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GENERALIZED BI-CIRCULAR PROJECTIONS ON $\mathcal{C}(\Omega, X)$

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ABSTRACT. Let Ω be a connected compact Hausdorff space and X a Banach space for which the Strong Banach-Stone property is valid. We give a complete characterization of the generalized bi-circular projections on Banach spaces of vector valued continuous functions. We also observe that generalized bi-circular projections on $\mathcal{C}(\Omega, X)$ are bi-contractive.

1. Introduction. Let $(X, \|\cdot\|)$ be a complex Banach space, and let $P : X \to X$ be a linear projection. A basic problem in Banach space theory is to determine the structure of the projections on a given Banach space and provide characterizations of their ranges. The existence of Hermitian projections on a Banach space and its connection with the geometric properties of the underlying space was investigated by Berkson, in [**2**]. Contractive and bi-contractive projections on L_p spaces and on spaces of continuous functions, as well as circular projections in a variety of settings, are among the standard problems addressed in the literature, see for example [**3, 6, 16**]. Recently, a new class of projections, namely bi-circular projections, was proposed and has been a focus of research interest, see [**17**]. A projection is called bi-circular if $e^{i\alpha}P + e^{i\beta}(I - P)$ is an isometry, for all real numbers α and β. These projections were studied in many different settings by Stacho and Zalar, see [**18**]. Furthermore, it was shown in [**9**] that these projections are norm Hermitian. As a consequence, many results on bi-circular projections follow from previously established results on Hermitian operators on Banach spaces. Fosner, Ilisevic and Li, in [**6**], have introduced a generalization of bi-circular projections by requiring $P + \lambda(I - P)$ to be an isometry, for some modulus 1 complex number $\lambda \neq 1$. They obtained interesting characterizations of these projections in the finite-dimensional case for both real and complex vector spaces. It is of interest to characterize these projections for other Banach spaces.

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In this paper, we investigate the structure of Fosner, Ilisevic and Li generalized bi-circular projections on the space of vector valued continuous functions, $C(\Omega, X)$. The characterization in the scalar case follows as a corollary. Our results are valid for both the real or complex cases.

2. Generalized bi-circular projections on spaces of vector-valued functions. We investigate the form of the generalized bi-circular projections on $\mathcal{C}(\Omega, X)$, with X representing a Banach space and Ω a connected compact Hausdorff space. The Banach space $\mathcal{C}(\Omega, X)$ consists of all continuous functions on Ω with values in X, equipped with $\|\cdot\|_\infty$, i.e., $\|f\|_\infty = \max_{x\in\Omega}\|f(x)\|$. Our characterization relies upon the form of surjective isometries of $\mathcal{C}(\Omega, X)$. This result is well known for $\mathcal{C}(\Omega, X)$, with X having the strong Banach Stone property, see Behrend's book [**1**]. In particular, this is the case if X is smooth or strictly convex, see [**1**, Theorem 8.10, page 147].

We review the definition of generalized bi-circular projection, cf. [**6**].

Definition 2.1. The operator Q on $\mathcal{C}(\Omega, X)$ is said to be a generalized bi-circular projection if and only if $Q^2 = Q$, and there exist $\lambda \in C$, $\lambda \neq 1$, and $|\lambda| = 1$ for which $Q + \lambda(I - Q)$ is an isometry of $\mathcal{C}(\Omega, X)$.

If X satisfies the strong Banach stone property, then a surjective isometry L on $\mathcal{C}(\Omega, X)$ is of the form

$$L(f)(\omega) = u_\omega(f \circ \phi(\omega)),$$

where ϕ is a homeomorphism of Ω and u is a continuous function on Ω with values in the space of invertible isometries of X $(\mathrm{Isom}\,(X))$, i.e., for every $\omega \in \Omega$, $u(\omega) = u_\omega \in \mathrm{Isom}\,(X)$. We observe that in the scalar case a surjective isometry L on $\mathcal{C}(\Omega)$ has the simplified form $L(f)(\omega) = u(\omega)f(\phi(\omega))$, with ϕ a homeomorphism of Ω and u a modulus 1 continuous function on Ω.

The next folklore lemma will be used in the proof of the forthcoming theorem.

Lemma 2.1. *If $\lambda \neq 1$, then R is a bounded operator on X satisfying*

$$\lambda\,\mathrm{Id} - (\lambda + 1)R + R \circ R = 0$$

if and only if there exist two complementary subspaces of X, X_1 and X_2, such that $X = X_1 \oplus X_2$ and orthogonal projections P_1 and P_2 onto X_1 and X_2, respectively, so that $R = P_1 + \lambda P_2$.

Proof. If $R = P_1 + \lambda P_2$, then $R \circ R = P_1 + \lambda^2 P_2$; hence, $\lambda \,\mathrm{Id} - (\lambda + 1)R + R \circ R = 0$. Conversely, if $\lambda \,\mathrm{Id} - (\lambda + 1)R + R \circ R = 0$, then $R(\mathrm{Id} - R) = \lambda(\mathrm{Id} - R)$ and also $R(\lambda \mathrm{Id} - R) = \lambda \mathrm{Id} - R$. Let $X_1 = \{v \in X : R(v) = v\}$ and $X_2 = \{v \in X : R(v) = \lambda v\}$. We show that $X = X_1 + X_2$. Clearly $X_1 \cap X_2 = \{0\}$. We observe that

$$(\mathrm{Id} - R) \circ (\mathrm{Id} - R) = (1 - \lambda)(\mathrm{Id} - R)$$

and

$$(\lambda \,\mathrm{Id} - R) \circ (\lambda \,\mathrm{Id} - R) = (\lambda - 1)(\lambda \,\mathrm{Id} - R).$$

On the other hand, given $x \in X$, we have the representation

$$x = \left[x - \frac{1}{1-\lambda}(\mathrm{Id} - R)(x)\right] + \frac{1}{1-\lambda}(\mathrm{Id} - R)(x).$$

Previous considerations imply that $(\mathrm{Id}-R)[x-1/(1-\lambda)(\mathrm{Id}-R)(x)] = 0$ and $(\lambda \,\mathrm{Id}-R)(1/(1-\lambda))(\mathrm{Id}-R)(x) = 0$; therefore, $[x-(1/(1-\lambda))(\mathrm{Id}-R)(x)] \in X_1$ and $(1/(1-\lambda))(\mathrm{Id} - R)(x) \in X_2$. □

Remark. The lemma above also asserts that if there exists a closed subspace of X, say X_1, and a projection onto that subspace $P_1 : X \to X_1$, then $R = P_1 + \lambda(\mathrm{Id} - P_1)$ satisfies the equation $\lambda \,\mathrm{Id} - (\lambda+1)R + R \circ R = 0$. Furthermore, if R is an isometry, then for every $x \in X$ we have $\|P_1(x) + \lambda(\mathrm{Id} - P_1)(x)\| = \|x\|$. Consequently, P_1 and $\mathrm{Id} - P_1$ are generalized bi-circular projections.

Theorem 2.1. *If X has the strong Banach stone property, then Q is a generalized bi-circular projection on $\mathcal{C}(\Omega, X)$ if and only if one of the following statements holds*:

1. *There exist a nontrivial homeomorphism $\phi : \Omega \to \Omega$ with $\phi^2 = \mathrm{Id}$ and a continuous function $u : \Omega \to \mathrm{Isom}\,(X)$ with $u_\omega \circ u_{\phi(\omega)} = \mathrm{Id}$ so that*

$$Q(f)(\omega) = \frac{1}{2}\left(f(\omega) + u_\omega(f \circ \phi(\omega))\right),$$

for every $\omega \in \Omega$.

2. *There exists a generalized bi-circular projection on* X, P_ω, *so that* $Q(f)(\omega) = P_\omega(f(\omega))$, *for each* $\omega \in \Omega$.

Proof. Since $Q + \lambda(I - Q)$ is an isometry, we have that

$$[Q + \lambda(I - Q)](f)(\omega) = u_\omega((f \circ \phi)(\omega)),$$

with u_ω an invertible isometry and ϕ a homeomorphism. Therefore,

$$Q(f)(\omega) = -\frac{\lambda}{1-\lambda} f(\omega) + \frac{1}{1-\lambda} u_\omega((f \circ \phi)(\omega)).$$

Moreover, Q being a projection ($Q^2 = Q$) implies that

$$\lambda f(\omega) - (\lambda + 1) u_\omega(f \circ \phi(\omega)) + u_\omega \circ u_{\phi(\omega)}(f \circ \phi^2\,(\omega)) = 0, \tag{1}$$

for every $f \in \mathcal{C}(\Omega,\, X)$ and $\omega \in \Omega$. If there exists an $\omega \in \Omega$ so that $\omega \neq \phi(\omega)$ and $\omega \neq \phi^2(\omega)$, then there exists a Urysohn's function g on Ω so that $g(\omega) = 1$ and $g(\phi(\omega)) = g(\phi^2(\omega)) = 0$, see [**12**]. Let $f(x) = g(x)v$, for every $x \in \Omega$ and for a given nonzero vector $v \in X$. The equation (1) reduces to $\lambda = 0$, contradicting the assumptions on λ. Hence, $\phi^2 = \mathrm{Id}$. If we assume that there exists an $\omega \in \Omega$ such that $\omega \neq \phi(\omega)$, we select a Urysohn's function g such that $g(\omega) = 1$ and $g(\phi(\omega)) = 0$. Letting $f(x) = g(x) \cdot v$, equation (1) yields $\lambda v + u_\omega \circ u_{\phi(\omega)}(v) = 0$ and thus $u_\omega \circ u_{\phi(\omega)} = -\lambda\,\mathrm{Id}$. Under this constraint, (1) becomes $(\lambda + 1) u_\omega(f \circ \phi(\omega)) = 0$. Therefore, $\lambda = -1$ and $u_\omega \circ u_{\phi(\omega)} = \mathrm{Id}$, for every $\omega \in \Omega$. Therefore $Q(f)(\omega) = (1/2)(f(\omega) + u_\omega(f \circ \phi(\omega)))$, as in statement 1.

If $\phi = \mathrm{Id}$, then for every $x \in X$, we have $\lambda x - (\lambda + 1) u_\omega(x) + u_\omega \circ u_\omega(x) = 0$. Lemma 2.1 implies that $u_\omega = P_\omega + \lambda(\mathrm{Id} - P_\omega)$ and, since u_ω is an isometry, P_ω is a generalized bi-circular projection on X. This case yields the formula in statement 2, i.e., $(Q(f))(\omega) = P_\omega(f(\omega))$. The remainder of the proof follows from straightforward computations. ◻

Remark. We state the scalar version as a particular case of Theorem 2.1, with $X = \mathbf{C}$ or $\mathbf{R}$. An operator Q is a generalized bi-circular projection on $\mathcal{C}(\Omega)$ if and only if there exists a homeomorphism $\phi : \Omega \to \Omega$, with $\phi^2 = \mathrm{Id}$ and a continuous function $u : \Omega \to C$, with $|u(\omega)| = 1$ and $u(\omega) = \overline{u(\phi(\omega))}$ such that

$$Q(f)(\omega) = \frac{1}{2}\left[f(\omega) + u(\omega) f(\phi(\omega))\right].$$

It also follows that the isometry associated with a generalized bi-circular projection is an isometric reflection. However, this is not true without the connectedness condition on Ω. For example, we consider Ω to be the Stone-Cech compactification of the set of positive integers, hence $C(\Omega)$ is isometric to l^∞ (over the complex numbers). We consider the isometry $T(x_1, x_2, x_3, \dots) = (x_1, ix_2, x_3, ix_4, \dots)$, and the projection $P(x_1, x_2, x_3, \dots) = (x_1, 0, x_3, 0, \dots)$. We have that $P + i(\mathrm{Id} - P) = T$ and $T \circ T \neq \mathrm{Id}$.

The following corollaries are straightforward consequences of Theorem 2.1.

Corollary 2.1. *If Q is a generalized bi-circular projection on $\mathcal{C}(\Omega, X)$, then both Q and* $\mathrm{Id} - Q$ *have norm* 1.

Corollary 2.2. *If Ω is topologically rigid, then any generalized bi-circular projection on $\mathcal{C}(\Omega, X)$ is of the form $Q(f)(\omega) = P_\omega(f(\omega))$ where P_ω is a generalized bi-circular projection of X.*

Proof. Rigid spaces have only the trivial homeomorphism, see [**8**]. If Q is a bi-circular projection, then $Q(f)(\omega) = -(\lambda/(1-\lambda))f(\omega) + (1/(1-\lambda))u_\omega(f(\omega))$, for every $\omega \in \Omega$. Therefore, by Lemma 2.1, $u_\omega = P_\omega + \lambda(I - P_\omega)$ and P_ω is a bi-circular projection on X. □

There are conditions on the range space X which ensure the existence of nontrivial generalized bi-circular projections on X. To this end we recall the definition of L^p projections, cf. [**1**].

Definition. Let X be a Banach space and R a projection on X with the property that, for every $x \in \mathcal{C}(\Omega, X)$,

$$\|x\|^p = \|Rx\|^p + \|(I-R)x\|^p$$

for $p \in [1, \infty)$ or

$$\|x\| = \max(\|Rx\|, \|(I-R)x\|)$$

for $p = \infty$. Such an operator is called an L^p projection on X.

Remark. $L^p(\mu)$ with $1 \leq p \leq \infty$ are standard examples of spaces admitting nontrivial L^p projections.

Corollary 2.3. *If X is a Banach space which admits a nontrivial L^p projection R, then the operator Q on $C(\Omega, X)$ given by $(Qf)(\omega) = R[f(\omega)]$ is a generalized bi-circular projection.*

Proof. Given $f \in C(\Omega, X)$ and a modulus 1 complex number λ, we have

$$\|R[f(\omega)] + \lambda(f(\omega) - R[f(\omega)])\|^p = \|R[f(\omega)]\|^p + \|\lambda(\mathrm{Id} - R)(f(\omega))\|^p = \|f(\omega)\|^p.$$

Consequently, $\|Q(f) + \lambda(\mathrm{Id} - Q)(f)\| = \|f\|$. □

Remark. Corollary 2.1 asserts that generalized bi-circular projections are bi-contractive on $\mathcal{C}(\Omega)$. Our results relate to those of Friedman and Russo, see [**7**]. Furthermore, their theorem also implies that every bi-contractive projection on $\mathcal{C}(\Omega)$ is a generalized circular projection. It would be interesting to know if every bi-contractive projection on $\mathcal{C}(\Omega, X)$ is a generalized bi-circular projection and to have a characterization of the bi-contractive projections on $C(\Omega, X)$.

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