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APPROXIMATION INDUCED BY A FOURIER SERIES

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1. **Introduction.** Let X be a normed vector space and S be a subset of X such that the vector space generated by S is dense in X. Suppose that f is a mapping from S into X and $V(f; S)$ is the vector space generated by the set $\{f(x) : x \in S\}$. We ask the following question: For what S and f is $V(f; S)$ dense in X? For instance, for the Banach space $C[0, 1]$ of continuous functions on the closed unit interval $[0, 1]$ with the supremum norm, we can take $S = \{1, t, t^2, \cdots\}$. If the mapping f from S into $C[0, 1]$ is given by $f(t^k) = t^{\alpha k}$, $k = 0, 1, \cdots$, where $\alpha > 0$, then by the Stone-Weierstrass theorem we see that $V(f; S)$ is dense in $C[0, 1]$. Other sets S and other mappings f for the Banach space $C[0, 1]$ have been considered by Korevaar [3] and Luxemburg [4]. In this note, we consider the Banach spaces $L^p = L^p(T)$ and $C = C(T)$, where T will always denote the unit circle $|z| = 1$ in the complex plane, and we take $S = \{1, e^{it}, e^{-it}, e^{i2t}, e^{-i2t}, \cdots\}$. Our f will be defined by some absolutely convergent Fourier series, and in particular, some exterior conformal maps. It should be mentioned that some related but a little different problems on Hilbert spaces have been considered by Hilding [1, 2] and Pollard [5].

2. **Approximation Induced by a Fourier Series.** Let $\mathcal{G}$ denote the class of all Fourier series $\sum_{-\infty}^{\infty} a_k e^{ikt}$ with

$$\sum_{k=2}^{\infty} (|a_k| + |a_{-k}|) \leqq \big|\,|a_1| - |a_{-1}|\,\big|. \tag{1}$$

Any Fourier series of class $\mathcal{G}$ converges uniformly to a continuous function on T, and we also say that this limit function belongs to class $\mathcal{G}$. Hence,

$$f(e^{it}) = \sum_{k=-\infty}^{\infty} a_k e^{ikt} \tag{2}$$

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belongs to $\mathcal{G}$ if and only if the coefficients a_k satisfy (1). If $f \in \mathcal{G}$, we denote by $V(f)$ the vector space of all continuous functions on T generated by the functions 1, $f(e^{it})$, $f(e^{-it})$, $f(e^{i2t})$, $f(e^{-i2t})$, $\cdots$. For instance, if $f(e^{it}) = e^{it} - e^{-it}$, $V(f)$ is the space of all odd trigonometric (or sine) polynomials; and if $f(e^{it}) = e^{it} + e^{-it}$, $V(f)$ is the space of all even (or cosine) ones. We have

THEOREM 1. *Let $f \in \mathcal{G}$ be given by the Fourier series* (2) *and different from a constant and let $1 \leqq p < \infty$. Then $V(f)$ is dense in $L^p = L^p(T)$ if and only if $a_1 \pm a_{-1} \neq 0$.*

If $a_1 + a_{-1} = 0$, then $a_n = 0$ for all n with $|n| \geqq 2$, so that $f(e^{it}) = a_1(e^{it} - e^{-it})$, and hence, $V(f)$ is the space of all sine polynomials. Similarly, if $a_1 - a_{-1} = 0$, $V(f)$ is the space of all cosine polynomials. We also remark that the above theorem does *not* hold for the Banach space $C(T)$ of all continuous functions on T with the supremum norm, as can be seen from the following

EXAMPLE 1. Let

$$f(e^{it}) = e^{-it} - \sum_{k=1}^{\infty} \frac{1}{2^k} e^{i(2k-1)t}.$$

Here, we have

$$\sum_{k=2}^{\infty} (|a_k| + |a_{-k}|) = \frac{1}{2} = \big|\, |a_1| - |a_{-1}| \,\big| ,$$

and hence $f \in \mathcal{G}$. Also, $a_1 \pm a_{-1} = -(1/2) \pm 1 \neq 0$. However, $f(1) = f(-1)$ so that every function g in $V(f)$ satisfies $g(1) = g(-1)$, and hence, uniform limits of sequences chosen from $V(f)$ also satisfy this condition.

However, if we modify the proof of Theorem 1 a little, we have the following

COROLLARY. *Let f be given by the Fourier series* (2) *such that*

$$\sum_{k=2}^{\infty} (|a_k| + |a_{-k}|) \leqq \alpha \big|\, |a_1| - |a_{-1}| \,\big| ,$$

for some $\alpha < 1$. Then $V(f)$ is dense in $C(T)$ if and only if $a_1 \pm a_{-1} \neq 0$.

We now prove Theorem 1. One direction is clear. Suppose now $a_1 \pm a_{-1} \neq 0$. We wish to show that $V(f)$ is dense in L^p. By the Hahn-Banach theorem, it is sufficient to prove that every bounded linear functional on L^p which vanishes on $V(f)$ is the zero functional; but by the Riesz representation theorem every bounded linear functional on L^p can be "represented" by a function in L^q where $1/p + 1/q = 1$. Let $g \in L^q$ such that

$$\int_T h\bar{g} = 0\,, \tag{3}$$

for all $h \in V(f)$. It is sufficient to prove that $g = 0$ a.e. Let

$$b_j = \int_0^{2\pi} e^{ijt}\bar{g}(e^{it})\,dt. \tag{4}$$

By the uniqueness property of the Fourier series of $\bar{g}$, in order to prove that $g = 0$ a.e., it is sufficient to prove that $b_j = 0$ for all $j = 0, \pm 1, \pm 2, \cdots$. By (3) we already know that $b_0 = 0$ and that $\int_0^{2\pi} f(e^{ijt})\bar{g}(e^{it})\,dt = 0$ for $j = \pm 1, \pm 2, \cdots$. Since $f \in \mathcal{G}$, its Fourier series converges to f uniformly on T. Hence, we have for all $j = \pm 1, \pm 2, \cdots$,

$$\sum_{k=-\infty}^{\infty} a_k b_{jk} = 0. \tag{5}$$

Since f is different from a constant, $|a_1| + |a_{-1}| \neq 0$. Let us take, say, $0 \neq |a_{-1}| \geqq |a_1|$. Let $c_k = -a_k/a_{-1}$. Then we have

$$b_j = c_1 b_{-j} + \sum_{k=2}^{\infty} (c_k b_{-jk} + c_{-k} b_{jk}), \tag{6}$$

for all $j = \pm 1, \pm 2, \cdots$, and since $f \in \mathcal{G}$, we have

$$|c_1| + \sum_{k=2}^{\infty} (|c_k| + |c_{-k}|) \leqq 1. \tag{7}$$

Let $M = \sup\{|b_j| : j = 0, \pm 1, \pm 2, \cdots\}$, and assume, on the contrary, that $M > 0$. Since $b_j \rightarrow 0$ as $|j| \rightarrow \infty$ by the Riemann-Lebesgue theorem, we can choose the largest $|j_0|$, say $j_0 > 0$, such that $|b_{j_0}| = M$. Hence, $|b_{j_0 k}| < M$ for all k with $|k| \geqq 2$. Suppose that $c_k \neq 0$ for some k with $|k| \geqq 2$. Then from (6) for $j = j_0$, we get $0 < M \leqq |c_1|M + \sum_{k=2}^{\infty} (|c_k b_{-j_0 k}| + |c_{-k} b_{j_0 k}|) < M\{|c_1| + \sum_{k=2}^{\infty} (|c_k| + |c_{-k}|)\}$, and this is a contradiction to (7). Hence, $c_k = 0$ for $k = \pm 2, \pm 3, \cdots$. That is, (6) becomes

$$b_j = c_1 b_{-j}\,, \tag{8}$$

for all $j = \pm 1, \pm 2, \cdots$. Therefore, $b_j = c_1(c_1 b_j) = c_1{}^2 b_j$. But $a_1 \pm$

$a_{-1} \neq 0$ means that $c_1 \neq \pm 1$ or $c_1{}^2 \neq 1$. Hence, $b_j = 0$ for all j. This contradicts the fact that $M > 0$.

3. **The Exterior Conformal Mapping.** Let

$$\Phi(z) = \rho z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots, \tag{9}$$

$\rho > 0$, be a univalent meromorphic function in $|z| > 1$. We say that Φ belongs to class ϑ if

$$\sum_{k=1}^{\infty} |a_k| \leqq \rho. \tag{10}$$

(By the Area Theorem, we know that $|a_1| \leqq \rho$ and $|a_1| = \rho$ if and only if $a_k = 0$ for all $k = 2, 3, \cdots$.) If $\Phi \in \vartheta$, the series (9) converges uniformly to a continuous function Φ on $1 \leqq |z| < \infty$; and we extend Φ to $1 \leqq |z| < \infty$ by using this limit and call this extension also by Φ. Let $G = G_\Phi$ be the image of $|z| > 1$ under Φ. Then G is a simply connected domain in the extended plane and contains the point at infinity. We also denote by $\partial G = \partial G_\Phi$ the boundary of G.

If Φ as given by (9) is univalent in $|z| > 1$ and satisfies $\sum_{k=1}^{\infty} k|a_k| \leqq \rho$, we say that Φ belongs to the class $\mathcal{G}$. Hence, $\mathcal{G} \subset \vartheta$. We now give some examples of functions in $\mathcal{G}$. It is clear that if ∂G_Φ is an ellipse or a straight segment, then $\Phi \in \mathcal{G}$. We also have

Example 2. Let Φ, as given in (9), be a univalent meromorphic function in $|z| > 1$ such that the coefficients a_k, $k = 1, 2, \cdots$, are nonnegative. Then $\Phi \in \mathcal{G}$.

Indeed, the function $f(x) = \Phi(x) - a_0$ is real-valued and one-one on the interval $(1, \infty)$ and $f(x) \to \infty$ as $x \to \infty$. Hence, it is a strictly increasing function. Therefore, $0 < f'(x) = \rho - \sum_{k=1}^{\infty} k a_k x^{k-1}$ on $(1, \infty)$. Taking $x \to 1$, we see that $\Phi \in \mathcal{G}$.

For a complex number a_0 and a $\rho > 0$, we let $L_1 = \{a_0 + x : -2\rho \leqq x \leqq 2\rho\}$ and $L_2 = \{a_0 + iy : -2\rho \leqq y \leqq 2\rho\}$ be the horizontal and vertical straight line segments with center at a_0 and length 4ρ. It is clear that if $\partial G_\Phi = L_1$ then $\Phi(z) = \rho z + a_0 + \rho z^{-1}$; and if $\partial G_\Phi = L_2$ then $\Phi(z) = \rho z + a_0 - \rho z^{-1}$. For $\Phi \in \vartheta$, we again let $V(\Phi)$ be the vector space of continuous functions on T generated by 1, $\Phi(e^{it})$, $\Phi(e^{-it})$, $\Phi(e^{i2t})$, $\Phi(e^{-it})$, $\cdots$. As a consequence of Theorem 1 and its corollary, we have

Theorem 2. *Let Φ be a univalent meromorphic function in $|z| > 1$ as given by* (9) *and let G be the image of $|z| > 1$ under Φ. If $\Phi \in \vartheta$ and $1 \leqq p < \infty$ then $V(\Phi)$ is dense in L^p if and only if $\partial G \neq L_1, L_2$.*

If Φ satisfies

$$|a_1| + \alpha \sum_{k=2}^{\infty} |a_k| \leqq \rho\,, \tag{11}$$

for some $\alpha > 1$, then $V(\Phi)$ is dense in $C(T)$ if and only if $\partial G \neq L_1$, L_2.

It is clear that $V(\Phi)$ cannot be dense in $C(T)$ if $\Phi(1) = \Phi(-1)$ or $\Phi(i) = \Phi(-i)$. We have

THEOREM 3. *Let Φ be a univalent meromorphic function in $|z| > 1$, given by* (9) *and satisfying* (11) *for some $\alpha > 1$. Let G be the image of $|z| > 1$ under Φ. Then*

(a) *if $\Phi(1) = \Phi(-1)$, $\partial G = L_2$ and*

(b) *if $\Phi(i) = \Phi(-i)$, $\partial G = L_1$.*

We only prove (a) since the proof of (b) is similar. Since $\Phi(1) = \Phi(-1)$, it is clear that $\rho + a_1 + a_3 + a_5 + \cdots = 0$, so that $|\rho + a_1| \leqq \sum_{k=2}^{\infty} |a_k|$. Suppose that $\partial G \neq L_2$; then by the Area Theorem, $a_1 \neq -\rho$, so that $\sum_{k=2}^{\infty} |a_k| > 0$. Hence, by the Area Theorem, $|a_1| < \rho$. Therefore, by (11) we have $0 < \rho - |a_1| \leqq |\rho + a_1| \leqq \sum_{k=2}^{\infty} |a_k| \leqq (\rho - |a_1|)/\alpha$. This is absurd since $\alpha > 1$. Hence, $\partial G = L_2$.

We conclude this note with the following open problem. Let Φ be a univalent meromorphic function in $|z| > 1$ and let G be the image of $|z| > 1$ under Φ. Suppose that ∂G is a Jordan curve. Is $V(\Phi)$ dense in $C(T)$? Here, although Φ may not be of class ϑ, it still satisfies $\sum_{n=1}^{\infty} n|a_n|^2 \leqq \rho^2$.

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