Y, W], \qquad [Y, Z] = Y = [X, W].$$

Then, according to the proof of Theorem 3.3, case iii) up to a compatibility we can choose (J_1, J_2) such that

$$J_2(X) = X, \quad J_2(Y) = Y, \quad J_2(Z) = -Z, \quad J_2(W) = -W,$$

and similarly for $(\widetilde{J}_1, \widetilde{J}_2)$. Then, because of the relations of *para-hypercomplex* structure and the integrability the structure J_1 has the form

$$J_1(X) = dZ - cW, \qquad J_1(Y) = cZ + dW, \quad c^2 + d^2 \neq 0,$$

and similarly for $\widetilde{J}_1$ and some $\widetilde{c}, \tilde{d}$. These two structures are equivalent. Namely, the equivalence Φ is given by

$$\Phi(Z) = Z, \quad \Phi(W) = W, \quad \Phi(X) = \delta X - \gamma Y, \quad \Phi(Y) = \gamma X + \delta Y,$$

with $\gamma = (\tilde{d}c - \tilde{c}d)/(\tilde{c}^2 + \tilde{d}^2)$, $\gamma = (\tilde{c}c + \tilde{d}d)/(\tilde{c}^2 + \tilde{d}^2)$. We have proved that any two such structures are equivalent to $J71B$

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \square$$

Theorem 5.7. *On the algebra $\mathfrak{g} = \mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R})$ there are two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to one of these two. With respect to one of them the derived algebra $\mathfrak{g}'$ is Lorencian and with respect to the other it is totally null.*

Proof. The Lorencian case appears in case ii) of the proof of Theorem 3.3. The proof that all such structures are equivalent is easy and we give only the structure $J72A$ in the canonical basis of $\mathfrak{aff}(\mathbf{R}) \oplus \mathfrak{aff}(\mathbf{R})$ (see Section 2)

$$J_1 = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The case of totally null $\mathfrak{g}'$ appears in case iii) or the proof of Theorem 3.3. Note that the case of algebra PHC8 is contained here. The proof that all the structures are equivalent is similar to one in Theorem 5.6. The structure $J72B$ in the canonical basis is

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \square$$

Theorem 5.8. *On the algebra $\mathfrak{g} = \mathfrak{d}_{4,1}$ there are two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to the one of this two. With respect to one of them* *the derived algebra $\mathfrak{g}'$ is degenerate (has exactly one null direction) and with respect to the other $\mathfrak{g}'$ is totally null.*

Proof. The case of totaly null derived algebra $\mathfrak{g}'$ appears in case iii) of the proof of Theorem 3.3 and can be proved in the same way as in Theorem 5.6. In the canonical basis of $\mathfrak{d}_{4;1}$ (see Section 2) the structure $J73A$ is

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Suppose that the derived algebra $\mathfrak{g}'$ is degenerate with respect to the integrable *para-hypercomplex* structure (J_1, J_2) of $\mathfrak{g}$. (here we follow case iv) of the proof of Theorem 3.3). For the null direction $N \in \mathfrak{g}'$, up to a compatibility we can suppose that $N = J_1X - J_2X$ for $X \in \mathfrak{g}'$. By scaling vectors N and X we could have achieved $a = 1$ in the proof of Theorem 3.3. In the basis (N, X, J_1N, J_X) the commutator relations are

$$[N, J_1X] = N = [X, J_1N], \qquad [X, J_1X] = bN + X, \quad b \in \mathbf{R}.$$

For another such structure $(\widetilde{J}_1, \widetilde{J}_2)$ in the corresponding basis $(N, \widetilde{X}, J_1N, \widetilde{J}_X)$ the similar commutator relations hold for some $\tilde{b} \in \mathbf{R}$. The automorphism Φ between the structures is given by

$$\Phi(N) = N, \qquad \Phi(X) = (b - \tilde{b})X + N.$$

Hence, all such structures are equivalent. In the canonical basis one of these structures $J73B$ is

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \qquad \square$$

5.7. Algebra PHC9. The algebra PHC9 corresponds to the algebras $\mathfrak{t}_{4,1,c}$ and $\mathfrak{r}_{4,1}$ as explained in Lemma 2.1. Note that the algebra

$\mathfrak{t}_{4,1,1}$ corresponds to the real hyperbolic space $\mathbf{R}H^4$. The algebra PHC9 appears twice in Theorem 3.4, once in its general form, with degenerate $\mathfrak{g}'$, and the other time with non-degenerate $\mathfrak{g}'$ in the form of $\mathfrak{t}_{4,1,1}$.

Theorem 5.9. *On the Lie algebra $\mathfrak{g} = \mathfrak{t}_{4,1,c}$, $c \neq 1$ there exist two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to the one of this two. In both cases the three-dimensional subalgebra $\mathfrak{g}'$ is degenerate with respect to the induced metric.*

Proof. From the proof of Theorem 3.4 one can see that this case happens when the subalgebra $\mathfrak{g}'$ is abelian and degenerated with respect to the metric induced by *para-hypercomplex* structure (J_1, J_2).

This part of the proof works for all algebras PHC9, with degenerated $\mathfrak{g}'$. As in the proof of Theorem 3.4 let $N = J_1X - J_2X$. From formulas (3.18) we have that

$$[N, J_1X] = N, \quad [J_1N, J_1X] = J_1N, \quad [X, J_1X] = aN + bJ_1N + cX,$$

for some numbers a, b, c with $c \neq 0$. If $(\widetilde{J}_1, \widetilde{J}_2)$ is an equivalent structure, for some $\widetilde{X} \in \mathfrak{g}'$, $N = \widetilde{J}_1\widetilde{X} - \widetilde{J}_2\widetilde{X}$ we have similar relations with coefficients $\widetilde{a}, \widetilde{b}, \widetilde{c}$, and $\widetilde{c} \neq 0$. One easily checks that the equivalence Φ between the structures (J_1, J_2) and $(\widetilde{J}_1, \widetilde{J}_2)$ has to be of the form $\Phi : \mathfrak{g} \to \mathfrak{g}$, $\Phi X = \alpha N + \beta \widetilde{J}_1 N + \widetilde{X}$, $\Phi N = (1 - 2\beta)N$, $\beta \neq 1/2$. From the compatibility with commutators we also have:

$$(5.17) \qquad c = \widetilde{c}, \quad \beta(1 - c + 2b) = b - \widetilde{b}, \quad \alpha(1 - c) + 2a\beta = a - \widetilde{a}.$$

Since we are interested in the algebra $\mathfrak{g} = \mathfrak{t}_{4,1,c}$, $c \neq 1$, fix some $c \neq 0, 1$ and let $2b = c - 1$.

Then $b = \widetilde{b}$ and all structures determined by any a are equivalent. To find one of them explicitly, let $a = 0$, $b = (c-1)/2$. By choosing $X' = 2X + J_1N$, $Y' = N$, $Z' = J_1N$, $W' = J_1X$ we get the commutator relations of $\mathfrak{t}_{4,1,c}$, $c \neq 1$,

$$[X', W'] = cX', \quad [Y', W'] = Y', \quad [Z', W'] = Z',$$

and the structure $J91A$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & -1/2 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/2 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

Now, fix some $c \neq 0, 1$ and let $2b \neq c - 1$. Then, the solution α, β exists and all such structures are equivalent. To get a particular one, we can choose the structure determined by $a = 0$ and $b = 0 \neq (c-1)/2$. We immediately get the relations of the algebra $\mathfrak{t}_{4,1,c}$, $c \neq 1$, and the structure $J91B$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.18}$$

This structure is not equivalent to the previous since otherwise we would have $\beta = 1/2$. □

Theorem 5.10. *On the Lie algebra* $\mathfrak{g} = \mathfrak{t}_{4,1,1}$ *there exist two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to the one of the following two. In one case the three-dimensional subalgebra* $\mathfrak{g}'$ *is degenerate with respect to the induced metric and in the other case it is non-degenerate.*

Proof. In the degenerate case, take $c = 1$ and $a = 0 = b$ in the proof of Theorem 5.9. We immediately get the commutator relations of $\mathfrak{t}_{4,1,1}$ and the structure (5.18) (which we also denoted by $J92A$).

In the non-degenerate case one can prove that the structure is unique up to equivalence, and the structure $J92B$ is given by:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ □

Theorem 5.11. *On the Lie algebra* $\mathfrak{g} = \mathfrak{r}_{4,1}$ *there exist* S^1*-family of non-equivalent integrable* para-hypercomplex *structures. Any other is* *compatibly equivalent to some structure from this family. The induced metric on $\mathfrak{g}'$ is degenerate.*

Proof. We start as in the proof of Theorem 5.9 and suppose that $c = 1$ and $a^2 + b^2 \neq 0$. Then from the relations (5.17) we get that the structures are equivalent if and only if (a, b) is proportional to $(\widetilde{a}, \widetilde{b})$. Hence, we have an S^1 family of non-equivalent structures. One can show that in the canonical basis of $\mathfrak{r}_{4,1}$ this family $J93(\phi)$ is given by

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -\sin 2\phi & \cos 2\phi & \sin\phi & \cos\phi \\ \cos 2\phi & \sin 2\phi & -\cos\phi & \sin\phi \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad \square$$

5.8. Algebras PHC10. The algebra PHC10 is one of the algebras $\mathfrak{d}_{4,\lambda}$ for $\lambda \neq 1, 0$ or $\mathfrak{h}_4$ as explained in Lemma 2.1. However, the algebra $\mathfrak{d}_{4,1/2}$ is considered separately since it admits many more non-equivalent structures.

Theorem 5.12. *On the Lie algebra $\mathfrak{g} = \mathfrak{d}_{4,1/2}$ there are five non-equivalent structures. Any other is compatibly equivalent to the one of this five structures.*

Proof. By study of the proof of Theorem 3.4 we see that there are four geometrically different cases with the last resulting in two non-equivalent structures. To describe them, denote a central element of the commutator subalgebra $\mathfrak{g}' \cong \mathfrak{h}_3$ by Z, and let $(\mathfrak{g}')^\perp$ be the one-dimensional space orthogonal to $\mathfrak{g}'$.

The first case: $\operatorname{sign}(Z) = \operatorname{sign}((\mathfrak{g}')^\perp)$ and $(\mathfrak{g}')^\perp \not\subset \mathfrak{g}'$, that is, $\mathfrak{g}'$ is not degenerated. By following the proof of Theorem 3.4 one can easily prove that all such structures are equivalent and one of them is $J101A$:

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

The second case: $\operatorname{sign}(Z) = -\operatorname{sign}((\mathfrak{g}')^{\perp})$ and $(\mathfrak{g}')^{\perp} \not\subset \mathfrak{g}'$, that is $\mathfrak{g}'$ is not degenerated. One can prove that all such structures are equivalent and one of them is $J101B$:

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The third case: $(\mathfrak{g}')^{\perp} \subset \mathfrak{g}'$, that is, $\mathfrak{g}'$ is degenerated and $Z \in (\mathfrak{g}')^{\perp}$. This is a generic case, so we calculate it for any algebra PHC10. From the formulas (3.19), up to a compatibility, the commutators in the basis (N, J_1N, X, J_1X) are:

$$\begin{aligned} [J_1X, N] &= N, & [J_1X, J_1N] &= (1-\lambda)J_1N, \\ [J_1N, X] &= \lambda N, & [J_1X, X] &= aN + bJ_1N + \lambda X, \end{aligned}$$

for some $a, b \in \mathbf{R}$, $\lambda \neq 0, 1$. Here, $N = Z$ is a central vector of $\mathfrak{g}'$, as well as normal to $\mathfrak{g}'$ and $X \in \mathfrak{g}'$. If $(\widetilde{J}_1, \widetilde{J}_2)$ is another such structure we have similar commutators for some $\widetilde{X} \in \mathfrak{g}'$ and some real numbers $\widetilde{a}, \tilde{b}, \widetilde{\lambda} \neq 0, 1$. The map Φ compatible with these two structures is of the form

$$\Phi(N) = (2k+n)N, \qquad \Phi(X) = nN - kJ_1N + qX,$$

for some n, k, $2k + n \neq 0$ and $q \neq 0$. Such Φ is an automorphism of $\mathfrak{g}$ if and only if:

$$\widetilde{\lambda} = \lambda, \quad q = 1, \quad n - 2ak = a - \widetilde{a}, \quad k(1 - 2b - 2c) = \tilde{b} - b. \tag{5.19}$$

Now, we specialize to the case of algebra $\mathfrak{d}_{4,1/2}$, i.e., $\lambda = 1/2$, $b = 0$. We see that equations (5.19) always have a solution, that is, all structures are equivalent. To get a particular one we can take $a = 0 = b$ and after scaling of the basis to match the commutators of $\mathfrak{d}_{4,1/2}$, we get the structure $J101C$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1/2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1/2 & -1 \\ 2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The fourth case: $(\mathfrak{g}')^\perp \subset \mathfrak{g}'$, that is, $\mathfrak{g}'$ is degenerated, but $Z \notin (\mathfrak{g}')^\perp$. From the proof of Theorem 3.4, formulas (3.20) the commutators are:

$[Z, J_1Z] = 2\alpha Z,\ [N, J_1N] = \lambda Z,\ [N, J_1Z] = \alpha N,\ [J_1N, J_1Z] = \alpha J_1N,$

$\alpha, \lambda \neq 0$, where $N \in (\mathfrak{g}')^\perp$. The automorphism Φ of the algebra $\mathfrak{g}$ compatible with the structure (J_1, J_2) is of the form $\Phi(N) = pN$, $\Phi(Z)qZ$ for p, q satisfying

$$q = \frac{\alpha}{\tilde{\alpha}}, \quad p^2 = q\frac{\lambda}{\tilde{\lambda}}.$$

Therefore, we have two non-equivalent structures for $(\alpha, \lambda) = (1, \pm 1)$. After scaling the basis (N, J_1N, Z, J_1Z) to match the commutators of $\mathfrak{d}_{4,1/2}$, we get the structures $J101D$ and $J101E$

$$J_1^\pm = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \mp 1 & 0 \end{pmatrix}, \qquad J_2^\pm = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & \mp 1 & 0 \end{pmatrix}. \qquad \square$$

Theorem 5.13. *On the Lie algebra $\mathfrak{g} = \mathfrak{d}_{4,c}$, $c \neq 1/2$, there are two non-equivalent integrable* para-hypercomplex *structures. Any other is compatibly equivalent to the one of the this two. With respect to both structures the commutator algebra $\mathfrak{g}'$ is degenerated and the center of the algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) = (\mathfrak{g}')^\perp$.*

Proof. This algebra is a special case of algebras PHC10 for $c \neq 1/2$, and it appears in the formulas (3.19). By analyzing the formulas (5.19) we find that there are two non-equivalent structures for $b = (2c-1)/2 = \tilde{b}$ and $b \neq (2c-1)/2$. In the first case we set $(a, b) = (0, (2c - 1/2))$ and get the structure $J102A$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -2c & -1/2 \\ 1/(2c) & 1/(2c) & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & -1/2 \\ 0 & 0 & -2c & -1/2 \\ 1/(2c) & -1/(2c) & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}.$$

In the case $b \neq (2c-1)/2$ we can set $(a,b) = (0,0)$ since $c \neq 1/2$, and the structure is $J102B$:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & 1/c & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -c & -1 \\ 1/c & -1/c & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

Theorem 5.14. *Integrable* para-hypercomplex *structure on the Lie algebra* $\mathfrak{g} = \mathfrak{h}_4$ *is unique up to compatible equivalence. With respect to that structure the commutator algebra* $\mathfrak{g}'$ *is degenerated and the center of the algebra* $\mathfrak{g}$ *is* $Z(\mathfrak{g}) = (\mathfrak{g}')^{\perp}$.

Proof. This algebra is a special case of algebras PHC10 for $c = 1/2$, $b = 0$ and it appears in the formulas (3.19). By analyzing formulas (5.19) we see that all such structures are equivalent. To get a particular one we choose $(a,b) = (0,1)$ and get the structure $J103$:

$$J_1 = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1/2 & -1 \\ 0 & 0 & 0 & -2 \\ 2 & -1 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \end{pmatrix}. \quad \square$$

6. Geometry related to phc-structure. Lemma 2.2 says that every integrable *para-hypercomplex* structure (J_1, J_2) on a four-dimensional Lie algebra $\mathfrak{g}$ defines the conformally unique scalar product on $\mathfrak{g}$. Moreover, it says that any compatible structure (J_1', J_2') defines the same scalar product. Possibly different conformal geometries may arise from non-equivalent structures (as defined at the beginning of Section 5).

This scalar product defines left-invariant metric on the corresponding four-dimensional Lie group G. This metric is anti self-dual (see [**8, 10**]). Note also that every such Lie group is a complex manifold admitting a left-invariant neutral metric.

The results were not complete, since only one structure for each Lie algebra was given, in order to prove the existence. However, the results have been already used by Ivanov and Zamkovoy [**8**] to study geometry of four-dimensional Lie groups in more details. They showed that some of this metrics are not conformally flat. In [**8**] it was also checked that the induced conformal structure $[g]$ is actually locally hyper-Kähler.

Moreover, a compact four-dimensional solve-manifold $M = G/\Gamma$ which admit anti-self-dual neutral metrics are considered. The examples obtained are locally conformally hyper-Kähler and most of them are not conformally flat. An important particular case is based on the Lie algebra $\mathfrak{d}_4$. The corresponding solvable Lie group is known as Sol_1^4. The geometric structure modeled on Sol_1^4 is one of the possible geometric structure on four manifolds [**13**]. Its compact quotients by discrete group Γ are the Inoe surfaces modeled on Sol_1^4. In [**8**] the following theorem was proved:

Theorem 6.1 [**8**]. *The Inoe surface* $N = \mathrm{Sol}_1^4/\Gamma$ *modeled on* Sol_1^4 *admit a locally conformally hyper-para-Kähler structure and do not admit any global one. The Lie form is parallel and the Weyl curvature is not zero. Therefore, the Inoe surfaces* $N = \mathrm{Sol}_1^4/\Gamma$ *modeled on* Sol_1^4 *have anti-self-dual not Weyl flat neutral metric.*

In this section we give some additional properties of the conformal geometries induced by the integrable *para-hypercomplex* structures. As first, we can calculate the curvature of the induced metric.

Recall, that the curvature tensor of a neutral four-dimensional manifold M can be seen as a self-adjoint map

$$R : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M$$

of six-dimensional space $\Lambda^2 T^*M$ of 2-forms on the manifold M. There is a decomposition

$$\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$$

of the space of 2-forms onto three-dimensional spaces of self-dual and anti-self-dual 2-forms. With respect to that decomposition the curvature tensor can be decomposed as:

$$R = \begin{pmatrix} W_+ & B \\ B^* & W_- \end{pmatrix} + \frac{s}{12} I,$$

where W_+ and W_- are self-dual and anti-self-dual parts of the Weil tensor W, B is the Einstein part and s is the scalar curvature. The Weil tensor $W = W_+ \oplus W_-$ is conformal invariant. In our case when the metric is induced by integrable *para-hypercomplex* structure, it is anti-self-dual, that is, $W_- = 0$ ([**8**]). The curvature of Lie groups associated to the Lie algebras form our classification is given in Table 1. However, we point out some interesting facts that one can prove by direct calculations.

Theorem 6.2. *The only Lie algebras admitting a flat integrable* para-hypercomplex *structure are*: $\mathbf{R}$, $\mathbf{R}\oplus\mathfrak{h}_3$, $\mathfrak{d}_{4,1}$, $\mathfrak{d}_{4,-1}$, $\mathfrak{r}_{4,1}$ *and* $\mathfrak{t}_{4,1,c}$, $c = \pm 1$.

Remark 6.1. Note that the algebras $\mathfrak{d}_{4,-1}$, $\mathfrak{r}_{4,1}$ and $\mathfrak{t}_{4,1,c}$, $c = \pm 1$ admit both flat and non-flat integrable *para-hypercomplex* structure.

Theorem 6.3. *The only Lie algebras admitting a Ricci flat, but not flat, integrable* para-hypercomplex *structure are*: $\mathfrak{r}_{4,1}$, $\mathfrak{t}_{4,1,-1}$, $\mathfrak{d}_{4,-1}$.

Most of the geometries arising from *para-hypercomplex* structures are conformally flat.

Theorem 6.4. *The only Lie algebras admitting non-conformally flat structures are* $\mathbf{R}\oplus\mathfrak{r}_{3,1}$, $\mathbf{R}^2\oplus\mathfrak{aff}(\mathbf{R})$, $\mathfrak{d}_4$, $\mathfrak{r}_{4,1}$, $\mathfrak{d}_{4,c}$, $\mathbf{R}\oplus\mathfrak{h}_3$ $\mathfrak{t}_{4,1,c}$, $c \neq 1/2$. *Among them only* $\mathfrak{d}_{4,1/2}$ *admits a structure with scalar non-flat metric.*

Theorem 6.5. *The only para-hypercomplex locally symmetric Lie groups of dimension 4 are determined by the algebras* $\mathbf{R}^4$, $\mathbf{R}\oplus\mathfrak{sl}_2(\mathbf{R})$, $\mathbf{R}\oplus\mathfrak{h}_3$, $\mathfrak{d}_4$, $\mathfrak{r}_{4,1}$, $\mathfrak{d}_{4,c}$, $c = \pm 1$, $\mathfrak{t}_{4,1,c}$, $c = \pm 1, -3$. *The induced metrics are biinvariant only on* $\mathbf{R}\oplus\mathfrak{sl}_2(\mathbf{R})$ *and* $\mathfrak{d}_4$.

Since a Lie group with a left invariant metric is an analytic manifold the holonomy algebra is generated by $\{R(x,y),\ \nabla_z R(x,y) \mid x,y,z \in \mathfrak{g}\}$. By a direct computation, using *Mathematica*, we calculated the holonomy algebras as given in Table 1. The holonomy algebra were also computed by H. Baum and A. Galaev another way. They used a theorem of Wang specializing in the holonomy algebra of naturally reductive spaces (see [**8**, II, p. 204, Corollary 4.2).

Theorem 6.6. *Let g be the left invariant metric on the Lie group G which is determined by an integrable para-hypercomplex structure (J_1, J_2) on G. Then, its holonomy algebra is one of the following:* 0, $\mathbf{R}$, $\mathbf{R}^2$, $\mathfrak{so}(1,2)$, $\mathfrak{so}(2,2)$, $\mathbf{R}\oplus\mathfrak{so}(1,2)$. *The complete list of the holonomy algebras is given in Table* 1.

Remark 6.2. We have confirmed the result of Andrada [**1**], that the only algebras admitting para-hyperKähler structure (that is *para-hypercomplex* structure with parallel structures) are: abelian $\mathbf{R}^4$, $\mathbf{R}\oplus\mathfrak{h}_3$, $\mathfrak{t}_{4,1,-1}$ and $\mathfrak{d}_{4,2}$.

Remark 6.3. The para-hypercomplex structure equivalent up to compatibility class induce the isometric metric (up to a homothety). If the induced metrics are isometric (up to a homothety) the structures are itself are not necessarily equivalent. They are equivalent up to a compatibility class.

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APPENDIX

7. The catalog of integrable, left invariant para-hypercomplex structures on four-dimensional Lie algebras (up to equivalent compatible class) is presented in the following Table. The commutator relations and the para-hypercomplex structures are given. Some properties of the naturally induced left-invariant compatible conformal class of metrics are systemized. We checked to see if the metrics are locally symmetric and biinvariant. Moreover, the Weyl curvature, Ricci curvature, the scalar curvature and the holonomy algebras are presented.

Notation	Relations	J_1	J_2	Symm./Biinv.	Curvature	PHK	hol
abelian PHC1				yes/yes	flat	yes	0
$\mathbb{R} \oplus \mathfrak{sl}_2(\mathbb{R})$ PHC2	$[X,Y]=W$, $[Y,W]=-X$, $[W,X]=Y$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	yes/yes	$W_+=0$ Scal=3/4 n. Einst.	no	$sl_2(\mathbb{R})$
$\mathbb{R} \oplus \mathfrak{t}_{3,1}$ PHC3	$[X,Y]=Y$, $[X,W]=W$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 1 & -\cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & -\cos\phi & 0 \\ 0 & \sin\phi & 1 & \cos\phi \end{bmatrix}$	no/no	$W_+\neq 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
		$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	no/no	$W_+=0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
$\mathbb{R} \oplus \mathfrak{h}_3$ PHC4	$[X,Y]=Z$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\cos\phi & \sin\phi & \sin 2\phi & \cos 2\phi \\ \sin\phi & \cos\phi & \cos 2\phi & -\sin 2\phi \end{bmatrix}$	yes/no	flat	yes	0
$\mathbb{R}^2 \oplus \mathfrak{aff}(\mathbb{R})$ PHC5	$[X,Y]=Y$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ -\cos 2\phi & \sin 2\phi & 0 & 0 \\ -\sin\phi & -\cos\phi & 0 & 1 \\ -\cos\phi & \sin\phi & 1 & 0 \end{bmatrix}$	no/no	$W_+\neq 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
		$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ -\cos 2\phi & \sin 2\phi & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	no/no	$W_+=0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
$\mathfrak{d}_4$ PHC6	$[X,W]=Z$, $[X,Y]=X$, $[Y,W]=W$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{n}{2} & 1 & 0 & -1 \\ 1 & \frac{n}{2} & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{n}{2}-1 & -1 & 0 & 1 \\ 1 & 1-\frac{n}{2} & 1 & 0 \end{bmatrix}$	no iff $n\neq 1$/no	$W_+\neq 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
		$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$	yes/yes	$W_+=0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
		$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	no/no	$W_+\neq 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$

$\mathfrak{aff}(\mathbb{C})$ PHC7 $4a < b^2$	$[X,Z]=X$, $[Y,Z]=Y$, $[X,W]=Y$, $[Y,W]=-X$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	no/no	$W_+=0$ Scal=3 n. Einst.	no	$sl_2(\mathbb{R})$
		$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	no/no	$W_+=0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
$\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$ PHC7 $4a > b^2$	$[X,Z]=X$, $[Y,W]=Y$	$\begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	no/no	$W_+=0$ Scal=-3/4 n. Einst.	no	$sl_2(\mathbb{R})$
		$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	no/no	$W_+=0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
$\mathfrak{d}_{4,1}$ PHC7 $4a=b^2$	$[X,Z]=X$, $[Y,Z]=Y$, $[X,W]=Y$	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	yes/no	flat	no	0
		$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	yes/no	flat	no	0
$\mathfrak{t}_{4,1,c}$, $c \neq 0,1$ PHC9 $c \neq 1$	$[X,W]=cX$, $[Y,W]=Y$, $[Z,W]=Z$	$\begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 2 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 2 & 0 & 0 & 0 \end{bmatrix}$	no iff $-c \neq 1, 3$/no	Scal $=0$, $W_+=0$ Not Ein. iff $c \neq -1$	iff $c=-1$	$\mathbb{R}^2, c \neq -1$ $0, c=-1$
		$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	no/no	Scal $=0$ $W_+ \neq 0$ iff $c \neq \frac{1}{2}$ Not Ein. iff $c \neq -1$	iff $c=-1$	$\mathbb{R}^2, c \neq -1$ $\mathbb{R}, c=-1$
$\mathfrak{t}_{4,1,1}$ PHC9 $c=1$, $a=b=0$	$[X,W]=X$, $[Y,W]=Y$, $[Z,W]=Z$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	yes/no	flat	no	0
		$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	no/no	$W_+=0$ Scal=-6 Einst.	no	$so(2,2)$

$\mathfrak{r}_{4,1}$ PHC9 $c=1$, $a^2 \neq -b^2$	$[X,W]=X$, $[Y,W]=Y$, $[Z,W]=Y+Z$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -\sin 2\phi & \cos 2\phi & \sin\phi & \cos\phi \\ \cos 2\phi & \sin 2\phi & -\cos\phi & \sin\phi \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	no iff $\sin\phi \neq 0$/no	Einst, Scal $=0$ $W_+ \neq 0$ iff $\phi \neq 0$	no	$\mathbb{R}^2$
$\mathfrak{d}_{4,\frac{1}{2}}$ PHC10 $c=\frac{1}{2}$, $b=0$	$[X,W]=\frac{1}{2}X$, $[Y,W]=\frac{1}{2}Y$, $[Z,W]=Z$, $[X,Y]=Z$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$	no/no	$W_+ \neq 0$ Scal=45/4 n. Einst.	no	$so(2,2)$
		$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 2 & 0 \end{bmatrix}$	no/no	$W_+ \neq 0$ Scal=45/4 n. Einst.	no	$so(2,2)$
		$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & -1 \\ 2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	no/no	$W_+ = 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$
		$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	no/no	$W_+ = 0$ Scal=0 n. Einst.	no	$\mathbb{R} \oplus sl_2(\mathbb{R})$
		$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	no/no	$W_+ = 0$ Scal=0 n. Einst.	no	$\mathbb{R} \oplus sl_2(\mathbb{R})$
$\mathfrak{d}_{4,c}$ $c \neq 1, \frac{1}{2}$ PHC10, $c \neq \frac{1}{2}$	$[X,W]=cX$, $[Y,W]=(1-c)Y$, $[Z,W]=Z$, $[X,Y]=Z$	$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -2c & -\frac{1}{2} \\ \frac{1}{2c} & \frac{1}{2c} & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -2c & -\frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$	no iff $c \neq -1$/no	$W_+ = 0$, Scal=0 Not Ein. iff $c \neq -1$	iff $c=-1$	$\mathbb{R}^2, c \neq -1$ $0, c=-1$
		$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & \frac{1}{c} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -c & -1 \\ \frac{1}{c} & -\frac{1}{c} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	no/no	$W_+ \neq 0$, Scal=0 Not Ein. iff $c \neq -1$	no	$\mathbb{R}^2$
$\mathfrak{h}_4$ PHC10 $c=\frac{1}{2}$ $b \neq 0$	$[X,W]=X$, $[Y,W]=X+Y$, $[Z,W]=2Z$, $[X,Y]=Z$	$\begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & -2 \\ 2 & -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}$	no/no	$W_+ \neq 0$ Scal=0 n. Einst.	no	$\mathbb{R}^2$

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Faculty of Mathematics, University of Belgrade, Serbia

Faculty of mathematics, University of Belgrade, Studenski trg 16, p.p. 550, 11001 Belgrade, Serbia
Email address: vsrdjan@matf.bg.ac.rs