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THE ROLE OF THE POLE IN RATIONAL APPROXIMATION

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The purpose of this paper is to present some old and some recent results that seem to indicate the current direction of growth of the theory of approximation by rational functions. An older result is in Walsh [**10**, § 8.7].

THEOREM 1. *In the z-plane, let the Jordan curve C_0 contain in its interior the Jordan curve C_1, and let the two curves bound the region B. Let the function $U(z)$ be harmonic in B, continuous on C_0 and C_1, equal to zero and unity on those respective curves. Let C_r denote generically the level locus $U(z) = r$, $0 < r < 1$, in B. Then there exist points α_{nk} $(k = 1, 2, \cdots, n)$, and points β_{nk} $(k = 1, 2, \cdots, n + 1)$ equally spaced on C_0 and C_1 respectively with regard to the conjugate of $U(z)$ in C, so that if $f(z)$ is a function analytic throughout the closed interior of C_μ, and if $r_n(z)$ denotes the rational function of degree n with poles in the α_{nk} and interpolating to $f(z)$ in the β_{nk}, then we have $(\mu' > \mu)$*

$$\limsup_{n\to\infty}\,[\max|f(z) - r_n(z)|, z \text{ on and within } C_{\mu'}]^{1/n} \leqq e^{-2\pi(\mu'-\mu)/\tau}, \tag{1}$$

where $\tau = \int_{C_r} (\partial U/\partial \nu)\, ds$, ν being the interior normal on C_r.

For approximation by polynomials in a Jordan arc we have [**11**, § 2.3].

THEOREM 2. *Let C be an analytic Jordan arc in the z-plane, let $f(z)$ be defined on C, and let $z = \phi(w)$ map C one-to-one and conformally onto the line segment S: $-1 \leqq w \leqq 1$. Then a necessary and sufficient condition that there exist polynomials $P_n(z)$ of respective degrees n satisfying*

$$|f(z) - p_n(z)| \leqq A/n^{p+\alpha},\ z \text{ on } C,\ 0 < \alpha < 1, \tag{2}$$

is that $f[\phi(\cos\theta)]$ possess on S a pth *derivative with respect to θ which satisfies a Lipschitz condition there with respect to θ.*

If C itself is S, Theorem 2 becomes the classical results in the real

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domain on polynomial approximation due to Jackson, Bernstein, and de la Vallée Poussin.

In 1964 D. J. Newman [4] published a surprising result on approximation by rational functions, in contrast to the classical theorems:

THEOREM 3. *For each n (>4) there exists a rational function $R_n(x)$ of degree n such that*

$$(3) \qquad ||x| - R_n(x)| \leqq 3e^{-\sqrt{n}}, \quad x \text{ on } [-1, +1],$$

but there exist no such rational functions such that

$$(4) \qquad ||x| - R_n(x)| \leqq e^{-9\sqrt{n}}/2, \quad x \text{ on } [-1, +1].$$

Newman's methods are based on explicit formulas involving the exponential function, and they have since been extended to include more general functions, by Szüsz, Turán, Freud, Gončar, and others.

Subsequent to Newman's paper, I proved [13] the indirect theorem that *if the function $f(z)$ is approximable on a Jordan arc C of the z-plane, to the order $n^{-\alpha}$ ($\alpha > 0$) by rational functions $Q_n(z)$ of respective degrees n whose poles have no limit on C, then $f(z)$ is also approximable on C to the order $n^{-\alpha}$ by polynomials of respective degrees n.*

Let us now turn to an examination of the Padé table, by contrast with the table of rational functions $R_{mn}(z)$ of respective types (m, n) of best approximation to a given function $f(z)$ defined on a closed bounded point set E:

$$(5) \qquad \begin{array}{l} R_{00}(z), R_{01}(z), R_{02}(z), \cdots, \\ R_{10}(z), R_{11}(z), R_{12}(z), \cdots, \\ R_{20}(z), R_{21}(z), R_{22}(z), \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots; \end{array}$$

$$R_{mn}(z) \equiv \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{c_0 z^n + c_1 z^{n-1} + \cdots + c_n}, \quad \sum_0^n |c_k| \neq 0$$

I first introduced this table in 1934 (see [9]); it is far more than an analog of the Padé table, as I showed ([12]) in 1964:

THEOREM 4. *Let the function $f(z) \equiv a_0 + a_1 z + \cdots$ be analytic at $z = 0$, and for ϵ (>0) sufficiently small and fixed (m, n) let $R_{mn}(\epsilon, z)$ denote the function of type (m, n) of best approximation to $f(x)$ in the sense of Tchebycheff on the disk δ: $|z| \leqq \epsilon$. Suppose*

$$\begin{vmatrix} a_m & a_{m-1} & \cdots a_{m-n+1} \\ a_{m+1} & a_m & \cdots a_{m-n+2} \\ \cdots & \cdots & \cdots \\ a_{m+n-1} & a_{m+n} & \cdots a_m \end{vmatrix} \neq 0;$$

then as ϵ approaches zero $R_{mn}(\epsilon, z)$ approaches the Padé function $P_{mn}(z)$ for $f(z)$ in 0 uniformly on any closed bounded set containing no pole of $P_{mn}(z)$.

There are other analogies between the theory of Padé approximants and the functions of Table (5). Perron [5, p. 466] has exhibited a function such that the *second row* of the Padé table consists of rational functions whose poles are everywhere dense in the circle of convergence of $f(z)$. It has more recently been shown ([8], [1]) that there exists an entire function such that the rational functions $R_{nn}(z)$ of best approximation on $|z| \leqq 1$ (that is, the diagonal functions in (5)) have poles everywhere dense in $|z| > 1$.

In somewhat the same vein, Rice [6] has computed numerically some functions $R_{mn}(z)$ of Table (5) for the given functions $\Gamma(z)$ and Erfc(z). Especially in the case of $\Gamma(z)$ the early $R_{mn}(z)$ possess poles not near the singularities of the given function. But the later $R_{mn}(z)$ show good agreement with the poles of the approximated function.

By way of contrast, Saff [7] has shown that the poles of the rational functions of Table (5) for the exponential function and the region $|z| \leqq \rho$ with Tchebycheff measure of approximation have no finite limit points. Degree of approximation plays here a large role. Saff's result is the first break-through in describing the global behavior of the totality of poles of functions of a Table (5).

Gončar [2] has shown the surprising fact that if the real function $f(z)$ is continuous on $J: [-1, 1]$, then the rational function of a given degree of best approximation to $f(x)$ on J need not be real; compare also Lungu [3].

I have presented in this short paper a few results to indicate some directions of the recent theoretical research on approximation by rational functions. Others are presenting results on applications to theoretical physics, by Padé functions. For the immediate future, the big riddle for all of us is: "Where are the poles of the approximating functions?"

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