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NOTES ON LEXIMORPHIC SPACES

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ABSTRACT. A leximorphic space is a linearly ordered topological space (LOTS) with first and last points that is isomorphic to its Cartesian product given the lexicographic ordering. In this paper, examples of both countable and uncountable leximorphic spaces are detailed. Cardinality and consistency conditions characterizing leximorphic spaces are described. Finally, this paper settles an open question concerning the connectivity of leximorphic spaces.

1. Introduction. A linearly ordered topological space (LOTS) S is a linearly ordered set with the order topology. Throughout this paper, M will be a LOTS with distinct first and last points, denoted 0 and 1, respectively.

$M \times M$ will denote the set of ordered pairs of elements of M, with the lexicographic ordering. $M \times M$ is also a LOTS with first and last points. More generally, for each natural number n, M^n is the LOTS of n-tuples of elements of M, given the lexicographic ordering.

If S and S' are two LOTS, a map $f : S \to S'$ is *order-preserving* if, for all $x, y \in S$ such that $x < y$, $f(x) < f(y)$. f is *nonorder-destroying* if for all $x, y \in S$ such that $x < y$, $f(x) < f(y)$ or $f(x) = f(y)$ ($f(x) \leq f(y)$). S is said to be *isomorphic* to S' (also called *order-isomorphic*) if there is an order-preserving, onto function f from S to S', with the function f called an *isomorphism* (or *order-isomorphism*). The inverse of such a function is also an isomorphism, so we would simply say that S and S' are isomorphic. Two LOTS are isomorphic if and only if they are homeomorphic topological spaces [**9**].

Standard notation for intervals will be used. In particular, if $x, y \in M$ with $x < y$, $[x, y] = \{m \in M \mid x \leq m \leq y\}$ is a closed interval. If $A, B \subset M$, then A *precedes* B ($A < B$) if $a < b$ for all $a \in A$ and $b \in B$.

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For each $x \in M$, define the *fiber* V_x by $\{x\} \times M$. Note that each fiber V_x is isomorphic to M.

In 1981 and 1982, Alexander and Plaut introduced the idea of leximorphic spaces:

Definition 1.1. A LOTS M is *leximorphic* if M and $M \times M$ are isomorphic.

2. Cardinality results. Before presenting examples of leximorphic spaces, it is important to identify some classes of ordered spaces which cannot be leximorphic.

Theorem 2.1. *If M is finite, then M is not leximorphic.*

Proof. Let $M = \{x_1, x_2, \dots, x_n\}$. The cardinality of M is the finite number n, and the cardinality of $M \times M$ is n^2. Since M has distinct first and last points, n is at least 2. So there is no onto function from M to $M \times M$. □

Lemma 2.2 [**4**]. *If S is a well-ordered set, then no proper initial segment of S is isomorphic to S.*

Theorem 2.3 [**1**]. *If M is well-ordered, then M is not leximorphic.*

Proof. The fiber V_0 is an initial segment of $M \times M$, and thus by Lemma 2.2, V_0 is not isomorphic to $M \times M$. As noted earlier, fibers are isomorphic to M, giving M not being isomorphic to $M \times M$. □

Definition 2.4. A set A is *order dense* in a LOTS S means that, for each pair $a, b \in S$ such that $a < b$, there exists a $c \in A$ such that $a < c < b$. Note that, if $A = S$, we simply say S is order dense.

The notion of order dense combined with countably infinite is quite restrictive, as is detailed in the following known result.

Theorem 2.5 [**9**]. *If M and M' are both countably infinite and order dense, then M and M' are isomorphic.*

Corollary 2.6. *If M is countably infinite and order dense, then every closed interval of M (as a subspace with the order topology) is isomorphic to M.*

Proof. If $[a,b]$ is a closed interval in M, then $[a,b]$ is also countably infinite and order dense with respect to itself as a subspace with the order topology. Thus, $[a,b]$ is isomorphic to M. □

A sub-interval condition implied by this corollary will be explored in more depth in the next section.

Alexander [**1**] describes an isomorphism from the set of dyadics in $[0,1] \subset \mathbf{R}$ to its product and notes that this means that all countably infinite, order dense spaces with distinct first and last points are leximorphic, since this closed interval of dyadics is countably infinite and order dense. Theorem 2.5 states that all such LOTS are isomorphic. An even simpler proof can be had, but a general constructive approach is also interesting.

Theorem 2.7 [**1**]. *If M is countably infinite and order dense, then M is leximorphic.*

Proof. First, a nonconstructive proof is given. Notice that, if M is a countably infinite, order dense, closed interval, then so too is $M \times M$. By Theorem 2.5, all such LOTS are isomorphic, and more specifically, M is isomorphic to $M \times M$.

Now let us explore a general constructive proof. For each $m \in M$, a closed interval I_m is constructed and mapped to the fiber $V_m = \{m\} \times M$. (Unfortunately, no independence from Theorem 2.5 can be claimed as Corollary 2.6 is used to guarantee the existence of these maps.) We use the fact that each V_m is isomorphic to M to build an isomorphism $f : M \to M \times M$.

Let $Q = \{q_0 = 0, q_1 = 1, q_2, q_3, \dots, q_n, \dots\}$ be any well-order of the elements of M indexed with the natural numbers. Let $a, b \in M$ such that $0 < a < b < 1$, and define $I_0 = [0, a]$ and $I_1 = [b, 1]$. Since the fibers are isomorphic to M, let $f(I_0) = V_0 = V_{q_0}$ and $f(I_1) = V_1 = V_{q_1}$ be detailed by the isomorphisms guaranteed by Corollary 2.6.

We have now built I_0 and I_1. We now inductively construct I_k for $k \geq 2$ as follows: Let q_α^k be the least element in $Q \setminus \cup_{l=0}^{k-1} I_l$ such that $q_\alpha^k < q_\alpha^l$ if and only if $q_k < q_l$ for all $l = 0, 1, \dots, k-1$. Let q_β^k be the least element in $Q \setminus \cup_{l=0}^{k-1} I_l$ such that $[q_\alpha^k, q_\beta^k] \cap I_l \neq \varnothing$ for $l = 0, 1, \dots, k-1$. Let $I_k = [q_\alpha^k, q_\beta^k]$ and define $f(I_k) = V_{q_k}$. Notice that by construction, f is order-preserving, well-defined (since M is order dense), and onto. □

Alexander [**1**] also conjectures with a picture that the subspace of rationals in the Cantor space is leximorphic. This would give an example of a leximorphic space which is not order dense. When speaking of the Cantor space, we mean the unit interval of the reals with successive middle thirds removed. The countable Cantor set and the unit interval of rationals can both be included in a large family of countably infinite LOTS that are all leximorphic. To generate this family, some machinery is needed.

Definition 2.8. M has a *dense n-partition*, $n > 0$, meaning that there exists a list of sets $\{A_0, A_1, \dots, A_n\}$ such that:

(1) $A_0 = \{0, 1\}$,

(2) $A_i \cap A_j = \varnothing$, for all $0 \leq i < j \leq n$,

(3) $\cup_{i=0}^n A_i = M$,

(4) A_i is order dense in M, for all $1 \leq i \leq n$.

Theorem 2.9. *Suppose M and M' are countably infinite with $\{A_0, \dots, A_n\}$ and $\{A'_0, \dots, A'_n\}$ respective dense n-partitions. Then there exists an isomorphism $f : M \to M'$ such that $f(A_i) = A'_i$ for all $0 \leq i \leq n$.*

Proof. For $n = 1$, this is Theorem 2.5. For $n = 2$, let $P_1 = \{p_1, p_2, \dots\}$ and $Q_1 = \{q_1, q_2, \dots\}$ be any well-ordering of the elements of A_1 and A_2, respectively. Likewise, construct well-ordered sets P_1' and Q_1' for A_1' and A_2'. To define an isomorphism from M to M', start with $f(0) = 0'$ and $f(1) = 1'$.

Let $f(p_1) = p_1'$, and define $P_2' = P_1' \setminus \{p_1'\}$. q_1 is between 0 and p_1 or p_1 and 1. If the first case holds, let ${}_1q'$ be the first element of Q_1' that lies between $0'$ and p_1'. In the latter case, let ${}_1q'$ be the first element of Q_1' that lies between p_1' and $1'$. In either case, the existence of such a point is guaranteed by the fact that A_2' is order dense in M'. Define $f(q_1) = {}_1q'$, $Q_2' = Q_1' \setminus \{{}_1q'\}$, and let $R_2 = \{0, 1, p_1, q_1\}$.

At stage k, all points $R_k = \{0,1\} \cup \{p_1, \ldots, p_{k-1}\} \cup \{q_1, \ldots, q_{k-1}\}$ have been mapped in an order-preserving manner, and we want to map p_k and q_k to appropriate points in the sets P_k' and Q_k'. R_k is finite and contains 0 and 1; thus, there exist points $x, y \in R_k$ such that $x < p_k < y$ and there is no point in R_k that is between x and y. Since A_1' is order dense, there exist points in P_k' between $f(x)$ and $f(y)$; let ${}_kp'$ be the first of these in P_k'. Define $f(p_k) = {}_kp'$ and $P_{k+1}' = P_k' \setminus \{{}_kp'\}$. Perform a similar procedure for mapping q_k to a point ${}_kq'$ using $R_k \cup \{p_k\}$ in selecting the bounding points x and y.

By construction, f is order-preserving, and by exhaustion, f is well-defined over the domain. It remains to be shown that f is onto. At any stage $N > 2$, let p_α' be the first point of P_N'. Since R_N is finite and contains 0 and 1, there exist points $x, y \in R_N$ such that $f(x) < p_\alpha' < f(y)$ and no point of R_N lies between x and y. f has been defined to be order-preserving over R_N; thus, no point of $f(R_N)$ lies between $f(x)$ and $f(y)$. A_1 is order dense in M; thus, there exist points in A_1 between x and y, and an infinite number of them are in P_1. The first of these in P_1 will map to p_α'. A similar argument applies to the first point of Q_N'. The conclusion is that, at any stage, the first points of the remaining sets in M' are guaranteed to be in the range of f. Thus, the process exhausts all points in M'.

For dense n-partitions where $n > 2$, the above process is easily extended by induction, using the fact that if A_1 and A_2 are order dense, then $A_1 \cup A_2$ is also order dense. $\square$

How does this theorem help? With it, the idea of a generalized Cantor space can be defined.

Definition 2.10. Let M be countably infinite with a dense n-partition $\{A_0, A_1, \dots, A_n\}$. Create a new space C by starting with M and, for each set A_i, $1 \le i \le n$, choose $m_i \ge 1$ and then replace each point $p \in A_i$ with points $p_1 < p_2 < \cdots < p_{m_i}$. Any space C constructed as such we will call a *countable generalized Cantor space* (CGCS) *of type* $(m_1, m_2, \dots, m_n)$. Note that the first and last points $A_0 = \{0, 1\}$ remain untouched.

Corollary 2.11. *If C and C' are both CGCS's of type $(m_1, m_2, \dots, m_n)$, then C and C' are isomorphic.*

Proof. Let C and C' be CGCS's constructed from respective spaces M and M' with respective dense n-partitions $\{A_0, \dots, A_n\}$ and $\{A'_0, \dots, A'_n\}$. Theorem 2.9 gives the existence of an isomorphism $f : M \to M'$ such that $f(A_i) = A'_i$, for all $0 \le i \le n$. It is straightforward to point-wise define an isomorphism $g : C \to C'$; consider a point $x \in C$. If $x \in A_0 = \{0, 1\}$, let $g(x) = f(x)$. For all other points, for each $x \in C$, there exists a $p \in A_i$ such that $x = p_k$; let $g(x) = f(p)_k$. The resulting isomorphism g is a composition of reversing the construction from C to M, following the isomorphism f from M to M', and then following the construction up to C', carefully matching replacement indices to maintain order. $\square$

It is worthwhile to reflect on some of the equivalence classes formed over the CGCS family. When constructing a CGCS from a space M, the order of indexing of the partition sets (other than A_0) is unimportant beyond matching the corresponding type numbers. Furthermore, nothing is gained by having multiple partition sets that have the same type number, as each partition set (other than A_0) is order dense in M. An equivalent space is constructed by having just one partition set corresponding to a given unique type number. Thus, no spaces (up to isomorphism) are lost by reducing our view to the following canonical spaces.

Definition 2.12. Let C be a CGCS. It may be assumed C is constructed from a space M with a dense n-partition such that C has type $(m_1, m_2, \dots, m_n)$ with $m_1 < m_2 < \cdots < m_n$.

This CGCS family contains several familiar spaces. If C is of type (1), then C is isomorphic to the unit interval of rationals with the usual ordering. If C is of type (2), then C is isomorphic to the endpoint subspace of the Cantor space. If C is of type $(1,2)$, then C is isomorphic to the rational subspace of the Cantor space. And finally, it is important to note that any CGCS that has a type number of 2 or greater is not order dense as a space, and if it has a type number of 3 or greater, then it will have a countably infinite number of isolated points. Many of these spaces are included in a large subfamily of countable spaces that are leximorphic.

Theorem 2.13. *If C is a CGCS of type (1) or $(1, 2, \ldots, m_n)$, then C is leximorphic.*

Proof. If C is of type (1), it is isomorphic to the underlying space M from which it was constructed, and in turn, M is the previously discussed countably infinite, order dense space (Theorem 2.7).

Suppose C is of the type $(1, 2, \ldots, m_n)$. The goal is to show that $C \times C$ is also a CGCS of type $(1, 2, \ldots, m_n)$, and thus isomorphic to all such CGCS's, including C. This can be done by finding a countably infinite space M' with a dense partition $\{A'_0, A'_1, \ldots, A'_n\}$ from which $C \times C$ can be constructed as a CGCS of type $(1, 2, \ldots, m_n)$.

Let M be a countably infinite LOTS with dense partition $\{A_0, A_1, \ldots, A_n\}$ from which C can be constructed. Start with the space $C \times M$; this is probably the most natural initial candidate for M', but there is a problem. There are gaps between points in C that create gaps between the corresponding fibers in $C \times M$. To fix this, we will combine the last points and first points of these consecutive fibers. For every point $p \in A_i$, $2 \leq i \leq n$, replace each pair of points $\{(p_k, 1), (p_{k+1}, 0)\}$, $1 \leq k \leq m_i - 1$, with a single point we will call $(p'_k, 1)$. The resulting LOTS M' is not equivalent to $M \times M$ or any other cartesian product. However, it is a space from which $C \times C$ can be constructed.

Define the following partition sets for M':

$$
\begin{aligned}
A'_0 &= \{(0,0), (1,1)\},\\
A'_1 &= \{(x,y) \in M' \mid y \in A_1\} \cup \{(0,1), (1,0)\}\\
&\quad \cup \{(p_1, 0) \mid p \in A_i,\ 1 \leq i \leq n\}\\
&\quad \cup \{(p_{m_i}, 1) \mid p \in A_i,\ 1 \leq i \leq n\},
\end{aligned}
$$

$$A_2' = \{(x,y) \in M' \mid y \in A_2\} \\ \cup \{(p_k', 1) \mid p \in A_i,\ 2 \leq i \leq n,\ 1 \leq k \leq m_i - 1\}, \text{ and}$$
$$A_i' = \{(x,y) \in M' \mid y \in A_i\},\ 2 < i \leq n.$$

All of the replaced points of $M \times \{0,1\}$ are in A_2'. All the nonreplaced points of $M \times \{0,1\}$ are in $A_0' \cup A_1'$. Otherwise, these sets are completely determined by the partition sets of M. This readily gives that $\{A_0', A_1', \dots, A_n'\}$ forms a partition of M'. M' is countable since $C \times M$ is countable. All that remains to be shown is that $C \times C$ can be constructed from M', and that the partition sets for M' are each order dense.

To see how $C \times C$ can be constructed from M', note that almost all of the points constructed from the A_i's map naturally to corresponding points in $C \times C$. The exceptions are the points $(p_k', 1)$ in A_2'. The p_k's are precisely the single points used to replace the consecutive points found in $C \times M$. This replacement is done to allow M' to be order dense. And by being in A_2', they are returned to being consecutive points in the Cantor construction.

It is easy to show that each A_i', $i \geq 1$, is order dense in M'. Let $(a,b) < (c,d)$ in M', and fix $1 \leq i \leq n$. There are three situations to consider:

(1) $a = c$. Let $e \in A_i \cap (b,d)$. Then

$$(a,b) < (a,e) < (c,d)$$

and $(e,f) \in A_i'$.

(2) $(a,c) \neq \varnothing$. Let $e \in (a,c)$ and $f \in A_i$. Then

$$(a,b) < (e,f) < (c,d)$$

and $(e,f) \in A_i'$.

(3) $(a,c) = \varnothing$ and $a \neq c$. Then $a = p_k$ and $c = p_{k+1}$ for some p_k. Thus, $b < 1$ or $d > 0$ (or else $(a,b) = (c,d)$ in M'). If $b < 1$, let $e = a$ and $f \in A_i \cap (b,1)$. If $d > 0$, let $e = c$ and $f \in A_i \cap (0,d)$. Then

$$(a,b) < (e,f) < (c,d)$$

and $(e,f) \in A_i'$.

We have shown that M' is a countably infinite LOTS with a dense partition that can be used to generate $C \times C$. Thus, $C \times C$ is a CGCS of type $(1, 2, \ldots, m_n)$ and is isomorphic to C. □

Remark 2.14. CGCS's of type $(2, \ldots, m_n)$ are *not* leximorphic. If A_1 is not type 1, then each point in $C \setminus \{0,1\}$ is in a consecutive pair. However, there are points in $(C \times C) \setminus \{(0,0),(1,1)\}$ which are not elements of consecutive pairs.

In particular, this means that the set of endpoints of the Cantor set (a CGCS of type (2)) is not a leximorphic space.

This theorem gives a variety of countably infinite subspaces of the real unit interval that are leximorphic, including spaces that aren't order dense and that have isolated points. Alas, as with the finite spaces, bigger subspaces of the unit interval are not possible. When moving from countably infinite to uncountable spaces, Alexander puts a limit to the possibilities with the following result:

Theorem 2.15 [**1**]. *If M is uncountable and separable, then M is not leximorphic.*

The short argument is that, if M is uncountable, $M \times M$ is not separable, and separability is preserved by a homeomorphism. This begs several questions, the first of which will be addressed in Section 5:

Question 1. Are there uncountable leximorphic spaces?

Question 2. Is Theorem 2.13 a characterization of all leximorphic subspaces of the unit interval?

3. Consistency and product results.

Theorem 3.1. *M is leximorphic if and only if all powers M^n of M, where n is a natural number, are isomorphic.*

Proof. Suppose M is leximorphic, and let $f : M \to M \times M$ be an isomorphism. Define $g : M^2 \to M^3 = M \times (M \times M)$ by $g(x, y) = (x, f(y))$. Then g is an isomorphism from M^2 to M^3. By induction, we can define isomorphisms between M^n and M^{n+1} for all $n \geq 1$. Thus, M is isomorphic to all its powers, and all its powers are isomorphic to each other. On the other hand, if all the powers of M are isomorphic, then in particular, M is isomorphic to M^2, and M is leximorphic. □

Definition 3.2. M is *consistent* if each closed interval in M is isomorphic to M.

Note that if M is consistent and contains more than two points, then it is infinite and order dense. Corollary 2.6 can be restated as countably infinite, order dense spaces with first and last points are consistent. The property of consistency was used strongly in the constructive proof of Theorem 2.7 and can be used to generate several nice results for products.

Lemma 3.3. *If M is consistent and M is isomorphic to M^n ($M \cong M^n$) for some natural number $n \geq 2$, then M is leximorphic.*

Proof. Let M be consistent and $M \cong M^n$ for some natural $n \geq 2$. If $n = 2$, M is leximorphic by definition, so suppose $n > 2$. Let $f : M \to M^n$ be an isomorphism.

Let $I = \{0\} \times M^{n-1} \subset M^n$. I is isomorphic to M^{n-1} and is a closed interval in M^n, so $f^{-1}(I)$ is a closed interval in M. By consistency, $f^{-1}(I) \cong M$. Thus, $M \cong M^{n-1}$. Repeating this process $n - 2$ times, we conclude that $M \cong M^2$. □

Theorem 3.4. *If M is consistent and $M^m \cong M^n$ for some distinct natural numbers m and n, then all powers of M are isomorphic.*

Proof. Without loss of generality, assume $m < n$. If $m = 1$, then M is leximorphic by Lemma 3.3. Then by Theorem 3.1, all powers of M would be isomorphic.

So assume $2 \leq m < n$, and let $f : M^m \to M^n$ be an isomorphism.

Let V be a one-dimensional fiber in M^m. That is, for some fixed x_1, $x_2, \dots, x_{m-1}$ in M, $V = \{(x_1, x_2, \dots, x_{m-1}, y) : y \in M\}$.

If $f(V)$ is not contained in a one-dimensional fiber of M^n, then it contains a two-dimensional interval of the form $R = \{y_1\} \times \{y_2\} \times \cdots \times \{y_{n-2}\} \times [a, b] \times M$. By consistency, $[a, b] \cong M$, and thus $R \cong M^2$. $f^{-1}(R)$ is a closed interval contained in V. V is isomorphic to M, so $f^{-1}(R) \cong M$ by consistency. So, in this case, M is leximorphic.

If $f(V)$ is properly contained in a one-dimensional fiber I of M^n, then $f^{-1}(I)$ contains a two-dimensional interval T of M^m containing V. T is isomorphic to M^2. $f(T)$ is a closed interval in I, so $f(T) \cong M$ by consistency. Since $M \times M \cong T \cong f(T) \cong M$, M is leximorphic.

This narrows the cases to the following: For M not to be leximorphic, f must map each one-dimensional fiber of M^m onto a one-dimensional fiber of M^n. Such an f can be used to define an isomorphism $g : M^{m-1} \to M^{n-1}$ as follows: $g(w) = z$, where $f(\{w\} \times M) = \{z\} \times M$.

In summary, if M is consistent, and $M^m \cong M^n$, then either M is leximorphic, and we're done, or $M^{m-1} \cong M^{n-1}$. If we continue this process, either we will conclude that M is leximorphic at some stage $1 \leq k \leq m-1$ or we will conclude that $M \cong M^{n-m+1}$. But if this is the case, then M is leximorphic by Lemma 3.3. □

The proof of Theorem 3.4 uses a technique first used by Alexander [**1**] to prove a special case: If M is consistent and M^2 is leximorphic, then M is leximorphic.

As demonstrated by the rational Cantor set, not all leximorphic spaces are consistent. Without consistency, there are still some product conditions which imply leximorphic.

Theorem 3.5. *If $M^m \cong M^n$ for some natural numbers m and n such that $m < n$, then $M^m \cong M^{(k+1)n-km}$ for all natural numbers $k \geq 0$.*

Proof. $M^m \cong M^n = M^m \times M^{n-m} \cong (M^m \times M^{n-m}) \times M^{n-m} = M^{2n-m}$. This process can be continued any finite number of steps k. □

Corollary 3.6. *If $M^m \cong M^n$ for some natural numbers m and n such that $n = (k+2)/(k+1)m$ for some natural number $k \geq 0$, then M^m is leximorphic.*

Proof. If $n = (k+2)/(k+1)m$ for some natural $k \geq 0$, then $(k+1)n - km = (k+2)m - km = 2m$, and $M^m \cong M^{2m}$ by Theorem 3.5. □

4. Connectivity results. Recall that an ordered space is *Dedekind complete* if each nonempty subset with an upper bound has a least upper bound. A linearly ordered topological space X is connected if and only if X is Dedekind complete and order dense [**9**].

So far, the only examples of leximorphic spaces described have been countable, and thus not connected. The following was an open question posed by Alexander [**1**]: *Can a leximorphic space be connected?*

The main result of this section is the following:

Theorem 4.1. *If M is leximorphic, then M is not Dedekind complete.*

This immediately leads to a negative answer to Alexander's question:

Corollary 4.2. *If M is leximorphic, then M is not connected.*

The proof of Theorem 4.1 makes use of the following theorem due to Plaut:

Theorem 4.3 [**7**]. *A linearly ordered topological space M is leximorphic if and only if there exists a nonorder-destroying onto function $\rho : M \to M$ such that, for each $m \in M$, $\rho^{-1}(\rho(m))$ is isomorphic to M.*

Theorem 4.3 guarantees a function (a *Plaut function*) that maps isomorphic copies of M onto points of M in the "correct" order. This is worth reflection, since it pinpoints how leximorphic spaces are "almost" consistent. With Theorem 4.3, a proof of Theorem 4.1 is now given:

Proof of Theorem 4.1. Suppose that M is a leximorphic space. Let

$$A = \{x \in M \mid \rho(x) < x \text{ or there exists a } y \in M \ (x < y \text{ and } \rho(y) < y)\},$$

where ρ is a Plaut function guaranteed by Theorem 4.3. Note that A is the set of all elements that "shift" left under ρ, as well as all those elements preceding any element which shifts left.

The set A is nonempty because it contains the closed interval mapped by ρ to the first element 0 of M. The complement $M \setminus A$ is nonempty, since $1 \notin A$.

Any point of $M \setminus A$ is an upper bound of A. It will be shown that A does not have a least upper bound. By way of contradiction, suppose that A has a least upper bound γ. If $\gamma \in A$, then γ must be the last point of A. If $\gamma \in M \setminus A$, then γ is the first point of $M \setminus A$.

Let $t \in \rho^{-1}(\gamma)$, and suppose that $t < \gamma$. Since $t < \gamma = \rho(t)$ implies that $t \in A$ and ρ does not shift t left, there exists an $s \in M$ such that $t < s$ and $\rho(s) < s$. Note that $s \in A$. Since $t < s$ and ρ is nonorder-destroying, $\rho(t) \leq \rho(s)$. Putting the inequalities together gives $\gamma = \rho(t) \leq \rho(s) < s \leq \gamma$, or $\gamma < \gamma$. This contradiction shows that no element of $\rho^{-1}(\gamma)$ precedes γ.

Now assume that $t \in \rho^{-1}(\gamma)$ and $\gamma < t$. Thus, $t \in M \setminus A$. However, $\rho(t) = \gamma < t$ implies that $t \in A$. This contradiction shows that γ does not precede any element of $\rho^{-1}(\gamma)$.

These last two results imply that $\rho^{-1}(\gamma) = \{\gamma\}$. But, by definition, ρ pulls back each element of M to an isomorphic copy of M. Thus, A cannot have a least upper bound, and M is not Dedekind complete. □

This result leads to the following question:

Question 3. Can a leximorphic space contain a nontrivial connected subset, or are all leximorphic spaces totally disconnected?

5. Uncountable leximorphic spaces. We now return to Question 1. If an uncountable leximorphic space exists, it is not well-ordered (Theorem 2.3), separable (Theorem 2.15), or connected (Corollary 4.2). In this section, examples of uncountable leximorphic spaces are constructed.

5.1. η_1**−sets.** The construction of an uncountable leximorphic space uses the following generalization of order dense:

Definition 5.1. An ordered set S is called an η_1-set if, for each pair of countable subsets A and B such that $A < B$, there exists an element $x \in S$ such that $A < x < B$.

Lemma 5.2 [**6**]. *The cardinality of any* η_1*-set is at least* $\mathfrak{c}$.

Lemma 5.3 [**6**]. *All* η_1*-sets of cardinality* $\aleph_1$ *are isomorphic.*

Remark 5.4. Note that this result is vacuous without the continuum hypothesis.

At this point, we have some properties of η_1-sets, but we have not proven that such a set even exists. Now we will construct an η_1-set of cardinality $\mathfrak{c}$ and modify it to demonstrate an example of an uncountable leximorphic space.

5.2. Construction of an η_1**-set.** Let $\mathcal{S}$ consist of all the $\{0,1\}$-valued sequences of length ω_1. Order $\mathcal{S}$ lexicographically: If $\vec{a} = \{a_\xi : \xi < \omega_1\}$ and $\vec{b} = \{b_\xi : \xi < \omega_1\}$ are elements of $\mathcal{S}$, $\vec{a} < \vec{b}$ if there exists $\sigma < \omega_1$ such that $a_\xi = b_\xi$ for all $\xi < \sigma$, and $a_\sigma = 0$ and $b_\sigma = 1$. Note that $|\mathcal{S}| = 2^{\aleph_1}$.

Now consider the following subset of $\mathcal{S}$: Let $\mathcal{Q}$ consist of the elements $\vec{a}$ of $\mathcal{S}$ such that there exists an ordinal $\sigma < \omega_1$ where $a_\sigma = 1$ and $a_\xi = 0$ for all $\xi > \sigma$. That is, the elements of $\mathcal{Q}$ have a last 1.

The elements of $\mathcal{Q}$ are the so-called "upper elements" of $\mathcal{S}$. Each element of $\mathcal{Q}$ is the immediate successor of an element of $\mathcal{S}$ with a last 0. These corresponding elements are called "lower elements."

Theorem 5.5 [**6, 8**]. $\mathcal{Q}$ *is an* η_1*-set of cardinality* $\mathfrak{c}$.

Together, Lemma 5.3 and Theorem 5.5 give us the following:

Corollary 5.6. *Each* η_1*-set of cardinality* $\mathfrak{c}$ *is isomorphic to* $\mathcal{Q}$.

5.3. An uncountable leximorphic space.

Theorem 5.7 (CH). *$\mathcal{Q} \times \mathcal{Q}$ is an η_1-set of cardinality $\mathfrak{c}$ and is thus isomorphic to $\mathcal{Q}$.*

Proof. First, note that $|\mathcal{Q} \times \mathcal{Q}| = \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. Now let A and B be countable subsets of $\mathcal{Q} \times \mathcal{Q}$ such that $A < B$. Let $A' = \{x \in \mathcal{Q} : A \cap (\{x\} \times \mathcal{Q}) \neq \varnothing\}$ and $B' = \{x \in \mathcal{Q} : B \cap (\{x\} \times \mathcal{Q}) \neq \varnothing\}$. A' and B' are both countable subsets of $\mathcal{Q}$.

If $A' \cap B' = \varnothing$, then $A' < B'$. Since $\mathcal{Q}$ is an η_1-set, there exists a $y \in \mathcal{Q}$ such that $A' < y < B'$. Then $A < (y, y) < B$ in $\mathcal{Q} \times \mathcal{Q}$.

If $A' \cap B' \neq \varnothing$, then $A' \cap B' = \{z\}$, where $z \in \mathcal{Q}$. Let $A_z = \{x \in \mathcal{Q} : (z, x) \in A\}$ and $B_z = \{x \in \mathcal{Q} : (z, x) \in B\}$. Then $A_z < B_z$ in $\mathcal{Q}$, so there exists a $y \in \mathcal{Q}$ such that $A_z < y < B_z$. Then $A < (z, y) < B$ in $\mathcal{Q} \times \mathcal{Q}$. □

Remark 5.8. Note that the proof shows that the product of any η_1-set with itself is still an η_1-set.

The preceding theorem shows that there is an example (assuming CH) of an uncountable LOTS which is isomorphic to its cross product. However, our definition of leximorphic requires that the spaces have first and last points. $\mathcal{Q}$, and η_1-sets, in general, do not have first and last points. There is a natural way to add a first and last point to $\mathcal{Q}$: Attach the points $\vec{0}$ and $\vec{1}$.

Theorem 5.9 (CH). *Let $\widetilde{\mathcal{Q}} = \mathcal{Q} \cup \{\vec{0}, \vec{1}\}$. Then $\widetilde{\mathcal{Q}}$ is an uncountable leximorphic space.*

Proof. Let $\mathcal{P} = (\widetilde{\mathcal{Q}} \times \widetilde{\mathcal{Q}}) \setminus \{(\vec{0}, \vec{0}), (\vec{1}, \vec{1})\}$. First, note that $|\widetilde{\mathcal{Q}}| = |\mathcal{Q}| = \mathfrak{c}$, so $|\mathcal{P}| = \mathfrak{c}$. We will show that $\mathcal{P}$ is an η_1-set. This proves that there is an isomorphism from $\mathcal{P}$ to $\mathcal{Q}$. Then we can map $(\vec{0}, \vec{0})$ to $\vec{0}$ and $(\vec{1}, \vec{1})$ to $\vec{1}$, and we have an isomorphism from $\widetilde{\mathcal{Q}} \times \widetilde{\mathcal{Q}}$ to $\widetilde{\mathcal{Q}}$, thus proving that $\widetilde{\mathcal{Q}}$ is leximorphic.

Let A and B be countable subsets of $\mathcal{P}$ such that $A < B$. Let $A' = \{x \in \widetilde{\mathcal{Q}} : A \cap (\{x\} \times \widetilde{\mathcal{Q}}) \neq \varnothing\}$ and $B' = \{x \in \widetilde{\mathcal{Q}} : B \cap (\{x\} \times \widetilde{\mathcal{Q}}) \neq \varnothing\}$. A' and B' are both countable subsets of $\widetilde{\mathcal{Q}}$.

If $A' \cap B' = \varnothing$, then $A' < B'$. If $A' \setminus \{\vec{0}\}$ and $B' \setminus \{\vec{1}\}$ are both nonempty, then the proof is just as in this case in Theorem 5.7.

If $A' = \{\vec{0}\}$, then B' is a countable subset of $\mathcal{Q} \cup \{\vec{1}\}$. Since $B' \setminus \{\vec{1}\}$ is a countable subset of $\mathcal{Q}$, there exists an element $y \in \mathcal{Q}$ such that $y < B'$. Then $A' < y < B'$ in $\widetilde{\mathcal{Q}}$ and $A < (y, y) < B$ in $\mathcal{P}$.

Similarly, if $B' = \{\vec{1}\}$, A' is a countable subset of $\mathcal{Q} \cup \{\vec{0}\}$, so there exists an element $y \in \mathcal{Q}$ such that $A' < y$. Then $A' < y < B'$ in $\widetilde{\mathcal{Q}}$ and $A < (y, y) < B$ in $\mathcal{P}$.

If $A' \cap B' \neq \varnothing$, then $A' \cap B' = \{z\}$, for some $z \in \widetilde{\mathcal{Q}}$. Let $A_z = \{x \in \widetilde{\mathcal{Q}} : (z, x) \in A\}$ and $B_z = \{x \in \widetilde{\mathcal{Q}} : (z, x) \in B\}$. If $A_z \neq \{\vec{0}\}$ and $B_z \neq \{\vec{1}\}$, then the proof is just as in Theorem 5.7.

If $A_z = \{\vec{0}\}$, then B_z is a countable subset of $\mathcal{Q} \cup \{\vec{1}\}$. There exists an element $y \in \mathcal{Q}$ such that $y < B_z$. Then $A_z < y < B_z$ in $\widetilde{\mathcal{Q}}$ and $A < (z, y) < B$ in $\mathcal{P}$.

If $B_z = \{\vec{1}\}$, then A_z is a countable subset of $\mathcal{Q} \cup \{\vec{0}\}$. There exists an element $y \in \mathcal{Q}$ such that $A_z < y$. Then $A_z < y < B_z$ in $\widetilde{\mathcal{Q}}$ and $A < (z, y) < B$ in $\mathcal{P}$.

Thus, $\mathcal{P}$ is an η_1-set of cardinality $\mathfrak{c}$, and $\widetilde{Q}$ is a leximorphic space. □

5.4. Larger leximorphic sets. The definitions of η_1-sets and $\mathcal{Q}$ can be generalized to larger cardinalities. Then, nearly identical arguments can be applied to show that there are uncountable leximorphic sets of cardinality larger than $\aleph_1$.

Definition 5.10. Let α be an ordinal. An ordered set S is called an η_α-set if, for each pair of subsets A and B of cardinality less than $\aleph_\alpha$ such that $A < B$, there exists an element $x \in S$ such that $A < x < B$.

Definition 5.11. (1) Let $\mathcal{S}_\alpha$ be the set of all $\{0, 1\}$-valued sequences of length ω_α.

(2) Let $\mathcal{Q}_\alpha$ be the set of all $\vec{a}$ in $\mathcal{S}_\alpha$ with a last 1.

Remark 5.12. Note that this definition agrees with our earlier definition when $\alpha = 1$. Also, η_0-sets are simply the order dense sets. $\mathcal{Q}_0$ is isomorphic to $\mathbf{Q}$.

In this section, some definitions regarding ordinal and cardinal numbers are useful:

Definition 5.13. (1) An ordinal α is called a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β.

(2) A *limit ordinal* is a nonzero ordinal which is not a successor ordinal.

(3) A cardinal number κ is *regular* if it is not the sum of fewer than κ cardinal numbers less than κ. The cardinal $\aleph_\alpha$ is regular if and only if the union of any collection of fewer than $\aleph_\alpha$ ordinals less than ω_α is less than ω_α. In particular, any cardinal of the form $\aleph_{\beta+1}$ is regular.

(4) A regular cardinal $\aleph_\alpha$ is *strongly inaccessible* if α is a nonzero limit ordinal and $2^\lambda < \aleph_\alpha$ for all $\lambda < \aleph_\alpha$.

The following theorem compiles results from Rosenstein, which give us what we need to adapt the techniques in the last section to larger cardinalities.

Theorem 5.14 [**8**]. (1) *An η_α-set has cardinality at least $\aleph_\alpha$.*

(2) *Any two η_α-sets of cardinality $\aleph_\alpha$ are isomorphic.*

(3) *If $\aleph_\alpha$ is a regular cardinal number, then $\mathcal{Q}_\alpha$ is an η_α-set.*

(4) $|Q_\alpha| = \sum\{2^{\aleph_\beta} : \beta < \alpha\}$.

(5) *If $\alpha = \gamma + 1$ is a successor ordinal, then $\mathcal{Q}_\alpha$ has cardinality $2^{\aleph_\gamma}$. Thus, if $2^{\aleph_\gamma} = \aleph_{\gamma+1}$, $\mathcal{Q}_\alpha$ is an η_α-set of cardinality $\aleph_\alpha$.*

(6) *If $\aleph_\beta < \aleph_\alpha$ implies that $2^{\aleph_\beta} < \aleph_\alpha$, then $\mathcal{Q}_\alpha$ has cardinality $\aleph_\alpha$. Thus, if $\aleph_\alpha$ is strongly inaccessible, then $\mathcal{Q}_\alpha$ is an η_α-set of cardinality $\aleph_\alpha$.*

This leads to the main result of the section. Here we assume the generalized continuum hypothesis: $2^{\aleph_\alpha} = \aleph_\alpha + 1$ for all ordinals α.

Theorem 5.15 (GCH). *Let α be an ordinal such that α is a successor ordinal or $\aleph_\alpha$ is strongly inaccessible. Then $\widetilde{\mathcal{Q}}_\alpha = \mathcal{Q}_\alpha \cup \{\vec{0}, \vec{1}\}$ is a leximorphic space of cardinality $\aleph_\alpha$.*

Proof. By the cardinality assumptions, $\mathcal{Q}_\alpha$ is an η_α-set of cardinality $\aleph_\alpha$.

Let $\mathcal{P} = (\widetilde{\mathcal{Q}}_\alpha \times \widetilde{\mathcal{Q}}_\alpha) \setminus \{(\vec{0}, \vec{0}), (\vec{1}, \vec{1})\}$. Then $|\widetilde{\mathcal{Q}}_\alpha| = |\mathcal{Q}_\alpha| = \aleph_\alpha$, so $|\mathcal{P}| = \aleph_\alpha$. A proof analogous to that of Theorem 5.9 shows that $\mathcal{P}$ is an η_α-set, thus demonstrating that $\widetilde{\mathcal{Q}}_\alpha$ is leximorphic. □

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