ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 4, 2010

POINTWISE CONVERGENCE ON THE BOUNDARY IN THE DENJOY-WOLFF THEOREM

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ABSTRACT. If ϕ is an analytic self-map of the disk (not an elliptic automorphism) the Denjoy-Wolff theorem predicts the existence of a point p with $|p| \leq 1$ such that the iterates ϕ_n converge to p uniformly on compact subsets of the disk. Since these iterates are bounded analytic functions, there is a subset of the unit circle of full linear measure where they are all well-defined. We address the question of whether convergence to p still holds almost everywhere on the unit circle. The answer depends on the location of p and the dynamical properties of ϕ. We show that when $|p| < 1$ (elliptic case), pointwise almost everywhere convergence holds if and only if ϕ is not an inner function. When $|p| = 1$ things are more delicate. We show that when ϕ is hyperbolic or nonzero-step parabolic, then pointwise almost everywhere convergence always holds. The last case, zero-step parabolic, remains open.

1. Introduction. Let ϕ be an analytic map defined on the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, and assume that $\phi(\mathbf{D}) \subset \mathbf{D}$ (we call ϕ a self-map of the disk from now on). The iterates of ϕ are $\phi_n = \phi \circ \cdots \circ \phi$, n times. The following result is classical (elliptic automorphisms are those that can be conjugated to a rotation).

Theorem 1.1 (Denjoy-Wolff). *If a self-map of the disk ϕ is not an elliptic automorphism, then there exist a point $p \in \overline{\mathbf{D}}$ such that the sequence $\phi_n(z)$ converges uniformly on compact subsets of $\mathbf{D}$ to p.*

Moreover, when $p \in \mathbf{D}$, $\phi(p) = p$ and $|\phi'(p)| < 1$, while when $p \in \partial\mathbf{D}$, then $\phi(p) = p$ and $0 < \phi'(p) \leq 1$ in the sense of nontangential limits.

The point p is referred to as the Denjoy-Wolff point of ϕ. When $p \in \mathbf{D}$, the map ϕ is called *elliptic*. When $p \in \partial\mathbf{D}$, ϕ is called *hyperbolic* if $\phi'(p) < 1$ and *parabolic* if $\phi'(p) = 1$.

2010 AMS *Mathematics subject classification.* Primary 30D05, 30C85.

Received by the editors on April 20, 2005, and in revised form on February 20, 2008.

DOI:10.1216/RMJ-2010-40-4-1275

Since the functions ϕ_n are bounded analytic functions, it is well known that one can define the corresponding boundary functions

$$\phi_n^*(e^{i\theta}) = \lim_{r\uparrow 1} \phi_n(re^{i\theta})$$

for almost every $e^{i\theta}$ on $\partial\mathbf{D}$. More precisely, for every $n = 1, 2, 3, \ldots$, there is a set $E_n \subset \mathbf{D}$ of linear measure zero, so that ϕ_n^* is well defined on $\partial\mathbf{D}\setminus E_n$. Then $W = \partial\mathbf{D}\setminus\cup_{n=1}^{\infty} E_n$ has full measure and every iterate ϕ_n extends to W.

It is natural to ask whether for almost every point $\zeta \in W$ the sequence $\phi_n^*(\zeta)$ still converges to p as $n \to \infty$. An answer to this question in the elliptic case (when p is not super-attracting) can be extracted from the proofs of our paper [**5**]. The purpose of this note is to do a more systematic study of this problem.

Before stating our result, we need to recall some definitions.

A bounded analytic function f on the unit disk $\mathbf{D}$ is an *inner function* if the corresponding boundary function $f^*(e^{i\theta})$ has modulus equal to 1 for almost every $e^{i\theta}$ on $\partial\mathbf{D}$. It follows from the Riesz theorem [**4**, page 205], that if f is inner then $f \circ f$ is inner as well.

Recall the hyperbolic distance, defined for $z, w \in \mathbf{D}$ as:

$$\rho(z, w) = \log \frac{1 + |z - w|/|1 - \bar{w}z|}{1 - |z - w|/|1 - \bar{w}z|}.$$

Given a self-map ϕ of parabolic type, pick a point $z_0 \in \mathbf{D}$, and let $z_n = \phi_n(z_0)$ be the corresponding orbit. Define $s_n := \rho(z_n, z_{n+1})$, i.e., the hyperbolic step of the orbit. By Schwarz's lemma, ϕ is a contraction with respect to the metric ρ, in particular, s_n is a nonincreasing sequence, and hence $s_\infty = \lim_{n\to\infty} s_n$ exists. There are two cases: ϕ is *nonzero-step* if $s_\infty > 0$ and *zero-step* if $s_\infty = 0$. It follows from the main theorem of [**8**] that this classification does not depend on the choice of z_0 (also see comments after Theorem 1.8 of [**7**]).

Theorem 1.2. *Suppose ϕ is an analytic self-map of the disk which is not an elliptic automorphism.*

(1) *If ϕ is elliptic, then $\phi_n^*(\zeta)$ converges to the Denjoy-Wolff point of ϕ, for almost every ζ in $\partial\mathbf{D}$, if and only if ϕ is not an inner function.*

(2) *If ϕ is hyperbolic or parabolic nonzero-step, then $\phi_n^*(\zeta)$ converges to the Denjoy-Wolff point of ϕ, for almost every ζ in $\partial\mathbf{D}$.*

Our proofs do not extend to the parabolic zero-step case, which in some sense is more similar to the elliptic case since the hyperbolic steps are tending to zero. In fact, we make the following conjecture.

Conjecture 1.3. *Suppose ϕ is an analytic self-map of the disk which is parabolic zero-step. Then $\phi_n^*(\zeta)$ converges to the Denjoy-Wolff point of ϕ, for almost every ζ in $\partial\mathbf{D}$, when ϕ is not an inner function.*

Bourdon, Matache, and Shapiro [**1**] have independently given a different proof of Theorem 1.2, and they have also given examples of inner functions of parabolic zero-step type for which pointwise convergence on $\partial\mathbf{D}$ to the Denjoy-Wolff point does occur.

In Section 2 we tackle the elliptic case. In Section 3 we deal with the hyperbolic and parabolic nonzero-step cases.

2. The elliptic case. Assume that ϕ is elliptic, i.e., the Denjoy-Wolff point p of ϕ is in $\mathbf{D}$ (and assume that ϕ is not an automorphism). If ϕ is inner, then for almost every $\zeta \in \partial\mathbf{D}$, $|\phi_n^*(\zeta)| = 1$ for all n, so the convergence to p does not occur.

The converse is less straightforward. Assume then also that ϕ is not an inner function. The argument revolves around showing that there cannot exist a set $E \subset \partial\mathbf{D}$ of positive linear measure on which all the iterates ϕ_n^* have modulus one.

2.1. An exhaustion of the unit disk. Given $n = 1, 2, 3, \ldots$, fix a parameter $t > 0$, and consider the open set $U_n(t) = \{z \in \mathbf{D} : \rho(\phi_n(z), p) < t\}$. Then, let $\Omega_n(t)$ be the connected component of $U_n(t)$ which contains p, and let $F_n(t)$ be $\partial\Omega_n(t) \cap \mathbf{D}$. Notice that $F_n(t)$ consists of at most countably many piecewise analytic Jordan arcs, and either there is only one closed arc, or all the arcs have the property that their two ends tend to $\partial\mathbf{D}$ (by the maximum principle). Let $C(r) = \{z \in \mathbf{D} : \rho(z, p) = r\}$; then there is an $r > 0$ such that $\phi(z) \neq p$, for all $z \in C(r)$. Therefore, we can find $t_0 > 0$ small enough so that $\Omega_1(t_0)$ is compactly contained in $\mathbf{D}$, and therefore $F_1(t_0)$ consists of one closed Jordan arc. From now on we write Ω_n for $\Omega_n(t_0)$ and F_n for $F_n(t_0)$.

By definition,

$$\phi_k(\Omega_{n+k}) \subset \Omega_n \quad \text{and} \quad \phi_k(F_{n+k}) \subset F_n \tag{2.1}$$

for $n, k = 1, 2, 3, \dots$. Moreover, we also have

$$\Omega_n \cup F_n \subset \Omega_{n+1}, \tag{2.2}$$

for $n = 1, 2, 3, \dots$. This is because whenever $\zeta \in \Omega_n \cup F_n$, there is a path $\gamma \subset \Omega_n \cup F_n$ connecting p to ζ, and by the invariant form of Schwarz's lemma and the fact that $\phi(p) = p$,

$$\rho(p, \phi_{n+1}(\gamma(s))) < \rho(p, \phi_n(\gamma(s))) \leq t_0, \quad 0 \leq s \leq 1.$$

2.2. Harmonic measure. If Ω is an open set and E a closed set, then we write

$$\omega(z, E, \Omega)$$

for the Perron solution of the Dirichlet problem, in the component U of $\Omega \setminus E$ containing z, with data χ_E (the characteristic function of E). Recall that this is obtained by taking the supremum of all the values $v(z)$, when v ranges among all subharmonic functions on U such that $\limsup_{z\to\zeta} v(z) \leq \chi_E(\zeta)$, for all $\zeta \in \partial\Omega \cup E$ (these functions v are often referred to as "candidates").

Write $\omega_n(z) = \omega(z, F_n, \Omega_n)$. We will need two results about harmonic measure. We refer to [**10**] for the potential theory background that is needed.

Lemma 2.1 (Schwarz-type lemma). *Let E be a closed set in* $\mathbf{D}$ *with* $\operatorname{Cap}(\phi^{-1}(E)) > 0$. *Then,*

$$\omega(z, \phi^{-1}(E), \mathbf{D} \setminus \phi^{-1}(E)) \leq \omega(\phi(z), E, \mathbf{D} \setminus E).$$

Proof of Lemma 2.1. This proof is similar to the proof of Lemma 3.1 in [**6**]. Let $G = \mathbf{D} \setminus \phi^{-1}(E)$. Let v be a candidate for the Dirichlet problem on G with data $\chi_{\phi^{-1}(E)}$, and let $u(z) = \omega(\phi(z), E, \mathbf{D} \setminus E)$. When $z \in G$, $\phi(z) \notin E$, and hence $v - u$ is subharmonic on G. Suppose now that $\zeta \in \partial G$. There are two cases. First, assume that $\zeta \in \partial\mathbf{D}$, i.e., $\zeta \notin \phi^{-1}(E)$. Then by the definition of v and, since u is positive, $\limsup_{z\to\zeta}[v(z) - u(z)] \leq 0$. When $\zeta \in \phi^{-1}(E)$, $\limsup_{z\to\zeta} v(z) \leq 1$. Also, $\operatorname{Cap} E > 0$ by Corollary 3.6.6 of [**10**], and at nearly every $\eta \in E$ we have $\lim_{z\to\eta} \omega(z, E, \mathbf{D} \setminus E) = 1$, by Theorem 4.2.5 of [**10**] (Kellogg's theorem), and by Theorem 4.3.4 (b) of [**10**]. Therefore, since ϕ is analytic, for nearly every $\zeta \in \phi^{-1}(E)$, $\lim_{z\to\zeta} u(z) = 1$. By the extended maximum principle for subharmonic functions, [**10**, Theorem 3.6.9 (b)], $v - u \leq 0$ on G, and the conclusion is reached by taking the supremum over all the candidates v. □

The second result is a well-known "conditional probability estimate."

Lemma 2.2 (Conditional probability). *Suppose that Ω is an open set and E is a nonempty Borel subset of $\partial\Omega$. Also suppose that F is a closed subset of Ω which separates a point $z \in \Omega$ from E in Ω, i.e., if U is the connected component of $\Omega \setminus F$ containing z, then $E \cap \partial U = \varnothing$. We have*

$$\omega(z, E, \Omega) \leq \omega(z, F, \Omega) \sup_{\zeta\in F} \omega(\zeta, E, \Omega).$$

Proof. With U as above, $u(w) := \omega(w, F, U) = \omega(w, F, \Omega)$ is harmonic in U. Let v be a subharmonic candidate for $\omega(z, E, \Omega)$. Then $f(w) := v(w) - u(w) \sup_{\zeta\in F} \omega(\zeta, E, \Omega)$ is subharmonic in U. First note that since F separates z from E in Ω, we must have $\operatorname{Cap} F > 0$, see Corollary 3.6.4 of [**10**]. Then, for nearly every $\xi \in F$, $\lim_{w\to\xi} u(w) = 1$, see [**10**, Theorem 4.3.4]. Moreover, since $\xi \notin E$ and v is upper semi-continuous, $\limsup_{w\to\xi} v(w) \leq v(\xi) \leq \omega(\xi, E, \Omega)$. Therefore, $\limsup_{w\to\xi} f(w) \leq 0$. On the other hand, if $\xi \in \partial U \setminus F$, then $\xi \in \partial\Omega \setminus E$; therefore, since $u(w) \geq 0$, and since by definition, $\limsup_{w\to\xi} v(w) \leq 0$, we again have $\limsup_{w\to\xi} f(w) \leq 0$. So by the extended maximum principle for subharmonic functions, [**10**, Theorem 3.6.9 (b), $f(w) \leq 0$ for $w \in U$. The conclusion is reached by taking the supremum over all candidates v. □

2.3. Noninner elliptic self-maps. Recall the exhaustion Ω_n defined in subsection 2.1, and that $\omega_n(z) := \omega(z, F_n, \Omega_n)$.

Lemma 2.3. *Assume that the elliptic self-map of the disk ϕ is not inner. Then, there is an integer $N > 1$ large enough such that $\omega_N(z) < 1$ for every $z \in \Omega_N$. In particular,*

$$\alpha := \sup_{\zeta \in F_1} \omega_N(\zeta) < 1. \tag{2.3}$$

Proof of Lemma 2.3. This argument is very similar to the one in [**5**, page 506]. We reproduce it here for convenience. Since ϕ is not inner, there is a set of positive measure $A \subset W$ (recall $W \subset \partial\mathbf{D}$ is the set of full-measure where all the iterates of ϕ are well defined) such that $\phi^*(A) \subset D$. By Lindelöf's theorem [**4**, page 75], it is well known that the radial limits of ϕ coincide with its nontangential limits. Therefore, for $\zeta \in \partial\mathbf{D}$, we define the nontangential region:

$$\Gamma(\zeta) = \left\{ z \in \mathbf{D} : |\zeta - z| < 2\frac{1+|p|}{1-|p|}(1-|z|) \right\}$$

(notice that $p \in \Gamma(\zeta)$ for all $\zeta \in \partial\mathbf{D}$).

By restricting ourselves to a subset of A of positive linear measure, we can assume that $\sup\{|\phi(z)| : z \in \Gamma(\zeta)\} \leq s < 1$, for some $0 < s < 1$. By uniform convergence of ϕ_n on $s\overline{\mathbf{D}}$, there is an $N \in \mathbf{N}$ such that $\rho(p, \phi_N^*(z)) < t_0$ for all $z \in \Gamma(\zeta)$ and all $\zeta \in A$. Thus, the region $G = \cup_{\zeta \in A}\Gamma(\zeta)$ is a Jordan domain contained in Ω_N. The boundary of G is locally Lipschitz, so harmonic measure on ∂G is absolutely continuous with respect to linear measure (this follows from McMillan's sector theorem, see [**9**, Section 6.6]). Hence, $\omega(z, A, G) > 0$ for $z \in G$. By the maximum principle, then, $\omega(z, A, \Omega_N) > 0$ as well, and since $\omega(z, \cdot, \Omega_N)$ is a probability measure on $\partial\Omega_N$, we must have $\omega(z, F_N, \Omega_N) < 1$ for $z \in \Omega_N$ (recall $F_N \subset \mathbf{D}$). $\square$

Proposition 2.4. *Assume that the elliptic self-map of the disk ϕ is not inner. With the notations above,*

$$\omega_n(p) \longrightarrow 0, \quad \textit{as } n \to \infty. \tag{2.4}$$

Proof. We apply Lemma 2.1 with $E = F_N$ (where N and α are as in Lemma 2.3), and $\psi_k := \phi_{(N-1)k}$, $k = 1, 2, 3, \ldots$, instead of ϕ, to obtain

$$\omega(z, \psi_k^{-1}(F_N), \mathbf{D} \setminus \psi_k^{-1}(F_N)) \leq \omega(\psi_k(z), F_N, \mathbf{D} \setminus F_N).$$

For notational simplicity, let $T_k = F_{N+(N-1)k}$ and $G_k = \Omega_{N+(N-1)k}$. Then, by (2.1), $T_k \subset \psi_k^{-1}(F_N)$ and $\psi_k^{-1}(F_N) \cap G_k = \varnothing$. Therefore, for $z \in G_k$,

$$\omega(z, \psi_k^{-1}(F_N), \mathbf{D} \setminus \psi_k^{-1}(F_N)) = \omega(z, T_k, G_k).$$

Taking the supremum for $z \in T_{k-1}$, which is a subset of G_k by (2.2), and since by (2.1) $\psi_k(T_{k-1}) \subset F_1$, we obtain:

$$\sup_{\zeta \in T_{k-1}} \omega(\zeta, T_k, G_k) \leq \sup_{\zeta \in F_1} \omega(\zeta, F_N, \Omega_N) = \alpha < 1.$$

Now, we use the conditional probability estimate of Lemma 2.2, for $n > N$,

$$\omega_n(p) \leq \alpha \sup_{\zeta \in F_N} \omega(\zeta, F_n, \Omega_n) \leq \alpha.$$

Likewise, for $n > 2N - 1$,

$$\begin{aligned}\omega_n(p) &\leq \alpha \sup_{\zeta \in F_N} \omega(\zeta, F_n, \Omega_n) \\ &\leq \alpha \sup_{\zeta \in F_N} \omega(\zeta, F_{2N}, \Omega_{2N}) \sup_{\zeta \in F_{2N}} \omega(\zeta, F_n, \Omega_{2N}) \leq \alpha^2.\end{aligned}$$

More generally, for $n > N + k(N-1)$,

$$\omega_n(p) \leq \alpha^k \longrightarrow 0$$

as k tends to infinity. Therefore, (2.4) is proved. □

2.4. Proof in the elliptic case. Observe that, given a point $\zeta \in W$, if $\phi_n^*(\zeta) \in \mathbf{D}$, then $\phi_{n+k}^*(\zeta) = \phi_k(\phi_n^*(\zeta)) \to p$, as $k \to \infty$. Thus, if the sequence ϕ_n^* does not converge pointwise to p, there is a set $A \subset W$ of positive linear measure such that, for any $\zeta \in A$, $|\phi_n^*(\zeta)| = 1$ for all $n = 1, 2, 3, \ldots$. We claim that

$$0 < \omega(p, A, \mathbf{D}) \leq \omega_n(p), \tag{2.5}$$

but letting n tend to infinity and using (2.4) we thereby reach a contradiction.

To prove (2.5), we use the fact that, although F_n may not separate A from p, it at least does so "radially." Fix an integer n, and for every $\zeta \in A$, since $|\phi_n^*(\zeta)| = 1$, we can find $0 < r(\zeta) < 1$ so that $\rho(\phi_n(r\zeta), p) > t_0$ for $r(\zeta) \le r < 1$. In particular, the slit $S_\zeta = [r(\zeta)\zeta, \zeta)$ does not intersect Ω_n. So, letting $\tilde{A} = \cup_{\zeta \in A} S_\zeta$, we find that

$$\omega(p, \tilde{A}, \mathbf{D} \setminus \tilde{A}) \le \omega_n(p), \tag{2.6}$$

as one can see from Lemma 2.2, for instance.

Finally, the proof of (2.5) is completed if we can show that

$$\omega(p, A, \mathbf{D}) \le \omega(p, \tilde{A}, \mathbf{D} \setminus \tilde{A}). \tag{2.7}$$

To see (2.7), let $v(z)$ be a subharmonic candidate for $\omega(z, A, \mathbf{D})$. By the maximum principle, $v(z) \le 1$ for all $z \in \mathbf{D}$. So v is also a candidate for $\tilde{A}$ in $\mathbf{D} \setminus \tilde{A}$, i.e., $v(z) \le \omega(z, \tilde{A}, \mathbf{D} \setminus \tilde{A})$, and (2.7) is proved by taking the supremum over the v's and evaluating at $z = p$.

This completes the proof of Theorem 1.2 in the case of an interior Denjoy-Wolff point. □

2.5. Remarks. We have just shown that the pointwise almost everywhere convergence on $\partial\mathbf{D}$ of the iterates to the Denjoy-Wolff point holds whenever the self-map is elliptic and noninner. As mentioned above, this fact at least in the case when the derivative of ϕ at p is nonzero was already contained in [**5, 6**]. However, there the main tool was the Koenigs' map $\sigma : \mathbf{D} :\to \mathbf{C}$, which solves the functional equation

$$\sigma \circ \phi(z) = \phi'(p)\sigma(z),$$

and the following dichotomy was proved: either ϕ is not inner and then σ has finite nontangential limits (in $H^p(\mathbf{D})$) almost everywhere on $\partial\mathbf{D}$, or ϕ is inner and then the radial maximal function of σ is infinite almost everywhere on $\partial\mathbf{D}$. In the case when $\phi'(p) = 0$, one would have to use a different conjugating map due to Böttcher, which is not even well defined in $\mathbf{D}$, but the logarithm of its modulus is, see [**3**, page 33].

What we have done here is purge the map σ from the arguments. We will see below in the hyperbolic and parabolic cases, that conjugations will again be useful.

3. The case when the Denjoy-Wolff point is on the boundary. When the Denjoy-Wolff point w is on $\partial\mathbf{D}$ it is customary to change variables with the Möbius transformation $i(w+z)/(w-z)$ so that ϕ becomes a self-map of the upper half-plane $\mathbf{H}$ with the Denjoy-Wolff point at infinity. Julia's lemma [**4**, page 57] then implies that ϕ can be written as

$$\phi(z) = Az + p(z) \tag{3.1}$$

for some $A \geq 1$ and some function p, with $\operatorname{Im} p(z) \geq 0$ for all $z \in \mathbf{H}$, such that

$$n.t. - \lim_{z\to\infty} \frac{p(z)}{z} = 0.$$

In particular, the horodisks $H(t) = \{z \in \mathbf{H} : \operatorname{Im} z > t\}$, $t > 0$, are mapped into themselves, and the map ϕ is classified as *hyperbolic* if $A > 1$ and *parabolic* if $A = 1$.

The proof of Theorem 1.2 in the elliptic case hinged on the fact that for noninner self-maps of the disk there cannot exist a set $E \subset \partial\mathbf{D}$ of positive linear measure on which the nontangential limits ϕ_n^* of each iterate ϕ_n all have modulus one. This, however, is quite possible for noninner self-maps of hyperbolic and parabolic type, as the following example shows.

Example 3.1. Let G be the upper half-plane minus the slits $L_n = \{z = x + iy : x = 2n, 0 < y \leq 2^n\}$ for $n = 1, 2, 3, \ldots$, and minus the rectangle $R = \{z = x + iy : -1 \leq x \leq 1, 0 < y \leq 1\}$. The domain G is simply connected, so let σ be the Riemann map of $\mathbf{H}$ onto G, such that $n.t. - \lim_{z\to\infty} \sigma(z) = \infty$. Defining $\phi(z) := \sigma^{-1}(2\sigma(z))$, one can check that ϕ is hyperbolic, noninner, and all of its iterates have zero imaginary part on $\sigma^{-1}(L_1) \subset \mathbf{R}$. A parabolic example can be obtained by letting $L_n = \{z = x + iy : x = n, 0 < y \leq 1\}$ for $n = 1, 2, 3, \ldots$, and $R = \{z = x + iy : x \leq 0, 0 < y \leq 1\}$.

Therefore, the proof in the hyperbolic and parabolic cases must necessarily be different. We begin with the hyperbolic case.

3.1. The hyperbolic case. We need the following conjugation due to Valiron; see also [**2**] for a recent exposition of this result.

Theorem 3.2 (Valiron). *Assume ϕ is as in* (3.1) *with $A > 1$, i.e., ϕ is hyperbolic. Then, there is an analytic map σ with $\sigma(\mathbf{H}) \subset \mathbf{H}$ such that n.t.* $-\lim_{z\to\infty} \sigma(z) = \infty$ *and actually σ is isogonal at infinity, i.e., n.t.* $-\lim_{z\to\infty} \operatorname{Arg} \sigma(z) = 0$, *which satisfies the functional equation*:

$$\sigma \circ \phi = A\sigma.$$

Since σ is bounded analytic (after a change of variables), it has nontangential limits almost everywhere on the real axis. We let σ^* be the boundary function. By the Riesz theorem ([**4**, page 205]) the set $Z = \{x \in \mathbf{R} : \sigma^*(x) = 0\}$ has measure zero. Suppose $x \in \mathbf{R} \setminus Z$ is a point at which $\sigma^*(x)$ and all the iterates $\phi_n^*(x)$ are well defined, and assume that

$$\liminf_{n\to\infty} |\phi_n^*(x)| = R < \infty.$$

Choose a sequence of integers n for which $|\phi_n^*(x)| < 2R$. For each such n, define $\gamma_n(t) = \phi_n(x + it)$, for $t > 0$. Then, γ_n is a curve which connects the ball $B(2R) = \{|z| \le 2R\}$ to infinity. By the Riesz theorem, we also have that the boundary function $\sigma^*(x)$ is finite almost everywhere. In particular, there exists an $s > 2R$ such that $|\sigma^*(s)|$, $|\sigma^*(-s)| < \infty$. Therefore, if $\Gamma = \{z \in \mathbf{H} : |z| = s\}$, then

$$M := \sup_{z\in\Gamma} |\sigma(z)| < \infty. \tag{3.2}$$

Let $z_n = \phi_n(x+it_n)$ be a point in $\Gamma \cap \gamma_n$, which is necessarily nonempty. Then,

$$M \ge |\sigma(z_n)| = |\sigma(\phi_n(x + it_n))| = A^n |\sigma(x + it_n)|.$$

Therefore, since $A > 1$, $\lim_{n\to\infty} |\sigma(x + it_n)| = 0$. However, since $\lim_{t\to+\infty} \sigma(x + it) = \infty$, and $|\sigma(x + it)| > 0$ for all $t > 0$, we must conclude that t_n tends to 0, i.e., that $\sigma^*(x) = 0$. But this contradicts our hypothesis that $x \notin Z$.

In conclusion, we find that, except for a set of linear measure zero, at all points x where the iterates $\phi_n^*(x)$ are well defined we have $\lim_{n\to\infty} |\phi_n^*(x)| = \infty$, which proves Theorem 1.2 in the hyperbolic case.

3.2. The parabolic nonzero-step case. The counterpart of Valiron's theorem in the parabolic nonzero-step case is the following result of Pommerenke.

Theorem 3.3 (Pommerenke [**8**, Theorem 1 and (3.17)]). *Let ϕ be an analytic self-map of* $\mathbf{H}$ *of parabolic type as in* (3.1) *with* $A = 1$*, and let* $\{z_n = \phi_n(i)\}_{n=0}^{\infty}$ *be a forward-iteration sequence. Then,*

$$\frac{\operatorname{Im} z_{n+1}}{\operatorname{Im} z_n} \longrightarrow 1$$

as n tends to infinity.

Moreover, if ϕ is nonzero step, i.e., $\rho(z_n, z_{n+1}) \downarrow s_\infty > 0$*, letting* $z_n = u_n + iv_n$ *and considering the automorphisms of* $\mathbf{H}$ *given by* $M_n(z) = (z - u_n)/v_n$*, the normalized iterates* $M_n \circ \phi_n$ *converge uniformly on compact subsets of* $\mathbf{H}$ *to a function σ which satisfies the functional equation*

$$\sigma \circ \phi = \sigma + b,$$

where

$$b := \lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_n} \neq 0. \tag{3.3}$$

The conjugation σ is a self-map of $\mathbf{H}$ by construction and Pommerenke also shows that $n.t.-\lim_{z\to\infty} \operatorname{Im} \sigma(z) = +\infty$ (and actually the region of convergence can be extended to a tangential one). However, this is not enough information about the behavior of σ at infinity; in particular, some information on the behavior of $\operatorname{Re}\sigma(z)$ at infinity is necessary if one wants to repeat the same argument as in the hyperbolic case.

Instead we modify the argument slightly. Let σ be the conjugation of Theorem 3.3 and assume without loss of generality that the constant b in (3.3) is positive, so that u_n is eventually an increasing sequence and since $v_n \geq v_0$:

$$\lim_{n\to\infty} u_n = +\infty. \tag{3.4}$$

Let σ^* be the boundary function. Suppose $x \in \mathbf{R}$ is a point at which all the iterates $\phi_n^*(x)$ are well defined, and where $\sigma^*(x)$ is finite. Assume also that

$$\liminf_{n\to\infty} |\phi_n^*(x)| = R < \infty.$$

Instead of considering the half-line $\{z = x+it, t > 0\}$, let $z_n = \phi_n(i) = u_n + iv_n$, and define P to be the polygonal curve

$$P = [x, i] \cup [i, z_1] \cup [z_1, z_2] \cup \cdots .$$

At one end, P tends nontangentially to x. Near infinity, P is a simple curve tending to infinity. Moreover, by (3.4), $\operatorname{Re} z \to +\infty$ as z tends to infinity along P. Also, as Pommerenke remarks, in [**8**, Remark 1],

$$\frac{v_n}{u_n} \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.5}$$

so the argument of z tends to zero as z tends to infinity along P.

Choose a sequence of integers n so that $|\phi_n^*(x)| < 2R$, and let $\gamma_n = \phi_n(P)$. By construction, if z tends to x along P, then $\phi_n(z)$ tends to $\phi_n^*(x)$ and hence intersects the ball $B(2R)$. If z tends to infinity along P, we claim that

$$\lim_{P\ni z\to\infty} |\phi_n(z)| = +\infty. \tag{3.6}$$

In fact, find k so that $z \in [z_k, z_{k+1}]$; then by Schwarz's lemma,

$$\rho(\phi_n(z), z_{k+n}) \le \rho(z, z_k) \le \rho(z_1, z_2) < \infty,$$

so $\phi_n(z)$ tends to infinity.

Also, we claim that

$$m := \inf_{z\in P} \operatorname{Re} \sigma(z) > -\infty. \tag{3.7}$$

This holds on $[x, i]$ by our choice of x. Also, for $k = 1, 2, 3, \ldots$, $\sigma(z_k) = \sigma(\phi_k(i)) = \sigma(i) + kb$, and for $z \in [z_k, z_{k+1}]$, Schwarz's lemma implies

$$\rho(\sigma(z), \sigma(i) + kb) \le \rho(z, z_k) \le \rho(z_1, z_2) < \infty,$$

which yields (3.7) at once.

Find $s > 2R$ such that $|\sigma^*(s)|, |\sigma^*(-s)| < \infty$, which can be done since σ is a self-map of $\mathbf{H}$. Then each curve γ_n must intersect the circle $\{|z| = s\}$ at a point of the form $\phi_n(w_n)$ for some $w_n \in P$. Also, $\sup_{|z|=s} |\sigma(z)| = M < \infty$, so by (3.7),

$$M \geq |\sigma(\phi_n(w_n))| = |\sigma(w_n) + nb| \geq \operatorname{Re} \sigma(w_n) + nb \geq m + nb \longrightarrow +\infty,$$

which is a contradiction.

Thus, except for a set of linear measure zero, at all points x where the iterates $\phi_n^*(x)$ are well defined, we have $\lim_{n\to\infty} |\phi_n^*(x)| = \infty$, which proves Theorem 1.2 in the parabolic nonzero step-case.

4. Remarks about the parabolic zero-step case. Here we collect some remarks on the parabolic zero step-case, or, in Pommerenke's terminology, the *identity case*. Recall that these are analytic self-maps ϕ of the upper half-plane $\mathbf{H}$ that can be written as in (3.1) with $A = 1$, and such that the hyperbolic steps $\rho(z_n, z_{n+1})$ of the forward iteration sequence $z_n = \phi_n(i)$ tend to zero. We have already mentioned in the introduction that the fact that the hyperbolic steps tend to zero does not depend on the choice of the starting point i. Moreover, given any point $z \in \mathbf{H}$, we also have that $\rho(\phi_n(z), \phi_n(i)) \to 0$, as n tends to infinity. This also follows from Pommerenke's Theorem 1 [**8**]. In fact, with the notations of Theorem 3.3, the normalized iterates $M_n \circ \phi_n$ converge uniformly on compact subsets of $\mathbf{H}$ to i in this case. So

(4.1)
$$\rho(\phi_n(z), \phi_n(i)) = \rho(M_n \circ \phi_n(z), M_n \circ \phi_n(i)) = \rho(M_n \circ \phi_n(z), i) \longrightarrow 0.$$

It also follows from (3.1) that for any given $z \in \mathbf{H}$ the sequence of imaginary parts $\operatorname{Im} \phi_n(z)$ is strictly increasing, hence it either has a finite limit or it tends to infinity. Again, this is a property that does not depend on z, and in the parabolic nonzero step-case, both cases arise. In [**7**], we left open the problem of producing examples in the parabolic zero-step case as well. However, we now realize that this question can be easily answered.

Proposition 4.1. *If ϕ is a parabolic zero step self-map of $\mathbf{H}$ as above, then given $z \in \mathbf{H}$,*

$$\lim_{n\to\infty} \operatorname{Im} \phi_n(z) = +\infty.$$

The proof of this proposition is immediate because otherwise one would have

$$\ell_\infty := \lim_{n\to\infty} \operatorname{Im} \phi_n(i) < \infty$$

and by (4.1) $\lim_{n\to\infty} \operatorname{Im} \phi_n(z) = \ell_\infty$ as well, for any $z \in \mathbf{H}$, which contradicts the fact that $\operatorname{Im} \phi_n(z)$ increases as soon as $\operatorname{Im} z > \ell_\infty$.

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