

## QUASIANALYTIC GELFAND-SHILOV SPACES WITH APPLICATION TO LOCALIZATION OPERATORS

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**ABSTRACT.** We study localization operators with symbols in spaces of quasi-analytic distributions. More precisely, it is shown that certain quasianalytic distributions, considered as symbols, give rise to trace-class localization operators. We give a new structure theorem for quasianalytic distributions which combines its local and global properties. In the proof we use the heat kernel and parametrix techniques, while in the study of localization operators we use the techniques of time-frequency analysis.

**1. Introduction.** Localization operators (anti-wick operators, Toeplitz operators, Gabor multipliers) are pseudo-differential operators  $A_a^{\varphi_1, \varphi_2}$ , where  $a$  is the symbol of the operator and  $\varphi_1, \varphi_2$  are the analysis and synthesis windows, respectively (see below for an explicit expression). With respect to the classical pseudodifferential calculus, one may consider singular symbols for localization operators and nevertheless obtain good properties, in particular  $L^2$ -boundedness, see [6–8, 25, 26]. As an example, it was observed in [8] that certain compactly supported ultra-distributions give rise to trace-class operators. In this paper we study localization operators in the framework of quasianalytic distributions.

The support of a quasianalytic distributions cannot be defined. Therefore, in order to give a reasonable generalization of the results from [6, 8, 26], we present a technique which may, in a certain sense, describe the local behavior of a quasianalytic distribution. More precisely, we give a new representation theorem for the class of quasi-

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analytic distributions, Theorem 3.1. A related result, under different assumptions, can be found in [4]. Another type of representation theorems, based on Hermite functions, was recently obtained in [18, 19], see also [21]. Note that we could not use these results since they give a global characterization, while we need information on the local behavior as well.

The paper is organized as follows. We define Gelfand-Shilov spaces and list their basic properties in Section 2. Representation theorem 3.1 for quasianalytic distributions is given in Section 3. In Section 4 we study time-frequency representations on Gelfand-Shilov spaces, Theorem 4.1. Then we introduce modulation spaces with weights of exponential growth and show that the Gelfand-Shilov spaces are actually projective and inductive limits of the modulation spaces, Proposition 4.5. In the last subsection, we prove Theorem 4.6 which describes the growth of the short-time Fourier transform of quasianalytic distributions and, finally, prove that the corresponding localization operators are trace-class.

**1.1. Notation.** We put  $xy = x \cdot y$ , the scalar product on  $\mathbf{R}^d$ . Given a vector  $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ , the partial derivative with respect to  $x_j$  is denoted by  $\partial_j = \partial/\partial x_j$ . Given a multi-index  $p = (p_1, \dots, p_d) \geq 0$ , i.e.,  $p \in \mathbf{N}_0^d$  and  $p_j \geq 0$ , we write  $\partial^p = \partial_1^{p_1} \dots \partial_d^{p_d}$ ; moreover, we write  $x^p = (x_1, \dots, x_d)^{(p_1, \dots, p_d)} = \prod_{i=1}^d x_i^{p_i}$ . We shall denote the Euclidean norm by  $\|x\|$ . We write  $h \cdot |x|^{1/\alpha} = \sum_{i=1}^d h_i |x_i|^{1/\alpha_i}$ . Moreover, for  $p \in \mathbf{N}_0^d$  and  $\alpha \in \mathbf{R}_+^d$ , we set  $(p!)^\alpha = (p_1!)^{\alpha_1} \dots (p_d!)^{\alpha_d}$ , while as standard  $p! = p_1! \dots p_d!$ . In the sequel, a real number  $r \in \mathbf{R}_+$  may play the role of the vector with constant components  $r_j = r$ , so for  $\alpha \in \mathbf{R}_+^d$ , by writing  $\alpha > r$  we mean  $\alpha_j > r$  for all  $j = 1, \dots, d$ .

$\Omega$  denotes an open set in  $\mathbf{R}^d$  and  $K \subset\subset \Omega$  means that  $K$  is a compact subset in  $\Omega$ .

For  $A = (A_1, \dots, A_d)$  and  $B = (B_1, \dots, B_d)$ ,  $A > 0$  and  $B > 0$  means  $A_1, \dots, A_d, B_1, \dots, B_d > 0$ .

For a multi-index  $\alpha \in \mathbf{N}_0^d$  we have  $|\alpha| = \alpha_1 + \dots + \alpha_d$ . For given  $h > 0$  and multi-index  $\alpha \in \mathbf{N}_0^d$  we will (sometimes) use the notation  $h^\alpha := h^{|\alpha|}$ .

The Schwartz class is denoted by  $\mathcal{S}(\mathbf{R}^d)$ , the space of tempered distributions by  $\mathcal{S}'(\mathbf{R}^d)$ . We use the brackets  $\langle f, g \rangle$  to denote the extension

of the inner product  $\langle f, g \rangle = \int f(t)\overline{g(t)} dt$  on  $L^2(\mathbf{R}^d)$  to any pair of dual spaces. The space of smooth functions with compact support on  $\mathbf{R}^d$  is denoted by  $\mathcal{D}(\mathbf{R}^d)$ . The Fourier transform is normalized to be  $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi it\omega} dt$ .

Throughout the paper, we shall use the notation  $A \lesssim B$  to indicate  $A \leq cB$  for a suitable constant  $c > 0$ , whereas  $A \asymp B$  means that  $c^{-1}A \leq B \leq cA$  for some  $c \geq 1$ . The symbol  $B_1 \hookrightarrow B_2$  denotes the continuous and dense embedding of the topological vector space  $B_1$  into  $B_2$ .

**2. Gelfand-Shilov type spaces.** Let  $(M_p)_{p \in \mathbf{N}_0}$  be a sequence of positive numbers increasing to infinity which satisfies:

(M.1)  $M_p^2 \leq M_{p-1}M_{p+1}, p \in \mathbf{N};$

(M.2) There exist positive constants  $A, H$  such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_{p-q}M_q, \quad p, q \in \mathbf{N}_0,$$

or, equivalently, there exist positive constants  $A, H$  such that

$$M_{p+q} \leq AH^{p+q}M_pM_q, \quad p, q \in \mathbf{N}_0.$$

We assume  $M_0 = 1$ , and that  $M_p^{1/p}$  is bounded below by a positive constant. Put  $m_p = M_p/M_{p-1}, p \in \mathbf{N}$ . The following condition is used in [4]:

(C)  $\liminf_{p \rightarrow \infty} \left( \frac{m_{kp}}{m_p} \right)^2 > k, \quad \text{for all } k \in \mathbf{N}.$

The so-called *associated function* of  $(M_p)$  is defined by

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{\rho^p M_0}{M_p}, \quad 0 < \rho < \infty.$$

We will also use the function

$$M^*(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{p! \rho^p M_0}{M_p}, \quad 0 < \rho < \infty.$$

Another way to describe (M.1) and (M.2) is the following. Let  $(s_p)_{p \in \mathbf{N}_0}$ ,  $(r_p)_{p \in \mathbf{N}_0}$  and  $(l_p)_{p \in \mathbf{N}_0}$  be increasing to infinity sequences so that for every  $p, q \in \mathbf{N}_0$  there exist  $A, H > 0$  such that

$$(2.1) \quad s_{p+1} \cdots s_{p+q} \leq AH^p s_1 \cdots s_q,$$

and the same holds for  $(r_p)_{p \in \mathbf{N}_0}$  and  $(l_p)_{p \in \mathbf{N}_0}$ . With this,  $(S_p)_{p \in \mathbf{N}_0}$ ,  $(R_p)_{p \in \mathbf{N}_0}$  and  $(L_p)_{p \in \mathbf{N}_0}$  where  $S_p = s_1 \cdots s_p$ ,  $R_p = r_1 \cdots r_p$ , and  $L_p = l_1 \cdots l_p$ ,  $p \in \mathbf{N}$  ( $S_0 = 1$ ,  $R_0 = 1$ ,  $L_0 = 1$ ) satisfy conditions (M.1) and (M.2). Conversely, if  $(S_p)_{p \in \mathbf{N}_0}$  where  $S_p = s_1 \cdots s_p$  satisfies (M.1), then  $(s_p)_{p \in \mathbf{N}_0}$  increases to infinity, and if it satisfies (M.2) then (2.1) holds.

We are interested in the case when  $s_p = p^{1/2}$  so  $S_p = p!^{1/2}$  and  $(r_p)_{p \in \mathbf{N}_0}$  and  $(l_p)_{p \in \mathbf{N}_0}$  satisfy: For every  $\alpha \in (0, 1]$  and every  $k > 1$  so that  $kp \in \mathbf{N}$ ,  $p \in \mathbf{N}$ ,

$$(2.2) \quad \max \left\{ \left( \frac{r_{kp}}{r_p} \right)^2, \left( \frac{l_{kp}}{l_p} \right)^2 \right\} \leq k^\alpha, \quad p \in \mathbf{N}.$$

With this we have that

$$\max\{R_p, L_p\} \leq p!^{\alpha/2}, \quad p \in \mathbf{N}$$

for every  $\alpha \in (0, 1]$ . Also one can show that the sequences  $p!^{1/2}r_p$  and  $p!^{1/2}l_p$  satisfy condition (C):

$$\liminf_{p \rightarrow \infty} \left\{ \left( \frac{(kp)!^{1/2}r_{kp}}{p!^{1/2}r_p} \right)^2, \left( \frac{(kp)!^{1/2}l_{kp}}{p!^{1/2}l_p} \right)^2 \right\} > k, \quad \text{for all } k \in \mathbf{N}.$$

**Definition 2.1.** Let there be given sequences of positive numbers  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  which satisfy (M.1) and (M.2). The Gelfand-Shilov space  $\mathcal{S}_{M_p, A}^{N_q, B}(\mathbf{R}^d)$  is defined by

$$\begin{aligned} \mathcal{S}_{M_p, A}^{N_q, B}(\mathbf{R}^d) = \left\{ f \in C^\infty(\mathbf{R}^d) \mid \|x^\alpha \partial^\beta f\|_{L^\infty} \right. \\ \left. \leq CA^\alpha M_{|\alpha|} B^\beta N_{|\beta|}, \quad \text{for all } \alpha, \beta \in \mathbf{N}_0^d \right\}, \end{aligned}$$

for some positive constant  $C$ , where  $A = (A_1, \dots, A_d)$ ,  $B = (B_1, \dots, B_d)$ ,  $A, B > 0$ . Their projective and inductive limits are denoted by

$$\Sigma_{M_p}^{N_q} := \text{proj} \lim_{\substack{A>0 \\ B>0}} \mathcal{S}_{M_p, A}^{N_q, B}; \quad \mathcal{S}_{M_p}^{N_q} := \text{ind} \lim_{\substack{A>0 \\ B>0}} \mathcal{S}_{M_p, A}^{N_q, B}.$$

For  $M_p = p!^r$ ,  $p \in \mathbf{N}_0$  and  $N_q = q!^s$ ,  $q \in \mathbf{N}_0$ , we use the notation  $\mathcal{S}_{M_p}^{N_q} = \mathcal{S}_r^s$  and  $\Sigma_{M_p}^{N_q} = \Sigma_r^s$ .

Let  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  be sequences which satisfy (M.1). We write  $M_p \subset N_q$  ( $(M_p) \prec (N_q)$ , respectively) if there are constants  $H, C > 0$  (for any  $H > 0$  there is a constant  $C > 0$ , respectively) such that  $M_p \leq CH^p N_p$ ,  $p \in \mathbf{N}_0$ . Also,  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  are said to be equivalent if  $M_p \subset N_q$  and  $N_q \subset M_p$  hold.

**Theorem 2.2.** *Let there be given sequences of positive numbers  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  which satisfy (M.1) and (M.2) and  $p! \subset M_p N_p$  ( $p! \prec M_p N_p$ , respectively). Then the following conditions are equivalent:*

- a)  $f \in \mathcal{S}_{M_p}^{N_q}$  ( $f \in \Sigma_{M_p}^{N_q}$ , respectively).
- b) *There exist constants  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$  (for every  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$ , respectively) and there exist  $C > 0$  such that*

$$\|x^p f\|_{L^\infty} \leq CA^p M_{|p|} \quad \text{and} \quad \|\omega^q \hat{f}\|_{L^\infty} \leq CB^q N_{|q|}, \quad \text{for all } p, q \in \mathbf{N}_0^d.$$

- c) *There exist constants  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$  (for every  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$ , respectively) and there exist  $C > 0$  such that*

$$\|x^p f\|_{L^\infty} \leq CA^p M_{|p|} \quad \text{and} \quad \|\partial^q f\|_{L^\infty} \leq CB^q N_{|q|}, \quad \text{for all } p, q \in \mathbf{N}_0^d.$$

- d) *There exist constants  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$  (for every  $A, B \in \mathbf{R}^d$ ,  $A, B > 0$ , respectively) such that*

$$\|f(x) \exp(M(|Ax|))\|_{L^\infty} < \infty \quad \text{and} \quad \|\hat{f}(\omega) \exp(N(|B\omega|))\|_{L^\infty} < \infty,$$

where  $M(\cdot)$  and  $N(\cdot)$  are the associated functions for the sequences  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$ , respectively.

*Proof.* The proof for the inductive limit case,  $\mathcal{S}_{M_p}^{N_q}$ , can be found in [3] and the proof for the projective limit case,  $\Sigma_{M_p}^{N_q}$  is analogous.  $\square$

By the above characterization  $\mathcal{F}\mathcal{S}_{M_p}^{N_q} = \mathcal{S}_{N_q}^{M_p}$ . Observe that  $\mathcal{S}_{1/2}^{1/2}$  is the smallest nonempty Gelfand-Shilov space invariant under the Fourier transform. Theorem 2.2 implies that  $f \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$  if and only if  $f \in C^\infty(\mathbf{R}^d)$  and constants  $h > 0$  and  $k > 0$  exist such that

$$(2.3) \quad \|f \exp(h|\cdot|^{1/2})\|_{L^\infty} < \infty \quad \text{and} \quad \|\hat{f} \exp(k|\cdot|^{1/2})\|_{L^\infty} < \infty.$$

Note that  $\Sigma_{1/2}^{1/2} = 0$  and  $\Sigma_s^s$  is dense in the Schwartz space whenever  $s > 1/2$ . We are interested here in “fine tuning,” that is, in spaces  $\Sigma_{M_p}^{N_q}$  such that  $\Sigma_{1/2}^{1/2} \subset \Sigma_{M_p}^{N_q} \subset \Sigma_s^s$ ,  $s > 1/2$ . From now on, we will exclusively use sequences  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  given by

$$(2.4) \quad \begin{aligned} M_p &:= p!^{1/2} \prod_{k=0}^p l_k = p!^{1/2} L_p, \quad p \in \mathbf{N}_0, \\ N_q &:= q!^{1/2} \prod_{k=0}^q r_k = q!^{1/2} R_q, \quad q \in \mathbf{N}_0 \end{aligned}$$

where the sequences  $(r_p)_{p \in \mathbf{N}_0}$  and  $(l_p)_{p \in \mathbf{N}_0}$  satisfy (2.2). Then we have  $m_p = p!^{1/2} l_p$ ,  $p \in \mathbf{N}_0$ ,  $n_q = q!^{1/2} r_q$ ,  $q \in \mathbf{N}_0$  and  $p! \prec M_p N_p$ . For  $p, q, k \in \mathbf{N}_0^d$  we have  $L_{|p|} = \prod_{|k| \leq |p|} l_{|k|}$  and  $R_{|q|} = \prod_{|k| \leq |q|} r_{|k|}$ .

**3. Representations via ultra-differential operators.** In this section we give a new representation of quasianalytic ultradistributions. We use the notation and assumptions from the previous section. In particular, we assume that sequences  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  are given such that (2.2) and (2.4) hold. Recall that the corresponding associated functions are

$$M(t) := \sup_{p \in \mathbf{N}_0} \ln \frac{t^p}{p!^{1/2} L_p}, \quad N(t) := \sup_{q \in \mathbf{N}_0} \ln \frac{t^q}{q!^{1/2} R_q}, \quad 0 < \rho < \infty,$$

We put

$$\overline{M}(t) = \sup_{p \in \mathbf{N}_0} \ln \frac{t^p}{L_p^2}, \quad \overline{N}(t) = \sup_{q \in \mathbf{N}_0} \ln \frac{t^q}{R_q^2}, \quad 0 < \rho < \infty.$$

These are the  $M^*$  and  $N^*$  functions for sequences  $(M_p^2)_{p \in \mathbf{N}_0}$  and  $(N_q^2)_{q \in \mathbf{N}_0}$ .

Furthermore, in the upcoming Theorem 3.1 we use spaces of ultradistributions of Beurling and Roumieu type, the strong duals of spaces of ultradifferentiable functions.

Let a sequence  $(N_q)_{q \in \mathbf{N}_0}$  be given which satisfies (M.1) and (M.2). The function  $\phi \in C^\infty(\Omega)$  is called an *ultradifferentiable* function of Beurling class  $(N_q)$  (respectively of Roumieu class  $\{N_q\}$ ) if, for any  $K \subset\subset \Omega$  and for any  $h > 0$  (respectively for some  $h > 0$ ),

$$\|\phi\|_{N_q, K, h} = \sup_{\substack{x \in K \\ \alpha \in \mathbf{N}_0^d}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} N_{|\alpha|}} < \infty.$$

We say that  $\phi \in \mathcal{E}^{N_q, K, h}(\Omega)$  if  $\|\phi\|_{N_q, K, h} < \infty$  for given  $K$  and  $h > 0$ .

The following spaces of ultradifferentiable test functions will be used

$$\begin{aligned} \mathcal{E}^{(N_q)}(\Omega) &:= \text{proj}_{K \subset\subset \Omega} \lim_{h \rightarrow 0} \text{proj} \mathcal{E}^{N_q, K, h}(\Omega); \\ \mathcal{E}^{\{N_q\}}(\Omega) &:= \text{proj}_{K \subset\subset \Omega} \lim_{h \rightarrow \infty} \text{ind} \mathcal{E}^{N_q, K, h}(\Omega). \end{aligned}$$

Let  $u \in (\Sigma_{M_p}^{N_q})'(\mathbf{R}^d)$  ( $u \in (\mathcal{S}_{M_p}^{N_q})'(\mathbf{R}^d)$ , respectively). Then we say that  $u$  can be extended on  $\mathcal{E}^{(N_q)}(\Omega)$  (on  $\mathcal{E}^{\{N_q\}}(\Omega)$ , respectively) if  $u \in (\mathcal{E}^{(N_q)})'(\Omega)$  ( $u \in (\mathcal{E}^{\{N_q\}})'(\Omega)$ , respectively) and

$$\begin{aligned} (\Sigma_{M_p}^{N_q})'(\mathbf{R}^d) \langle u, \phi \rangle_{\Sigma_{M_p}^{N_q}(\mathbf{R}^d)} &=_{(\mathcal{E}^{(N_q)})'(\Omega)} \langle u, \phi|_\Omega \rangle_{\mathcal{E}^{(N_q)}(\Omega)} \\ (\mathcal{S}_{M_p}^{N_q})'(\mathbf{R}^d) \langle u, \phi \rangle_{\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)} &=_{(\mathcal{E}^{\{N_q\}})'(\Omega)} \langle u, \phi|_\Omega \rangle_{\mathcal{E}^{\{N_q\}}(\Omega)}, \text{ respectively).} \end{aligned}$$

Note that if  $(\phi_n)_{n \in \mathbf{N}}$  is a sequence in  $\Sigma_{M_p}^{N_q}(\mathbf{R}^d)$  (in  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ , respectively) and if  $\phi_n \rightarrow \phi$  in  $\Sigma_{M_p}^{N_q}(\mathbf{R}^d)$  (in  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ , respectively) then  $\phi_n|_\Omega \rightarrow \phi|_\Omega$  in  $\mathcal{E}^{(N_q)}(\Omega)$  (in  $\mathcal{E}^{\{N_q\}}(\Omega)$ , respectively).

Let there be given a sequence of positive numbers  $(N_q)_{q \in \mathbf{N}_0}$ . The operator  $P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha$  is called an *ultra-differential operator* of Roumieu class  $\{N_q\}$  (of Beurling class  $(N_q)$ , respectively) if for all  $L > 0$  there exists a  $C > 0$  (there exists an  $L > 0$  and there exists a  $C > 0$ ), respectively, such that

$$|a_\alpha| \leq C \frac{L^{|\alpha|}}{N_{|\alpha|}}, \text{ for all } \alpha \in \mathbf{N}_0^d.$$

In the proof of the following Theorem we will use the  $d$ -dimensional heat kernel

$$E(x, t) = \begin{cases} (4\pi t)^{-d/2} \exp(-|x|^2/4t) & t > 0, x \in \mathbf{R}^d, \\ 0 & t \leq 0, x \in \mathbf{R}^d. \end{cases}$$

It is an entire function for every  $t > 0$ ,  $\int E(x, t) dx = 1$ ,  $t > 0$ , and

$$|\partial_x^\alpha E(x, t)| \leq C^{|\alpha|} t^{-(d+|\alpha|)/2} |\alpha|!^{1/2} \exp[-a|x|^2/4t], \\ t > 0, \quad x \in \mathbf{R}^d, \quad \alpha \in \mathbf{N}_0^d$$

where  $a$  can be chosen as close as desired to 1 and  $0 < a < 1$  [4, 20].

The following theorem can be compared to [4, Theorem 2.5]. In [4] it is assumed that a given sequence  $(M_p)_{p \in \mathbf{N}_0}$  satisfies the condition

(M.0) : for all  $A > 0$ , exists  $C > 0$  such that  $p! \leq CA^p M_p$ ,  $p \in \mathbf{N}_0$ .

This condition does not hold for  $M_p = p!^s$  if  $s \leq 1$  and it holds for  $s > 1$ . However, if  $s > 1$ , then condition (C) holds not only for  $m_{k_p}^2/m_p^2$  but for  $m_{k_p}/m_p$  as well. Since we are interested in the case  $1 > s > 1/2$ , we essentially use sequences which satisfy (2.2) and (2.4). Since this implies (C), our Theorem 3.1 is less restrictive than [4, Theorem 2.5] in that sense.

The following condition, which is satisfied by  $M_{|p|} = |p|!^s$ , for  $s > 1/2$ , is used in [18]: there exists an  $A > 0$  such that, for any  $C > 0$  there is a  $B > 0$  (there are  $C > 0$  and  $B > 0$ , respectively) such that

$$|p|^{p/2} M_{|q|} \leq BC^{|p|} A^{|p|+|q|} M_{|p|+|q|}, \quad p, q \in \mathbf{N}_0^d.$$



However, representation theorems from [18, 19] cannot be used for our purposes where behavior on open subsets of  $\mathbf{R}^d$  is essential.

**Theorem 3.1.** *Let sequences  $(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  be given such that (2.2) and (2.4) hold. Let  $u \in (\mathcal{S}_{M_p}^{N_q})'$  ( $u \in (\Sigma_{M_p}^{N_q})'$ , respectively) such that it can be extended continuously to  $\mathcal{E}^{\{N_q\}}(\Omega)$  ( $\mathcal{E}^{\{N_q\}}(\Omega)$ , respectively) for some open set  $\Omega \subseteq \mathbf{R}^d$ . Then there exists  $P(\partial)$ , ultra-differential operator of Roumieu class  $\{N_q\}$  (of Beurling class  $(N_q)$ , respectively) and there exist bounded continuous functions  $g$  and  $h$  on  $\mathbf{R}^d$  such that  $u = P(\partial)g + h$ , i.e.,*

$$\begin{aligned} \langle u, \phi \rangle &= \langle g, P(-\partial)\phi \rangle + \langle h, \phi \rangle \\ &= \int_{\mathbf{R}^d} g(t)P(-\partial)\phi(t) dt + \int_{\mathbf{R}^d} h(t)\phi(t) dt, \end{aligned}$$

$\phi \in \mathcal{S}_{M_p}^{N_q}$  ( $\phi \in \Sigma_{M_p}^{N_q}$ , respectively).

*Proof.* We give the proof for the Roumieu case. The Beurling case can be proved analogously. Let  $u \in (\mathcal{S}_{M_p}^{N_q})'$ . Define

$$U(x, t) = \langle u(y), E(x - y, t) \rangle = \langle u(y), E(x - y, t)|_{\Omega} \rangle, \quad x \in \mathbf{R}^d, t > 0.$$

This is an entire function for every  $t > 0$ , and

$$(3.1) \quad \frac{\partial}{\partial t}U(x, t) - \Delta U(x, t) = 0 \text{ in } \mathbf{R}^d \times \mathbf{R}_+.$$

Let us show that

$$\lim_{t \rightarrow 0^+} \int_{\mathbf{R}^d} U(x, t)\phi(x) dx = \langle u, \phi \rangle, \text{ for all } \phi \in \mathcal{S}_{M_p}^{N_q}.$$

By the assumption,  $u \in (\mathcal{E}^{\{N_q\}})'(\Omega)$ , so there exists a  $K \subset\subset \Omega$  such that for every  $k > 0$  there exists a  $C > 0$  such that

$$\langle u, \phi \rangle \leq C\|\phi\|_{N_q, K, k}, \quad \phi \in \mathcal{E}^{\{N_q\}}(\Omega).$$

Let  $K_\delta$  be a delta neighborhood of  $K$ ,  $K_\delta = \{x \in \mathbf{R}^n; d(x, K) \leq \delta\}$  such that  $K_\delta \in \Omega$ . Since  $u \in (\mathcal{E}^{\{N_q\}}(\Omega))'$ , for every  $h > 0$  there exists a  $C_0 > 0$  such that

$$|U(x, t)| \leq C_0 \sup_{\substack{y \in K_\delta \\ \alpha \in \mathbf{N}_0^d}} \left| \frac{(\partial^\alpha / \partial y^\alpha) E(x - y, t)}{h^{|\alpha|} |\alpha|!^{1/2} R_{|\alpha|}} \right|, \quad x \in \mathbf{R}^d, \quad t > 0,$$

where  $C_0$  and  $h$  do not depend on  $x$  and  $t$ .

Note that

$$\frac{R_{|\alpha|+d}}{R_{|\alpha|}} \leq CH^{|\alpha|+d}, \quad \text{for all } \alpha \in \mathbf{N}_0^d.$$

By the estimates for  $(\partial^\alpha / \partial y^\alpha) E(x - y, t)$ , we have (with  $a \in (0, 1)$ )

$$\begin{aligned} |U(x, t)| &\leq C_0 \sup_{\substack{y \in K_\delta \\ \alpha \in \mathbf{N}_0^d}} \frac{C^{|\alpha|} t^{-(d+|\alpha|)/2} e^{-(a|x-y|^2)/4t}}{h^{|\alpha|} R_{|\alpha|}} \\ &\leq C_0 \sup_{\alpha \in \mathbf{N}_0^d} \frac{(Ct^{-1/2})^{d+|\alpha|} (C^{-1}h)^d R_{|\alpha|+d}}{h^{d+|\alpha|} R_{|\alpha|+d} R_{|\alpha|}} \sup_{y \in K_\delta} e^{-(a|x-y|^2)/4t} \\ &\leq C_3 \sup_{\alpha \in \mathbf{N}_0^d} \frac{(HCt^{-1/2})^{d+|\alpha|}}{h^{d+|\alpha|} R_{|\alpha|+d}} e^{-d(x, K_\delta)^2(a/4t)} \\ &\leq C_3 \left( \sup_{\alpha \in \mathbf{N}_0^d} \left( \frac{H^2 C^2}{h^2 t} \right)^{d+|\alpha|} \frac{1}{R_{|\alpha|+d}^2} \right)^{1/2} e^{-d(x, K_\delta)^2(a/4t)} \end{aligned}$$

and with  $\varepsilon = (H^2 C^2)/h^2$  we have that for every  $\varepsilon > 0$  there exists a  $C_\varepsilon > 0$  such that

$$(3.2) \quad |U(x, t)| \leq C_\varepsilon e^{(1/2)\overline{N}(\varepsilon/t) - d(x, K_\delta)^2(a/4t)}, \quad x \in \mathbf{R}^d, \quad t > 0,$$

where  $\overline{N}$  is the associated function for the sequence  $(N_q^2)_{q \in \mathbf{N}_0}$ . Clearly,

$$\lim_{t \rightarrow 0} \frac{1}{2} \overline{N}\left(\frac{\varepsilon}{t}\right) - d(x, K_\delta)^2 \frac{a}{4t} = -\infty.$$

Thus,  $U$  is a bounded function on  $\mathbf{R}^d \times [0, t_0]$  for every  $t_0 > 0$  (we put  $U(x, 0) = 0, x \in \mathbf{R}^d$ ).

We will use the fact that, for any  $\phi \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ , and

$$\phi_t(x) = \int_{\mathbf{R}^d} E(x - y, t)\phi(y) dy, \quad t > 0, x \in \mathbf{R}^d,$$

we have  $\lim_{t \rightarrow 0} \phi_t = \phi$  in  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ . If  $\phi \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ , then

$$\langle U(x, t), \phi(x) \rangle = \langle \langle u(y), E(x - y, t)|_{\Omega} \rangle, \phi(x) \rangle.$$

On the other hand,

$$\langle u(y), \langle E(x - y, t)\phi(x) \rangle \rangle = \langle u(y), \langle E(x - y, t)\phi(x) \rangle|_{\Omega} \rangle, \quad t > 0.$$

Since  $\langle E(x - \cdot, t)\phi(x) \rangle \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ ,  $t > 0$ , and  $\langle E(x - \cdot, t)\phi(x) \rangle \rightarrow \phi$  in  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$  (thus  $\langle E(x - \cdot, t)\phi(x) \rangle|_{\Omega} \rightarrow \phi|_{\Omega}$  in  $\mathcal{E}^{(N_q)}(\Omega)$ ), we have

$$\lim_{t \rightarrow 0^+} \int_{\mathbf{R}^d} U(x, t)\phi(x) dx = \lim_{t \rightarrow 0^+} \langle U(x, t)\phi(x) \rangle = \langle u, \phi \rangle.$$

By the parametrix method we have that there exists an ultra-differential operator of Roumieu class  $\{N_q^2\}_{q \in \mathbf{N}_0}$ :

$$(3.3) \quad \tilde{P}\left(\frac{d}{dt}\right) := \sum_{q \in \mathbf{N}_0} \tilde{a}_q \frac{d^q}{dt^q},$$

such that

$$|\tilde{a}_q| \leq C \frac{L^q}{q!R_q^2}, \quad q \in \mathbf{N}_0,$$

for every  $L > 0$  and for some  $C > 0$ , and  $v, w \in C_0^\infty(\mathbf{R})$ , with the properties

$$(3.4) \quad \text{supp } v \subset [0, 2], \quad \text{supp } w \subset [1, 2];$$

(for all  $L > 0$ ) (exists  $C_L > 0$ )

$$(3.5) \quad |v(t)| \leq C_L e^{-\bar{N}(L/t)}, \quad t > 0,$$

and

$$(3.6) \quad \tilde{P}\left(\frac{d}{dt}\right)v(t) = \delta(t) + w(t), \quad t \in \mathbf{R}.$$

We refer to [17, Lemma 11.4] for this version of the parametrix. As in [4], we put

$$\tilde{U}(x, t) = \int_0^\infty U(x, t + s)v(s) ds.$$

By (3.2) one can show that  $\tilde{U}(x, t)$  is uniformly bounded in  $\mathbf{R}^d \times [0, \infty)$  and thus continuous on  $\mathbf{R}^d \times [0, \infty)$ . It follows that  $g(x) := \tilde{U}(x, 0)$ ,  $x \in \mathbf{R}^d$  is a bounded continuous function. Let

$$H(x, t) := - \int_0^\infty U(x, t + s)w(s) ds.$$

Clearly,  $H$  is analytic in  $\{(x, t) \in \mathbf{R}^{d+1}; x \in \mathbf{R}^d, t > -1\}$  and satisfies the same growth condition as  $U$  on  $\mathbf{R}^d \times [0, t_0]$  for every  $t_0 > 0$  (see (3.2)). Now, by (3.1) and (3.3) we have

$$\begin{aligned} \tilde{P}(-\Delta)\tilde{U}(x, t) &= \tilde{P}\left(-\frac{d}{dt}\right) \int_0^\infty U(x, t + s)v(s) ds \\ &= \int_0^\infty U(x, t + s)\tilde{P}\left(\frac{d}{ds}\right)v(s) ds \\ &= \int_0^\infty U(x, t + s)(\delta(s) + w(s) ds \\ &= U(x, t) + \int_0^\infty U(x, t + s)w(s) ds, \end{aligned}$$

that is,

$$\tilde{P}(-\Delta)\tilde{U}(x, t) = U(x, t) - H(x, t).$$

Thus, by letting  $t \rightarrow 0^+$ , in  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$ ,

$$u = \tilde{P}(-\Delta)g + h,$$

where  $g = \tilde{U}(\cdot, 0)$  and  $h = H(\cdot, 0)$  are continuous bounded functions on  $\mathbf{R}^d$ . Since, for  $q \in \mathbf{N}_0$  and  $\beta \in \mathbf{N}_0^d$ ,

$$\Delta^q = \sum_{|\beta|=q} \binom{q}{\beta_1} \binom{q - \beta_1}{\beta_2} \dots \binom{q - \sum_{k=1}^{d-1} \beta_k}{\beta_d} \frac{\partial^{2\beta_1}}{\partial x_1^{2\beta_1}} \dots \frac{\partial^{2\beta_d}}{\partial x_d^{2\beta_d}},$$

we have

$$P(\partial) := \tilde{P}(-\Delta) = \sum_{q \in \mathbf{N}_0} (-1)^q \tilde{a}_q \sum_{|\beta|=q} \binom{q}{\beta_1} \binom{q-\beta_1}{\beta_2} \dots \binom{q-\sum_{k=1}^{d-1} \beta_k}{\beta_d} \frac{\partial^{2\beta_1}}{\partial x_1^{2\beta_1}} \dots \frac{\partial^{2\beta_d}}{\partial x_d^{2\beta_d}},$$

where  $|\tilde{a}_q| \leq C(L^q/q!R_q^2)$ , for every  $L > 0$  and some  $C > 0$ .

Put  $P(\partial) = \sum_{|\gamma|=0}^\infty a_\gamma(\partial^\gamma/\partial x^\gamma) := \tilde{P}(-\Delta)$ . We see that  $a_\gamma = 0$  if  $\gamma \neq 2\beta$ ,  $\beta \in \mathbf{N}_0^d$ . Furthermore, for a given  $\beta = (\beta_1, \dots, \beta_d) \in \mathbf{N}_0^d$  and for  $\gamma = 2\beta$ , the following estimate holds:

$$|a_\gamma| = \left| \tilde{a}_{|\beta|} \binom{|\beta|}{\beta_1} \binom{|\beta|-\beta_1}{\beta_2} \dots \binom{|\beta|-\sum_{k=1}^{d-1} \beta_k}{\beta_d} \right| \leq |\tilde{a}_{|\beta|}| d^{|\beta|} \leq C \frac{L^{|\beta|} d^{|\beta|}}{|\beta|! R_{|\beta|}^2},$$

for every  $L > 0$  and for some  $C > 0$ . Condition (M.2) implies  $R_{2|\beta|} \leq AH^{2|\beta|} R_{|\beta|}^2$  for some  $A, H > 0$ . This, together with  $|2\beta|! \leq 2^{2|\beta|} |\beta|!^2$ , gives

$$|a_\gamma| \leq C \frac{(\sqrt{2ALdH})^{2|\beta|}}{|2\beta|!^{1/2} R_{|2\beta|}} = C \frac{h^{|\gamma|}}{|\gamma|!^{1/2} R_{|\gamma|}}$$

for every  $h > 0$  and some  $C > 0$ , that is,  $P(\partial)$  is an ultradifferential operator of Roumieu class  $\{N_q\}$  as claimed. Thus, we obtained the desired representation  $u = P(\partial)g + h$ .  $\square$

**4. Time-frequency analysis and localization operators.**

Translation and modulation operators are defined by

$$(4.1) \quad T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

For a fixed nonzero  $g \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ , the short-time Fourier transform (STFT) of  $f \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$  with respect to the window  $g$  is given by

$$(4.2) \quad V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbf{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt, \quad x, \omega \in \mathbf{R}^d$$

and can be extended to  $f \in (\mathcal{S}_{1/2}^{1/2})'(\mathbf{R}^d)$  by duality. For a fixed  $g \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ , the following characterization of  $\mathcal{S}_{1/2}^{1/2}$  holds [15]:

$$f \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d) \iff V_g f \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^{2d}).$$

Another time-frequency representation we shall use in the sequel is the *cross-Wigner distribution*  $W(f, g)$ , defined by

$$(4.3) \quad W(f, g)(x, \omega) = \int f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega t} dt, \quad x, \omega \in \mathbf{R}^d.$$

Since

$$W(f, g)(x, \omega) = 2^d e^{4\pi i x \omega} V_{\tilde{g}} f(2x, 2\omega), \quad x, \omega \in \mathbf{R}^d$$

we conclude that, for a fixed  $g \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ ,

$$W(f, g) \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^{2d}) \iff f \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d).$$

In what follows, we need the fact that also Gelfand-Shilov type spaces  $\mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$  and  $\Sigma_{N_q}^{N_q}(\mathbf{R}^d)$  enjoy properties as above.

For the sake of simplicity, let us observe only the inductive limit case and note that the same conclusion holds in the projective limit case.

**Theorem 4.1.** *Let the sequence  $(N_q)_{q \in \mathbf{N}_0}$  be given such that (2.2) and (2.4) hold.*

i) *Let  $f, g \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$ . Then  $W(f, g)(x, \xi) \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$  and extends to a map from  $(\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d) \times (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d)$  into  $(\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^{2d})$ . The same is true for the short-time Fourier transform.*

ii) *Conversely, if  $W(f, g)$  (or  $V_g f$ ) belongs to  $\mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ , then  $f, g \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$ .*

*Proof.* The proof in the Schwartz space can be found in [10, 12].

i) Let another sequence  $(M_p)_{p \in \mathbf{N}_0}$  be given such that (2.2) and (2.4) hold. Obviously,  $f, g \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$  implies that  $f(x)g(t) \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d \times \mathbf{R}^d)$ .

$\mathbf{R}^d$ ), and also by the measure preserving change of variables  $(x, t) \mapsto (x + (t/2), x - (t/2))$  one can conclude that  $\varphi(x, t) := f(x + (t/2))g(x - (t/2))$  belongs to  $\mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d \times \mathbf{R}^d)$ .

Now we have to restrict our attention to the case when  $(M_p)_{p \in \mathbf{N}_0} = (N_q)_{q \in \mathbf{N}_0}$ . Let  $\varphi \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ . By a straightforward calculation it follows that the partial Fourier transform of  $\varphi$  with respect to the second variable is continuous bijection on  $\mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ , that is,

$$\Phi(x, \omega) = \int e^{-2\pi i t \omega} \varphi(x, t) dt \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d \times \mathbf{R}^d)$$

if and only if  $\varphi \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d \times \mathbf{R}^d)$ .

Thus, i) is proved in a general case for transforms of the type  $\int e^{-2\pi i t \omega} \varphi(x, t) dt$  with  $\varphi \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ . In particular, the assertion holds for the cross Wigner distribution and the short-time Fourier transform. The extensions to a map from  $(\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d) \times (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d)$  into  $(\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^{2d})$  are obtained by duality.

ii) Denote by  $\mathcal{F}_2\varphi$  the partial Fourier transform of  $\varphi$  with respect to the second variable and bijections  $\tau(x, t) = (x + (t/2), x - (t/2))$  and  $\theta(x, t) = (t, t - x)$ . Then  $\mathcal{F}_2$  and the pullback operators  $U \mapsto \tau^*U$  and  $U \mapsto \theta^*U$  are continuous bijections on  $\mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ . Furthermore,  $W(f, g) = (\mathcal{F}_2 \circ \tau^*)(f \otimes \bar{g})$  and, similarly,  $V_g(f) = (\mathcal{F}_2 \circ \theta^*)(f \otimes \bar{g})$ . Consequently,

$$\begin{aligned} V_g(f) \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d}) &\iff W(f, g) \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d}) \\ &\iff f \otimes \bar{g} \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d}) \\ &\iff f, g \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d). \quad \square \end{aligned}$$

*Remark 4.2.* Alternatively, Theorem 4.1 can be proved following the proof of [23, Theorems 3.8, 3.9], for example. Results similar to Theorem 4.1 can be found in [2, 16] where the case  $f = g$ , i.e., the Wigner distribution  $W(f, f)$  is observed. More precisely, it is proved that  $f \in \mathcal{S}_{M_p}^{N_q}(\mathbf{R}^d)$  if and only if  $|W(f, f)(x, \omega)| \leq C \exp\{-M(a|x|) - N(b|\omega|)\}$ , for some positive constants  $C, a$  and  $b$ . It is assumed that

$(M_p)_{p \in \mathbf{N}_0}$  and  $(N_q)_{q \in \mathbf{N}_0}$  satisfy (M.1), (M.2) and (C), and that  $M$  and  $N$  are the corresponding associated functions, see [2].

**4.1. Modulation spaces.** The modulation space norm  $M_m^{p,q}(\mathbf{R}^d)$  of a function  $f$  on  $\mathbf{R}^d$  is given by the  $L_m^{p,q}(\mathbf{R}^{2d})$  norm of its STFT  $V_g f$ , defined on the time-frequency space  $\mathbf{R}^{2d}$ , with respect to a *suitable* window function  $g$  on  $\mathbf{R}^d$ . Depending on the growth of the weight function  $m$ , different Gelfand-Shilov classes may be chosen as fitting test function spaces for modulation spaces, see [8, 23]. The widest class of weights allowing to define modulation spaces is the weight class  $\mathcal{N}$ . A weight function  $m$  on  $\mathbf{R}^{2d}$  belongs to  $\mathcal{N}$  if it is a continuous, positive function such that

$$(4.4) \quad m(z) = o(e^{cz^2}), \text{ for } |z| \rightarrow \infty, \text{ for all } c > 0,$$

with  $z \in \mathbf{R}^{2d}$ . For instance, every function  $m(z) = e^{s|z|^b}$ , with  $s > 0$  and  $0 \leq b < 2$ , is in  $\mathcal{N}$ . Thus, the weight  $m$  may grow faster than exponentially at infinity. We notice that there is a limit in enlarging the weight class for modulation spaces, imposed by Hardy's theorem: if  $m(z) \geq Ce^{cz^2}$ , for some  $c > \pi/2$ , then the corresponding modulation spaces are trivial [14].

Modulation spaces having weights with at most sub-exponential growth were first introduced by Feichtinger in [9]. We define them using the Gelfand-Shilov class  $\mathcal{S}_{1/2}^{1/2}$  as test function spaces hereafter in a similar way as [5].

**Definition 4.3.** Let  $m \in \mathcal{N}$ , and let  $g$  be a nonzero *window* function in  $\mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ . For  $1 \leq p, q \leq \infty$  and  $f \in (\mathcal{S}_{1/2}^{1/2})'(\mathbf{R}^d)$ , we define the modulation space norm (on  $\mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ ) by

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left( \int_{\mathbf{R}^d} \left( \int_{\mathbf{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q}$$

(with obvious changes if either  $p = \infty$  or  $q = \infty$ ). If  $p, q < \infty$ , the modulation space  $M_m^{p,q}$  is the norm completion of  $\mathcal{S}_{1/2}^{1/2}$  in the  $M_m^{p,q}$ -norm. If  $p = \infty$  or  $q = \infty$ , then  $M_m^{p,q}$  is the completion of  $\mathcal{S}_{1/2}^{1/2}$  in the



weak\* topology. If  $p = q$ ,  $M_m^p := M_m^{p,p}$ , and, if  $m \equiv 1$ , then  $M^{p,q}$  and  $M^p$  stand for  $M_m^{p,q}$  and  $M_m^{p,p}$ , respectively.

Notice that: (i) If  $f, g \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$ , the above integral is convergent thanks to (4.4) and (2.3). Namely, if  $m \in \mathcal{N}$  we choose  $c = h - \varepsilon > 0$  in (4.4), for a suitable  $\varepsilon > 0$ , with  $h$  being such that  $\|V_g f e^{h|\cdot|^2}\|_{L^\infty} < \infty$ . Hence,

$$\begin{aligned} & \int_{\mathbf{R}^d} \left( \int_{\mathbf{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \\ & \leq C \| (V_g f) e^{h|\cdot|^2} \|_{L^\infty} \int_{\mathbf{R}^d} \left( \int_{\mathbf{R}^d} |m(x, \omega)|^p e^{-hp|(x, \omega)|^2} dx \right)^{q/p} d\omega < \infty. \end{aligned}$$

(ii) By definition,  $M_m^{p,q}$  are Banach spaces.

(iii) It was shown in [5] that the Gelfand-Shilov class  $\mathcal{S}_{1/2}^{1/2}$  is densely embedded in  $M_m^1$ , with  $m \in \mathcal{N}$ .

(iv) Of course, the definition of  $M_m^{p,q}$  may depend on the choice of the window function  $g$ . So, we choose the Gaussian window  $\varphi(x) = e^{-\pi x^2} \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d)$  once and for all to define modulation spaces, and we shall always work with it in the sequel.

It is straightforward to check that [6, Proposition 2.4] can be rephrased in our context as follows.

**Proposition 4.4.** *Let  $\nu \in \mathcal{N}(\mathbf{R}^d)$  be a weight function only in the frequency variables  $\nu(x, \omega) = \nu(\omega)$  and  $1 \leq p, q, r, s, t \leq \infty$ . If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \quad \text{and} \quad \frac{1}{t} + \frac{1}{t'} = 1,$$

then

$$(4.5) \quad M_{1 \otimes \nu}^{p, st}(\mathbf{R}^d) * M_{1 \otimes \nu^{-1}}^{q, st'}(\mathbf{R}^d) \hookrightarrow M^{r, s}(\mathbf{R}^d)$$

with norm inequality  $\|f * h\|_{M^{r, s}} \lesssim \|f\|_{M_{1 \otimes \nu}^{p, st}} \|h\|_{M_{1 \otimes \nu^{-1}}^{q, st'}}$ .

Gelfand-Shilov type spaces can be characterized by modulation spaces in the following way:

**Proposition 4.5.** *Let there be given a sequence  $(N_q)_{q \in \mathcal{N}_0}$  such that (2.2) and (2.4) hold. Let  $1 \leq p, q \leq \infty, p' = (p/p - 1), q' = (q/q - 1)$ . We have*

$$\Sigma_{N_q}^{N_q}(\mathbf{R}^d) = \text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^{p,q}(\mathbf{R}^d),$$

$$\mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d) = \text{ind} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^{p',q'}(\mathbf{R}^d)$$

and, by duality,

$$(\Sigma_{N_q}^{N_q})'(\mathbf{R}^d) = \text{ind} \lim_{s \rightarrow \infty} M_{e^{-N(s|\cdot)}}^{p',q'}(\mathbf{R}^d),$$

$$(\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d) = \text{proj} \lim_{s \rightarrow \infty} M_{e^{-N(s|\cdot)}}^{p,q}(\mathbf{R}^d),$$

in the set theoretical sense.

*Proof.* The proof is similar to the proof of the non-quasianalytic case given in [24, Theorem 5.1]. Let us show  $\Sigma_{N_q}^{N_q}(\mathbf{R}^d) = \text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^{p,q}(\mathbf{R}^d)$ . We first show that  $\text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^\infty = \text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^{p,q}, 1 \leq p, q \leq \infty$ . For a fixed  $g \in \mathcal{S}_{N_q}^{N_q}$  and any weight  $m \in \mathcal{N}'$ , we have

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{M_m^{p,q}} \leq \|V_g f e^{N(s|\cdot)}\|_{L^\infty} \|e^{-N(s|\cdot)}\|_{L_m^{p,q}},$$

and therefore  $\text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^\infty \subset \text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot)}}^{p,q}$ . In order to prove the opposite inclusion we use

$$V_g f(x, \xi) e^{N(s|(x,\omega))} = \int_{-\infty}^x \int_{-\infty}^\xi \frac{\partial^2}{\partial t \partial \eta} (V_g f(t, \eta) e^{N(s|(t,\eta))}) dt d\eta,$$

which implies

$$|V_g f(x, \xi) e^{N(s|(x,\omega))}| \leq \left\| \frac{\partial^2}{\partial t \partial \eta} V_g f(t, \eta) \cdot e^{N(s|(t,\eta))} \right\|_{L^1} + \left\| V_g f \cdot \frac{\partial^2}{\partial t \partial \eta} e^{N(s|(t,\eta))} \right\|_{L^1}.$$

Now, the estimates similar to the ones given in the proof of [24, Theorem 5.1] give

$$|V_g f(x, \xi) e^{N(s|(x, \omega|)}| \leq C(\|f\|_{M_{e^{N(s_1|\cdot|)}}^{p,q}} + \|f\|_{M_{e^{N(s_2|\cdot|)}}^{p,q}} + \|f\|_{M_{e^{N(s_3|\cdot|)}}^{p,q}})$$

for certain  $s_1, s_2, s_3 > s$ .

Now,  $\Sigma_{N_q}^{N_q}(\mathbf{R}^d) = \text{proj} \lim_{s \rightarrow \infty} M_{e^{N(s|\cdot|)}}^\infty$  follows from Theorem 4.1 together with  $|\mathcal{F}V_g f(\cdot)| \leq C e^{-N(s|\cdot|)}$  which holds true since  $\Sigma_{N_q}^{N_q}(\mathbf{R}^{2d})$  is invariant under the action of the Fourier transform.  $\square$

**4.2. Localization operators.** Formally, the time-frequency localization operator  $A_a^{\varphi_1, \varphi_2}$  with symbol  $a$  and windows  $\varphi_1, \varphi_2$  is defined to be

$$(4.6) \quad A_a^{\varphi_1, \varphi_2} f(t) = \int_{\mathbf{R}^{2d}} a(x, \omega) V_{\varphi_1} f(x, \omega) M_\omega T_x \varphi_2(t) dx d\omega,$$

or, in a weak sense,

$$(4.7) \quad \langle A_a^{\varphi_1, \varphi_2} f, g \rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g \rangle = \langle a, \overline{V_{\varphi_1} f} V_{\varphi_2} g \rangle,$$

where the brackets express a suitable duality between a pair of dual spaces. Indeed, if  $a \in (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^{2d})$  and  $\varphi_1, \varphi_2 \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$ , then the weak definition (4.7) shows that  $A_a^{\varphi_1, \varphi_2}$  is a well-defined continuous operator from  $\mathcal{S}_{N_q}^{N_q}$  to  $(\mathcal{S}_{N_q}^{N_q})'$ .

**Theorem 4.6.** *Let sequence  $(N_q)_{q \in \mathbf{N}_0}$  be given such that (2.2) and (2.4) holds. Let  $u \in (\mathcal{S}_{N_q}^{N_q})'$  ( $u \in (\Sigma_{N_q}^{N_q})'$ , respectively) such that it can be extended continuously to  $\mathcal{E}^{\{N_q\}}(\Omega)$  ( $\mathcal{E}^{(N_q)}(\Omega)$ , respectively) for some open bounded set  $\Omega \in \mathbf{R}^d$ . If  $\varphi \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$  (if  $\varphi \in \Sigma_{N_q}^{N_q}(\mathbf{R}^d)$ , respectively) then*

$$(4.8) \quad |V_\varphi u(x, \omega)| \lesssim e^{-N(a|x|)} e^{N(\tilde{a}|\omega|)},$$

for some  $a, \tilde{a} > 0$  (respectively for every  $a, \tilde{a} > 0$ ).

*Proof.* Let us prove the case  $u \in (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^d)$ . By Theorem 3.1 we have

$$\begin{aligned} |V_\varphi u(x, \omega)| &= |V_\varphi(P(\partial)g + h)(x, \omega)| \\ &\leq |V_\varphi(P(\partial)g)(x, \omega)| + |V_\varphi(h)(x, \omega)|, \end{aligned}$$

where  $g$  and  $h$  are bounded continuous functions on  $\mathbf{R}^d$  and  $P(\partial)$  is the corresponding ultradifferentiable operator. We begin by studying the first term on the righthand side. Note that, for every  $\varphi \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$ , we have

$$\partial^\alpha(M_\omega T_x \varphi)(t) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi i \omega)^\beta M_\omega T_x \partial_t^{\alpha-\beta} \varphi(t),$$

and for any given  $\gamma \in \mathbf{N}_0$  there exist positive constants  $a$  and  $b$  such that

$$(4.9) \quad \sup_{x \in \mathbf{R}^d} \left| \frac{b^{|\gamma|} \partial^\gamma \varphi(x)}{|\gamma|!^{1/2} R_{|\gamma|}} e^{N(a|x|)} \right| < \infty.$$

We have

$$\begin{aligned} |V_\varphi(P(\partial)g)(x, \omega)| &\leq \sum_{\alpha \in \mathbf{N}_0^d} |a_\alpha| |\langle \partial^\alpha g, M_\omega T_x \varphi |_\Omega \rangle| \\ &= \sum_{\alpha \in \mathbf{N}_0^d} |a_\alpha| |\langle g, \partial^\alpha (M_\omega T_x \varphi) |_\Omega \rangle| \\ &\leq \sum_{\alpha \in \mathbf{N}_0^d} |a_\alpha| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |(2\pi i \omega)^\beta| \\ (4.10) \quad &\times \int_\Omega |(M_\omega T_x \partial_t^{\alpha-\beta} \varphi)(t)| |g(t)| dt \\ &\leq \sum_{\alpha \in \mathbf{N}_0^d} |a_\alpha| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi)^{|\beta|} |\omega|^{|\beta|} \\ &\times \int_\Omega |(\partial_t^{\alpha-\beta} \varphi)(t-x)| |g(t)| dt. \end{aligned}$$

By Theorem 3.1,

$$|a_\alpha| \leq C \frac{h^{|\alpha|}}{|\alpha|!^{1/2} R_{|\alpha|}}$$

for any  $h > 0$ . Using  $(n-m)!m! \leq n!$  and  $R_{n-m}R_m \leq R_n$ ,  $m \leq n$ ,  $m, n \in \mathbf{N}$ , together with  $\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|}$ , we obtain

$$\begin{aligned}
 |V_\varphi(P(\partial)g)(x, \omega)| &\leq C \sum_{\alpha \in \mathbf{N}_0^d} \left(\frac{2}{3}\right)^{|\alpha|} \sum_{\beta \leq \alpha} \\
 &\quad \times \int_{\Omega} \frac{(3h)^{|\alpha-\beta|}}{|\alpha-\beta|!^{1/2} R_{|\alpha-\beta|}} |(\partial_t^{\alpha-\beta} \varphi)(t-x)| \\
 &\quad \times (2\pi)^{|\beta|} |\omega|^{|\beta|} \frac{(3h)^{|\beta|}}{|\beta|!^{1/2} R_{|\beta|}} |g(t)| dt \\
 &\leq C \sum_{\alpha \in \mathbf{N}_0^d} \left(\frac{2}{3}\right)^{|\alpha|} \sum_{\beta \leq \alpha} \\
 &\quad \times \int_{\Omega} \frac{(3h)^{|\alpha-\beta|}}{|\alpha-\beta|!^{1/2} R_{|\alpha-\beta|}} |(\partial_t^{\alpha-\beta} \varphi)(t-x)| e^{N(a|t-x|)} \\
 &\quad \times e^{-N(a|x|)} e^{N(a|t|)} (2\pi)^{|\beta|} |\omega|^{|\beta|} \frac{(3h)^{|\beta|}}{|\beta|!^{1/2} R_{|\beta|}} |g(t)| dt.
 \end{aligned}$$

Now, by (4.9) with  $h := b/3$  and since  $\int_{\Omega} e^{N(a|t|)} |g(t)| dt \leq C < \infty$ , for some  $C > 0$ , the last term is less than or equal to

$$\begin{aligned}
 &C e^{-N(a|x|)} \sum_{\alpha \in \mathbf{N}_0^d} \left(\frac{2}{3}\right)^{|\alpha|} \sum_{\beta \leq \alpha} \frac{(2\pi)^{|\beta|}}{(2\pi+1)^{|\beta|}} \frac{((2\pi+1)b)^{|\beta|}}{|\beta|!^{1/2} R_{|\beta|}} |\omega|^{|\beta|} \\
 &\leq C e^{-N(a|x|)} \sum_{\alpha \in \mathbf{N}_0^d} \left(\frac{2}{3}\right)^{|\alpha|} e^{N((2\pi+1)b|\omega|)} \sum_{\beta \leq \alpha} \frac{(2\pi)^{|\beta|}}{(2\pi+1)^{|\beta|}}.
 \end{aligned}$$

Putting  $\tilde{a} := (2\pi+1)b$  we obtain

$$|V_\varphi(P(\partial)g)(x, \omega)| \leq C e^{-N(a|x|)} e^{N(\tilde{a}|\omega|)}.$$

For the term  $V_\varphi h$ , we use (4.9) to obtain

$$\begin{aligned}
 |V_\varphi h(x, \omega)| &= |\langle h, M_\omega T_x \varphi \rangle| \\
 &\leq \int_{\Omega} |T_x \varphi(t)| |h(t)| dt
 \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Omega} e^{-N(a|t-x|)} e^{N(a|t-x|)} |T_x \varphi(t)| |h(t)| dt \\ &\leq C e^{-N(a|x|)} \int_{\Omega} e^{N(a|t|)} |h(t)| dt \\ &\leq C_{K,h} e^{-N(a|x|)}, \end{aligned}$$

and the theorem is proved.  $\square$

Let  $L_{\sigma}$  be the Weyl pseudodifferential operator with symbol  $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ , defined weakly by

$$(4.11) \quad \langle L_{\sigma} f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbf{R}^d),$$

where  $W(\varphi_2, \varphi_1)$  is the cross-Wigner distribution. Then we have the operator equality  $A_a^{\varphi_1, \varphi_2} = L_{\sigma}$ , provided that [1, 10, 22]:

$$(4.12) \quad \sigma = a * W(\varphi_2, \varphi_1).$$

We shall focus now on a trace-class result for localization operators. First, let us recall that the singular values  $\{s_k(L)\}_{k=1}^{\infty}$  of a compact operator  $L \in B(L^2(\mathbf{R}^d))$  are the eigenvalues of the positive self-adjoint operator  $\sqrt{L^*L}$ . For  $p = 1$ , the trace class  $S_1$  is the space of all compact operators whose singular values enjoy  $\sum_{k=1}^{\infty} |s_k(L)| < \infty$ . To prove the main result of this section, we shall use a trace-class result for the Weyl calculus in terms of modulation spaces. For the proof we refer to [13].

**Theorem 4.7.** *If  $\sigma \in M^1(\mathbf{R}^{2d})$ , then  $L_{\sigma} \in S_1$  and  $\|L_{\sigma}\|_{S_1} \lesssim \|\sigma\|_{M^1}$ .*

Our result reads as follows.

**Theorem 4.8.** *Let the sequence  $(N_q)_{q \in \mathbf{N}_0}$  be given such that (2.2) and (2.4) hold. Let  $a \in (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^{2d})$  ( $a \in (\Sigma_{N_q}^{N_q})'(\mathbf{R}^{2d})$ , respectively) such that it can be extended continuously to  $\mathcal{E}^{\{N_q\}}(\Omega)$  ( $\mathcal{E}^{(N_q)}(\Omega)$ , respectively) for some open bounded set  $\Omega \subseteq \mathbf{R}^{2d}$ . Furthermore, let  $\varphi_1, \varphi_2 \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$  (respectively  $\varphi_1, \varphi_2 \in \Sigma_{N_q}^{N_q}(\mathbf{R}^d)$ ). Then  $A_a^{\varphi_1, \varphi_2}$  is a trace-class operator.*

*Proof.* Assume that  $a \in (\mathcal{S}_{N_q}^{N_q})'(\mathbf{R}^{2d})$ . The other case can be treated in an analogous way. As already mentioned, in the definition of modulation spaces, we fix Gaussian window  $g(x) = e^{-\pi x^2} \in \mathcal{S}_{1/2}^{1/2}(\mathbf{R}^d) \subset \Sigma_{N_q}^{N_q}(\mathbf{R}^d)$ , see [8, Lemma 2.3]. By Theorem 4.6, we have

$$|V_g a(x, \omega)| \leq C e^{-N(h|x|)} e^{N(k|\omega|)}$$

for arbitrary  $h, k > 0$ . Then, for a given  $b > 0$ , we choose  $k < b$  to obtain

$$\begin{aligned} \sup_{\omega \in \mathbf{R}^d} \int_{\mathbf{R}^d} |V_g a(x, \omega)| e^{-N(b|\omega|)} dx \\ \leq \sup_{\omega \in \mathbf{R}^d} \int_{\mathbf{R}^d} e^{-N(h|x|)} e^{N(k|\omega|)} e^{-N(b|\omega|)} dx < \infty. \end{aligned}$$

Therefore,  $a \in M_{1 \otimes e^{-N(b|\cdot|)}}^{1, \infty}(\mathbf{R}^{2d})$ , where  $b > 0$  can be chosen arbitrary.

If  $\varphi_1, \varphi_2 \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^d)$ , the characterization in Theorem 4.1 i) gives  $W(\varphi_2, \varphi_1) \in \mathcal{S}_{N_q}^{N_q}(\mathbf{R}^{2d})$ , and therefore, by Proposition 4.5,  $h, k > 0$  exist such that

$$W(\varphi_2, \varphi_1) \in M_{e^{N(h|x|)} \otimes e^{N(k|\omega|)}}^1(\mathbf{R}^{2d}) \subset M_{1 \otimes e^{N(k|\omega|)}}^1(\mathbf{R}^{2d}).$$

Now, we choose  $b = k$  and use the convolution relations of Proposition 4.4 to obtain

$$M_{1 \otimes e^{-N(k|\cdot|)}}^{1, \infty}(\mathbf{R}^{2d}) * M_{1 \otimes e^{-N(k|\cdot|)}}^1(\mathbf{R}^{2d}) \hookrightarrow M^1(\mathbf{R}^{2d});$$

hence,  $\sigma = a * W(\varphi_2, \varphi_1) \in M^1(\mathbf{R}^{2d})$ . Theorem 4.7 yields the desired result.  $\square$

For example, our result holds for  $f = \sum_{n \in \mathbf{N}} a_n \delta^{(n)}$ , where  $|a_n| \leq (C_h h^n / n!^s)$ , for every  $h > 0$  and corresponding  $C_h > 0$  and  $s > 1/2$ . Actually, one can show that  $f \in (\mathcal{E}^{p^s})'(\mathbf{R}^{2d})$  for any  $s > 0$ . (Note that  $f = e^{x^2}$  belongs to  $\mathcal{E}^{p^s}(\mathbf{R}^{2d})$  for any  $s > 0$ .)

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