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THE GROUPS OF REAL GENUS $\rho \leq 16$

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ABSTRACT. Let G be a finite group. The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts. Here we classify the groups of real genus ρ, for all values of ρ such that $9 \leq \rho \leq 16$, $\rho \neq 12$, and also for the remaining even values such that $\rho \leq 24$. We also determine the real genus of each group G with $32 < |G| < 48$. In addition, we obtain some general results about the partial presentations that groups acting on bordered surfaces must have.

1. Introduction. Let G be a finite group. The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts. There is a growing body of results about the real genus parameter. The groups with real genus $\rho \leq 8$ have been classified [**7, 11, 12, 15**], and Mockiewicz has shown that there are no groups of real genus 12 [**21**]. Genus formulas have been obtained for several families of groups, including the family of abelian groups [**20**]; for other results, see, for example, [**11, 13, 15**]. In addition, there are results about the real genus of groups of odd order in [**16**], and 2-groups were considered in [**17**].

Here we classify the groups of real genus ρ for all values of ρ such that $9 \leq \rho \leq 16$, $\rho \neq 12$, and also for the remaining even values such that $\rho \leq 24$. We also determine the real genus of each group G with $32 < |G| < 48$; the genus of each group with $|G| \leq 32$ has already been calculated [**11, 14, 15, 17**]. As part of this effort, we obtain some general results about the partial presentations that groups acting on bordered surfaces must have. The software package MAGMA [**1**] was employed to perform most of the calculations, and the MAGMA library of small groups was essential to the classification. We gratefully acknowledge our debt to this system.

We use the standard representation of a group G as a quotient of a non-Euclidean crystallographic (NEC) group Γ by a bordered surface

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group K; then G acts on the Klein surface U/K, where U is the open upper half-plane.

2. Preliminaries. We shall assume that all surfaces are compact. A bordered surface X can carry a dianalytic structure and be considered a Klein surface or a nonsingular real algebraic curve (with real points). Thus, the surface X has an algebraic genus g, and the real genus of a group is defined in terms of the algebraic genus.

Associated with the NEC group Γ is its *signature*, which has the form

$$(p; \pm; [\lambda_1, \dots, \lambda_r];\ \{(\nu_{11}, \dots, \nu_{1s_1}), \dots, (\nu_{k1}, \dots, \nu_{ks_k})\}). \tag{1}$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and nonorientable if the minus sign is used. The integers $\lambda_1, \dots, \lambda_r$, called the *periods*, are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The integers $\nu_{i1}, \dots, \nu_{is_i}$, called the *link periods*, are the ramification indices in fibers above points on the ith boundary component of X.

Associated with the signature (1) is a presentation for the NEC group Γ, although the form of the presentation depends upon whether the plus or minus sign is present. These presentations are in the monograph [**2**], for instance.

Let Γ be an NEC group with signature (1), and assume $k \geq 1$ so that the quotient space U/Γ is a bordered surface. The non-Euclidean area $\mu(\Gamma)$ of a fundamental region for Γ is given by [**22**, page 235]:

$$\mu(\Gamma)/2\pi = \gamma - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{\lambda_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{\nu_{ij}}\right),$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma). \tag{2}$$

An NEC group K is called a *bordered surface group* if the quotient map from U to the quotient space U/K is unramified and, further, U/K

has a nonempty boundary. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$. Then X can represented as U/K, where K is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Klein surface X. Then there are an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$. The group $G \cong \Gamma/K$, so that from (2) the algebraic genus g of the bordered surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/2\pi.$$

3. Group actions. Let the finite group G act on the bordered surface X of genus $g \geq 2$. Then represent $X = U/K$, where K is a bordered surface group and obtain an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = \text{kernel}\,\phi$. It is basic that each period and each link period of Γ divides $|G|$.

The following result requires only a careful consideration of the possible signatures for an NEC group Γ with $\mu(\Gamma)/2\pi < 1/2$ such that, further, there is a bordered surface group $K \subset \Gamma$ (so that the quotient group Γ/K acts on a bordered surface). A complete list of signatures with $\mu(\Gamma)/2\pi < 1$ is in [**17**, Theorem 2]; the labels in the following correspond to the labels there.

Lemma 1. *Let G be a group with $\rho(G) \geq 2$. Either $\rho(G) \geq 1+|G|/2$ or G is a quotient of an* NEC *group Γ by a bordered surface group, where the signature of Γ is in the following list.*

A1) $(0; +; [\,]; \{(2, 2, \ell, m)\})$, where $\ell \leq m$ and $m \geq 3$.

A2) $(0; +; [\,]; \{(2, 2, \ell, m, n)\})$, where $1/\ell + 1/m + 1/n > 1$.

B1) $(0; +; [\lambda]; \{(2, 2, m)\})$, where either $\lambda = 3$ and $m = 2$ or $\lambda = 2$.

B2) $(0; +; [\lambda]; \{(2, 2)\})$, where $\lambda \geq 3$.

C1) $(0; +; [\kappa, \lambda]; \{(\,)\})$, where either $\kappa = 2$ and $\lambda \geq 3$ or $\kappa = 3$ and $3 \leq \lambda \leq 5$.

E1) $(0; +; [\,]; \{(\,), (\ell)\})$.

The arithmetic restriction in signature A2) says that some of the link periods ℓ, m, n must be small. At least one of the three must be 2, and if only one is 2, then a second is 3 and the third is 3, 4 or 5.

Associated with each signature is a canonical presentation for the NEC group Γ. The canonical presentation for Γ typically involves redundant generators, and some simplification is possible. Further, the rank of G provides a minimum requirement on the number of generators in the simplified presentation for Γ; here see [**17**, Section 3].

It follows from Lemma 1 that a group G with $\rho(G) < 1+|G|/2$ must have a partial presentation of one of five types. These are determined by considering the simplified canonical presentation for each NEC group with signature in the list of the lemma. The goal here was to establish results in the spirit of [**8**, subsection 6.3.3], where partial presentations are given for all groups with symmetric genus one; similar results for 2-groups are in [**17**]. We indicate the general procedure in the proof of the following. In all partial presentations, we mean that the stated relations are "fulfilled," that is, $X^\lambda = 1$ means that the generator X has order λ and not merely a divisor of λ.

Theorem 1. *Let G be a group with $\rho(G) \geq 2$. Then $\rho(G) < 1+|G|/2$ if and only if G has a partial presentation* (*with the relations fulfilled*) *of one of the following types.*

(q) $T^2 = U^2 = V^2 = (TU)^\ell = (TV)^m = 1$, where $\ell \leq m$ and $m \geq 3$.

(p) $T^2 = J^2 = V^2 = W^2 = (JV)^\ell = (VW)^m = (WT)^n = 1$, where $1/\ell + 1/m + 1/n > 1$.

(b) $X^\lambda = B^2 = D^2 = (BD)^m = 1$, where either $\lambda = 3$ and $m = 2$ or $\lambda = 2$.

(c) $X^\kappa = Y^\lambda = 1$, where either $\kappa = 2$ and $\lambda \geq 3$ or $\kappa = 3$ and $3 \leq \lambda \leq 5$.

(e) $D^2 = [D, F]^\ell = 1$.

Proof. First suppose that the group G has one of the partial presentations. Then it is necessary to construct a homomorphism from the appropriate NEC group onto G such that the kernel is a bordered surface group. This is not hard. The partial presentation (q) comes from the extended quadrilateral group with signature A1); here see [**2**, page 99] for a complete proof. The partial presentation (p) arises from the extended pentagonal group with signature A2), which has presentation

$$(3)\qquad \begin{aligned} t^2 &= u^2 = j^2 = v^2 = w^2 = (tu)^2 = (uj)^2 \\ &= (jv)^\ell = (vw)^m = (wt)^n = 1. \end{aligned}$$

Constructing a homomorphism onto a group with presentation (p) is easy; just leave u in the kernel. Next, the partial presentation (c) comes from the signature C1; here see [**11**, Theorem 1]. If G has partial presentation (b) or (e), then G is a quotient of an NEC group with signature B1) or E1) by [**17**, Lemma 3] or [**17**, Lemma 2], respectively.

Establishing the converse is harder. Let G be a group with $\rho(G) \geq 2$. Assume $\rho(G) < 1 + |G|/2$. Then G is a quotient of an NEC group Γ by a bordered surface group K, where Γ has one of the six signatures listed in Lemma 1. If Γ has signature A1), B1) or E1), then G has partial presentation (q), (b) or (e) by [**2**, page 99], [**17**, Lemma 3] or [**17**, Lemma 2], respectively.

If Γ has either signature B2 or C1, then it is not hard to see that G is generated by two elements, one of which is an involution, and thus G has partial presentation (c).

Finally, assume Γ has signature A2) and presentation (3); this is the hardest case. The bordered surface group K contains at least one generating reflection and no analytic elements of finite order. As a consequence, it is easy to see that no more than two generating reflections can be in K, but both one and two are possible. First suppose that all the link periods are 2, that is, in (3) $\ell = m = n = 2$. In this case all the generating reflections are essentially the same, and G has partial presentation (b) if there are two generating reflections in K and partial presentation (p) (with all parameters 2) if there is one.

Next suppose that exactly four of the link periods are 2. We may assume that in (3) $\ell = m = 2$ and $n > 2$. If there are two generating reflections in K, then they must be u and v and then G has partial presentation (b). If only one generator is in K, then G has partial presentation (p) with parameters 2, 2, and n (possibly reordered 2, n, 2).

Next suppose that exactly three of the link periods are 2. One possibility is that these three are consecutive, so that we may assume that in (3) $\ell = 2$, $m > 2$ and $n > 2$. In this case there can be only one generating reflection in K, either j or u. It is not hard to see that

G has partial presentation (p) with parameters 2, m, n if $j \in K$ and parameters 2, n, m if $u \in K$.

The other possibility is that $\ell > 2$, $m = 2$ and $n > 2$. In this case u must be in K and G has partial presentation (p) with parameters ℓ, 2, n.

Finally suppose that exactly two of the link periods are 2, that is, in (3) we have $\ell > 2$, $m > 2$ and $n > 2$. Then $u \in K$ and G has partial presentation (p) with parameters ℓ, m, n.

Further, the real genus $\rho(G)$ is obtained by representing G as a quotient of the "natural" NEC group, that is, the one with signature A1), A2), B1), C1) or E1) giving the minimum non-Euclidean area (where ties are possible, of course). This is important, but also quite clear, since there are only a finite number of possibilities for the order of an element in a finite group. Thus, Theorem 1 yields a computational procedure for determining the real genus of a group G with $2 \leq \rho(G) < 1 + |G|/2$. Routines were programmed in MAGMA to check whether the group G has partial presentations of the types in the theorem. These routines are important in the classification of the groups of real genus $\rho \leq 16$.

If a group G has a partial presentation (with the relations fulfilled) of one of these types, we will say that G is in the *class* of that type.

Refined versions of Theorem 1 can be obtained, of course, by reducing the allowed area and thus making the group action "larger." Of special interest is the case in Lemma 1 in which the area restriction is reduced to $\mu(\Gamma)/2\pi < 1/6$. Then, in Theorem 1, $\rho(G) < 1 + |G|/6$ or, alternately, $|G| > 6[\rho(G) - 1]$. But this is quite limiting.

Theorem 2. *Let G be a group with real genus $\rho(G) \geq 2$. Then $\rho(G) < 1 + |G|/6$ if and only if G has a partial presentation* (*with the relations fulfilled*) *of one of the following types.*

(*q*) $T^2 = U^2 = V^2 = (TU)^\ell = (TV)^m = 1$, *where* $\ell = 2$ *and* $3 \leq m \leq 5$.

We will call a group of one of these three types a *large* group of automorphisms (of the surface on which it acts). We will consider these groups in Section 4.

Another useful idea is to consider whether or not the group G is generated by involutions. Obviously, if the NEC group Γ is generated by involutions, then G must be as well. Hence, if $\rho(G) < 1+|G|/2$ and G is not generated by involutions (and this applies to many groups), then G cannot be in class (p), class (q) or class (b) with $\lambda = 2$.

This idea has particular relevance for the family of direct product groups $Z_k \times D_n$, and we briefly consider the case in which both k and n are odd. It is not hard to establish the genus formula for these groups, and we take this opportunity to correct an old mistake in [**14**]. For $n \geq 3$, let the dihedral group D_n have generators A, B and defining relations

$$A^2 = B^n = (AB)^2 = 1.$$

We will refer to powers of B as "rotations" and the other elements AB^i as "reflections" (thinking of the action of the dihedral group on a regular n-sided polygon).

Now let W be a generator for Z_k, and construct the direct product $Z_k \times D_n$, where both k and n are odd.

First let $X = (1, BA)$ and $Y = (W, A)$. Then X and Y generate G, with $o(X) = 2$, $o(Y) = 2k$ and $o(XY) = lcm(k, n)$. Thus, G is a 2-generator group in class (c) with $\kappa = 2$ and $\lambda = 2k$ and also in class (c) with $\kappa = 2$ and $\lambda = lcm(k, n)$. If Γ is the related NEC group in either case, then $\mu(\Gamma)/2\pi = 1/2 - 1/\lambda$.

Next the commutator $[X, Y] = (1, B^2)$, an element of order n in G, since n is odd. Thus $[X, Y]^n = 1$ and G is also in class (e). Here $\mu(\Gamma)/2\pi = 1/2 - 1/2n$, where Γ is the related NEC group.

For example, $G = Z_5 \times D_3$ is in class (c) with $\kappa = 2$ and $\lambda = 10$ and also with $\kappa = 2$ and $\lambda = 15$. Further, G is in class (e) with $\ell = 3$, and this produces the genus action on a surface of genus 11. In general, we have the following.

Theorem 3. *Let $G = Z_k \times D_n$, where both k and n are odd, $k \geq 3$, $n \geq 3$. Then*

$$\rho(G) = \begin{cases} 1 + k(n-1) & \text{if } k > n \\ 1 + k(k-2) & \text{if } k = n \\ 1 + n(k-1) & \text{if } k < n. \end{cases}$$

Proof. Write $\ell = lcm(k,n)$. As we have seen, G has three actions, two as a class (c) group and one as a class (e) group. The minimum genus among the three related surfaces depends on the relative sizes of k and n. It is simple to check the three possibilities, and this gives an upper bound for the genus action of G. In particular, we have $\rho(G) < 1 + |G|/2$ in any case.

Now let G be a quotient of an NEC group Γ by a bordered surface group K. Since G is not generated by involutions, by Theorem 1, G is in class (b) with $\lambda = 3$ and $m = 2$, class (c) or class (e).

Suppose first that G is in class (b) with $\lambda = o(X) = 3$ and $m = 2$. Since Z_k is a quotient of G, this forces $k = 3$ (and $k \leq n$). Then, if Γ is the corresponding NEC group with signature B1), then $\mu(\Gamma)/2\pi = 5/12$, which is greater than $1/3 = 1/2 - 1/2k$, the non-Euclidean area in one of the class (c) actions giving the upper bound.

Suppose next that G is in class (c) with generators X and Y, with $o(X) \leq o(Y)$. First assume $o(X) = 2$. Since k is odd, $X = (1, R)$ for some involution R in the dihedral group. Write $Y = (V, S)$. Then V must generate Z_k so that V has order k. Also R and S generate the full dihedral group D_n. Hence, either R and S are both reflections with RS a rotation of order n or R is a reflection and S is a rotation of order n. If R and S are both reflections, then $o(Y) = 2k$. If R is a reflection and S is a rotation of order n, then $o(Y) = \ell = lcm(k,n)$. But these are the two actions already considered in the upper bound for $\rho(G)$.

Suppose now that G is in class (c) with $o(X) = 3$ and $3 \leq o(Y) \leq 5$. Since k is odd, there is a natural homomorphism $\alpha : G \to Z_{2k}$ of G onto Z_{2k}, and $\alpha(X)$ and $\alpha(Y)$ generate Z_{2k}. But Z_{2k} cannot be generated by two elements of odd order. Thus, $o(Y)$ must be even, and then here $o(Y) = 4$ and $k = 3$. Then, if Γ is the corresponding NEC group with signature C, $\mu(\Gamma)/2\pi = 5/12 > 1/3$.

Suppose finally that G is in class (e) with generators D and F. Then, as above, write the involution $D = (1, R)$ for some refection R in the dihedral group and write $F = (V, S)$. There are the same two possibilities to consider. But in either case the commutator $[D, F]$ has the form $(1, T)$ where T is a rotation of order n. Hence, the only class (e) action is the one already considered. Thus, the real genus is given by the minimum of the three actions, and the genus formula holds.

In connection with Theorem 3, see the recent paper by Gordejuela and Martinez [**5**].

Next, we obtain a companion result to Theorem 1 by relaxing the area restriction in Lemma 1 (allowing more signatures) but adding the restriction that the group G is not generated by involutions. In the following we consider signatures with $\mu(\Gamma)/2\pi < 2/3$ but with Γ not generated by involutions. As we shall see, Theorem 4, together with Theorem 1, suffices to handle the groups of orders 36 and 40, with three exceptions.

Lemma 2. *Let G be a group with $\rho(G) \geq 2$. Suppose G is not generated by involutions. Either $\rho(G) \geq 1 + 2|G|/3$ or G is a quotient of an* NEC *group Γ by a bordered surface group, where the signature of Γ is in the following list.*

B1) $(0; +; [\lambda]; \{(2, 2, m)\})$, *where* $\lambda \geq 3$ *and* $1/\lambda + 1/2m > 1/3$.

B2) $(0; +; [\lambda]; \{(2, 2)\})$, *where* $\lambda \geq 3$.

C1) $(0; +; [\kappa, \lambda]; \{(\ )\})$, *where* $\kappa \leq \lambda$, $\lambda \geq 3$, *and* $1/\kappa + 1/\lambda > 1/3$.

E1) $(0; +; [\ ]; \{(\ ), (\ell)\})$.

E2) $(0; +; [\ ]; \{(\ ), (2, \ell)\})$.

F1) $(0; +; [\lambda]; \{(\ ), (\ )\}$, *where* $\lambda = 2$.

G1) $(1; -; [\ ]; \{(2, 2)\})$.

H) $(1; -; [\lambda]; \{(\ )\}$, *where* $\lambda = 2$.

It follows in this case that the group G with $\rho(G) < 1 + 2|G|/3$ must have one of four partial presentations, one of which does not appear in Theorem 1. We only outline the general approach, but see the proof of the similar result in [**17**, Theorem 5], where more details are given.

Theorem 4. *Let G be a group with $\rho(G) \geq 2$. Suppose G is not generated by involutions. Then $\rho(G) < 1 + 2|G|/3$ if and only if G has a partial presentation* (*with the relations fulfilled*) *of one of the following types.*

(b) $X^\lambda = B^2 = D^2 = (BD)^m = 1$, *where* $\lambda \geq 3$ *and* $1/\lambda + 1/2m > 1/3$.

(c) $X^\kappa = Y^\lambda = 1$, *where* $\kappa \leq \lambda$, $\lambda \geq 3$, *and* $1/\kappa + 1/\lambda > 1/3$.

(e) $D^2 = [D, F]^\ell = 1$.

(f) $X^\lambda = D^2 = 1, FD = DF$, *where* $\lambda = 2$.

Proof. The first four signatures in Lemma 2 have already been considered; these signatures produce the first three partial presentations. It is easy to construct a homomorphism from an NEC group with signature F1) onto a group with partial presentation (f) such that the kernel is a bordered surface group.

Now assume the group G is not generated by involutions and $\rho(G) < 1 + 2|G|/3$. By Lemma 2, G is a quotient of an NEC group Γ by a bordered surface group K, where Γ has one of the eight signatures. If Γ has signature B1), B2), C1) or E1), then, as in Theorem 1, G has partial presentation (b), (c) or (e).

Let the group G be a quotient of an NEC group Γ by a bordered surface group K. First, assume that Γ has signature F1). Then there are two reflections in the simplified presentation for Γ. If only one generating reflection is in K, then G is in class (f). If both reflections are in K, then G is class (c).

If Γ has either signature G1) or signature H), then it is not hard to see that, again, G has class (c).

Now assume Γ has signature E2). Then, after eliminating redundant generators, Γ generates a reflection c and associated connecting generator e together with reflections b and d and connecting generator f with presentation

$$c^2 = b^2 = d^2 = (bd)^2 = (d \cdot fbf^{-1})^\ell = 1, cf = fc.$$

First assume $\ell > 2$. Then c must be in K, and G is in class (b), where $m = 2$ and $\lambda = \ell$.

The case with $\ell = 2$ is harder, since now one of b and d (but not both) can be in K as well as c. If only c is in K, then, again, G has partial presentation (b), where $m = 2$ and $\lambda = \ell$. If c and one of the others are in K, then G has partial presentation (c), where $\kappa = 2$ and λ is the order of the image of f in G. If c is not in K, then one of b and d must be, and in this case G has partial presentation (f), where $\lambda = 2$.

4. Large groups. The large groups of automorphisms are especially interesting. There are three types of these groups, all in class (q). If G is a large group acting on a bordered surface X of genus $g \geq 2$, then $|G|$ is either $12(g-1)$, $8(g-1)$ or $20(g-1)/3$.

The groups of largest possible order are called *M^*-groups* [**9**]. A surface X on which an M^*-group acts is said to have *maximal symmetry* [**6**]. These groups have received the most attention; here see [**6**].

For a particular value of the integer $g \geq 2$, Theorem 2 gives a procedure for determining the large groups of genus g. This procedure was implemented using the MAGMA library of small groups. First each group of order $12(g-1)$ is checked to see whether it is in class (q) with $\ell = 2$ and $m = 3$. If so, then the group is in general an M^*-group (although the smallest M^*-group D_6 has genus zero). Then the groups of order $8(g-1)$ are checked for groups in class (q) with $\ell = 2$ and $m = 4$. Each such group has real genus g unless it acts on a surface of lower genus. This can happen. For example, $Z_2 \times S_4$ acts on a surface of genus 7, but this M^*-group has real genus 5. Finally, the groups of order $20(g-1)/3$ are checked (if 3 divides $g-1$).

In the MAGMA library the groups are defined as power commutator groups. However, we regard a group as "identified" only if it is presented in terms of known (preferably familiar) groups or, alternately, there is complete presentation of the group. This identification can present difficulties, of course, especially the first type of recognition. However, there is a connection between large groups and groups of regular maps that will be useful here.

There is an important correspondence between bordered Klein surfaces with maximal symmetry and reflexible regular maps [**6**, Section 6] (note that "regular" in [**6**] is used in a strong sense and means, in more usual terminology, that the regular map is also "reflexible".) A bordered surface X of genus $g \geq 2$ with maximal symmetry corresponds to a reflexible regular M of type $\{3, q\}$ on a surface with topological genus p, where the number k of vertices of M is equal to the number of boundary components of X. Further, the automorphism group G of X is isomorphic to the full group of the map M, and $|G| = 2qk$.

There is a similar correspondence between bordered Klein surfaces of genus $g \geq 2$ with $8(g-1)$ automorphisms and reflexible regular maps of type $\{4, q\}$. This was pointed out in [**6**, Section 6]. Also, it is not hard to see that the same type of correspondence exists between bordered Klein surfaces of genus $g \geq 2$ with $20(g-1)/3$ automorphisms and reflexible regular maps of type $\{5, q\}$.

There is an important classification of the regular maps of small genus [**3**]. This classification includes all reflexible regular maps on orientable surfaces of genus 2 to 15 and all regular maps on nonorientable surfaces of topological genus 4 to 30. In [**3**] the maps are labeled with a symbol of the form $Rp.i$ or $Np.i$ for the ith map of (topological) genus p in the respective list. A presentation is given for each map group. This provides a method of identifying the large groups of automorphisms here, and we will write $G(M)$ to indicate the full group of the map M.

5. The groups of order 36. Here we give the real genus of each group of order 36. There are 14 of these groups; four are abelian. Of course, the real genus of some of these groups has already been determined. In particular, the groups of genus 0 and 1 are well known.

Let G be a group of order 36. We indicate the overall procedure used to find $\rho(G)$. The first easy check is whether $\rho(G)$ is 0 or 1. If $\rho(G) \geq 2$, then Theorem 1 and Theorem 4 are applied. Here MAGMA was used to check whether G has one of the partial presentations in the Theorems. If it does, then the real genus of G is given by the NEC group with minimal non-Euclidean area (ties are possible, of course). This procedure only leaves one of the 14 groups to consider.

Table 1 gives the real genus of each group of order 36 and, if relevant, its class (one class that gives the genus action). We also give a reference to a previous result about the specific group where possible. The group G_{18} is the non-Abelian group of order 18 that is not D_9 and not $Z_3 \times D_3$. The notation $Gp(i)$ is from [**18**, page 123], where presentations for these three groups are given. The general reference for the abelian groups is [**20**].

The exceptional group not handled by the procedure is $Gp(3)$, which has presentation

$$A^3 = B^3 = C^4 = [A, B] = 1, \quad C^{-1}AC = A^{-1}, \quad C^{-1}BC = B^{-1}.$$

The group $Gp(3)$ is a semi-direct product $(Z_3)^2 \times_\phi Z_4$. This group G has rank 3 and a single involution that is in the Frattini subgroup. Further, G is not generated by elements of order 3. Then the general lower bound [**13**, Theorem 1] gives $\rho(G) \geq 40$. But it is easy to see that G is a quotient of an NEC group Λ with signature $(0; +; [3,3,4]; \{(\ )\})$. Since $\mu(\Lambda)/2\pi = 13/12$, we have $\rho(G) = 40$. More details about this group are in [**18**, Section 7].

TABLE 1. Groups of order 36.

Group	ρ	Class	Reference	Group	ρ	Class	Reference
Z_{36}	0		[**11**, Thm. 3]	$Z_3 \times A_4$	13	c	
D_{18}	0		[**11**, Thm. 3]	$Z_6 \times D_3$	13	c	[**5**, Thm. 1]
$Z_2 \times Z_{18}$	1		[**11**, Thm. 4]	$Z_3 \times DC_3$	16	c	
$S_3 \times S_3$	4	q	[**12**, Thm. 1]	DC_9	19	c	[**11**, Thm. 9]
$Z_2 \times G_{18}$	10	q		$Z_3 \times Z_{12}$	22	c	
$Gp(1)$	10	c		$Z_6 \times Z_6$	25	c	
$Gp(2)$	10	e		$Gp(3)$	40		

TABLE 2. Groups of order 40.

Group	ρ	cl	Reference	Group	ρ	cl	Reference
Z_{40}	0		[**11**, Thm. 3]	G_{40}	11	c	
D_{20}	0		[**11**, Thm. 3]	DC_{10}	21	c	[**11**, Thm. 9]
$Z_2 \times Z_{20}$	1		[**11**, Thm. 4]	$Z_2 \times DC_5$	21	c	[**15**, Thm. 8]
$Z_2 \times D_{10}$	1		[**11**, Thm. 4]	$(Z_2)^2 \times Z_{10}$	21	f	[**13**, Thm. 4]
$Z_4 \times D_5$	11	c	[**5**, Thm. 1]	$Z_5 \times_\phi Z_8$	26	c	
$Z_5 \times D_4$	11	e	[**5**, Thm. 1]	$Z_5 \times_\theta Z_8$	28	c	
$Z_2 \times \langle 5,4,2\rangle$	11	c		$Z_5 \times Q$	29	c	

6. The groups of order 40. There are 14 groups of order 40; 3 are abelian. To find the real genus of these groups, the same general procedure was used. Here it is helpful to strengthen Theorem 4 by considering all signatures with $\mu(\Gamma)/2\pi < 3/4$; this is easy since the group order is not divisible by 3. Table 2 gives the real genus of the groups of this order. The group G_{40} has presentation

$$A^2 = B^4 = C^5 = (A,B)^2 = [A,B] = 1, \quad B^{-1}CB = C^{-1}.$$

Two groups are semi-direct products of Z_5 by Z_8. In the group of genus 26, the kernel of the action ϕ is Z_2; in the other group, the kernel is Z_4.

TABLE 3. The remaining groups.

Group	ρ	cl	Reference	Group	ρ	cl	Reference
$\langle 13, 3, 3 \rangle$	14	c	[**14**, Thm. 4]	$Z_7 \times D_3$	15	e	Theorem 3
$\langle 7, 6, 5 \rangle$	8	c	[**14**, Thm. 2]	$Z_2 \times \langle 7, 3, 2 \rangle$	22	c	
$Z_3 \times D_7$	15	c	Theorem 3	G_{44}	23	c	

7. The remaining groups with order less than 48. The real genus of each group G with $|G| \leq 32$ has already been determined. The groups with order less than 16 were considered in [**11**]. In [**14**] the calculation was extended for G with $16 < |G| < 24$. All remaining groups with order less than 32 were handled in [**15**]. Finally, the real genus of each group of order 32 was found in [**17**]. Here we consider each remaining group G with $32 < |G| < 48$.

There are six groups of order 42; two of these are the cyclic and dihedral groups. The real genus of each of the other four is in Table 3.

Each remaining group with order less than 48 is either cyclic, dihedral or abelian (of rank 2 with a Z_2 factor) with two exceptions, the metacyclic group $\langle 13, 3, 3 \rangle$ of order 39 and a group of order 44. This group G_{44} has presentation

$$A^{11} = C^4 = 1, \qquad C^{-1}AC = A^{-1}.$$

The group G_{44} is a semi-direct product $Z_{11} \times_\phi Z_4$. These two groups are included in Table 3.

8. The groups of real genus 9. To classify the groups of real genus 9, we first determine the large groups. We use MAGMA to determine which groups of order 96 = 12(9 - 1) are in class (q) with $m = 3$; there is only one of these. Then we check which groups of order 64 = 8(9 - 1) are in class (q) with $m = 4$; there are four of these. One is the group $G^{4,4,4}$, which is the group of a regular map of type $\{4, 4\}$ on the torus [**4**, page 138].

Each remaining group G with $\rho(G) = 9$ must have $|G| \leq 6(9-1) = 48$. But Theorem 1 applies in case $|G| > 2(9 - 1) = 16$. Of course, all groups with $|G| < 48$ have already been considered. MAGMA was used to check the remaining groups with order in this range (here only order 48 was necessary) for presentations in Theorem 1. This yields two interesting groups of order 48 that have real genus 9; these are $GL(2,3)$ and P_{48}, another group that contains $SL(2,3)$ as a subgroup. The notation P_{48} is from [**18**, page 116] where there is a presentation for this group. These three groups have also appeared as groups of low genus for other genus parameters. The groups $SL(2,3)$, $GL(2,3)$ and P_{48} are the three exceptional groups that have graph-theoretic genus one but do not have symmetric genus one [**8**, subsections 6.4.3, 6.4.6]. These groups have symmetric genus 2 [**18**, Theorem 4]. The M^*-group of order 96 [**10**, page 377] is in fact the group T_{96}, the unique group of graph theoretic genus 2 [**23**]. This group also has symmetric genus 2 [**18**, Theorem 4] and is the full group of the regular map $\{3, 4+4\}$ [**18**, page 127]; this map is denoted $R2.1$ in [**3**].

Table 4 gives the 24 groups of real genus 9. One of these groups is the quasi-abelian group QA_5 of order 32; the genus formula for the family of quasi-abelian groups is in [**17**, Theorem 3]. This result in [**17**] corrects the genus formula for these groups given in [**15**, Theorem 10].

For each group in a table, we also give its class and a reference to a previous result, generally to a genus calculation but sometimes to information about the specific group. In particular, a reference to [**3**] identifies a large group as the full group of a map.

TABLE 4. Groups of real genus 9.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
$Z_4 \times Z_4$	16	c	[**13**]	$\Gamma_3 b$	32	q	[**17**]	QD_5	32	c	[**15**]
$(Z_2)^2 \times Z_4$	16	f	[**13**]	$\Gamma_3 c_1$	32	c	[**17**]	$GL(2,3)$	48	c	
DC_4	16	c	[**11**]	$\Gamma_3 e$	32	c	[**17**]	P_{48}	48	c	
$\langle 2,2 \mid 4;2 \rangle$	16	c	[**15**]	$\Gamma_4 a_2$	32	q	[**17**]	$G^{4,4,4}$	64	q	
$SL(2,3)$	24	c	[**15**]	$\Gamma_4 b_1$	32	q	[**17**]	$G(R3.5)$	64	q	[**3**]
$(Z_2)^2 \times D_4$	32	p	[**13**]	$\Gamma_5 a_1$	32	p	[**17**]	$G(R3.6)$	64	q	[**3**]
QA_5	32	e	[**17**]	$\Gamma_7 a_1$	32	e	[**17**]	$G(R4.5)$	64	q	[**3**]
$\Gamma_2 j_1$	32	e	[**17**]	$\Gamma_7 a_2$	32	e	[**17**]	T_{96}	96	q	

9. The groups of real genus 10 and 11. To classify the groups of real genus 10, we first determine the large groups. We use MAGMA to determine which groups of order $108 = 12(10-1)$ are in class (q) with

$m = 3$; the only one is the M^*-group $G^{3,6,6}$, which is the group of a regular map of type $\{3, 6\}$ on the torus [**4**, page 138]. Next we find that there are two groups of order $72 = 8(10 - 1)$ that are in class (q) with $m = 4$. One interesting thing here is that these two groups are groups of maps on non-orientable surfaces. We also must check whether there are any groups of order $60 = 20(10 - 1)/3$ in class (q) with $m = 5$; the only such group is A_5, which has real genus 6.

Each remaining group G with $\rho(G) = 10$ must have $|G| \leq 6(10-1) = 54$. But Theorem 1 applies in case $|G| > 2(10 - 1) = 18$. Again MAGMA was used to check all groups with order in this range for presentations in Theorem 1; there are seven groups with genus 10, including two of order 54. Together with the three large groups, this gives a total of ten groups of real genus 10. These groups are in Table 5.

The two groups G_1 and G_2 of order 54 are semi-direct products of the non-Abelian group $K = (Z_3)^2 \times_\phi Z_3$ of order 27 by Z_2. The group K has presentation

$$(4) \quad A^3 = B^3 = C^3 = 1, \quad AB = BA, AC = CA, \quad C^{-1}BC = AB.$$

Then the group G_1 has presentation (4) together with

$$X^2 = 1, \quad XAX = A^{-1}, \quad XBX = B^{-1}, \quad XC = CX,$$

and G_2 is the group defined by (4) and

$$Y^2 = 1, \quad YA = AY, \quad YBY = B^{-1}, \quad YCY = C^{-1}.$$

The same general procedure was used to classify the groups of real genus 11. The large group of order 80 is the group of two different maps, the map $R4.4$ of type $\{4, 10\}$ and the map $R5.8$ of type $\{4, 20\}$. These maps are on different topological surfaces but have isomorphic groups; these maps are doubtless related. The M^*-group $Z_2 \times A_5$ acts on a sphere with 12 holes [**9**, page 9] and is also the group of each of the maps $N6.1$ and $N6.2$.

By the way, applying the procedure confirms the result of Mockiewicz [**21**] that there are no groups of real genus 12.

TABLE 5. Groups of real genus 10.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
$Z_3 \times Z_6$	18	c		$Gp(2)$	36	e		$G(N5.2)$	72	q	[**3**]
$(Z_3)^2 \times_\phi Z_3$	27	c	[**13**]	G_1	54	c		$G(N7.2)$	72	q	[**3**]
$Z_2 \times G_{18}$	36	q		G_2	54	q		$G^{3,6,6}$	108	q	
$Gp(1)$	36	c									

TABLE 6. Groups of real genus 11.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
DC_5	20	c	[**11**]	$Z_5 \times D_4$	40	e	[**5**]	$D_3 \times D_5$	60	q	
$Z_5 \times D_3$	30	e	Thm. 3	$Z_2 \times \langle 5,4,2 \rangle$	40	c		$G(R4.4)$	80	q	[**3**]
$Z_3 \times D_5$	30	c	Thm. 3	G_{40}	40	c		$Z_2 \times A_5$	120	q	[**9**]
$Z_4 \times D_5$	40	c	[**5**]								

10. The groups of real genus 13, 14, 15 and 16. Continuing the same approach yields a classification of the groups of real genus 13, 14, 15 and 16. In genus 16, for example, there are three large groups, and each remaining group G with this genus has $|G| \leq 6(16-1) = 90$. Then Theorem 1 and the computational procedure applies in case $|G| > 2(16-1) = 30$.

Among the 25 groups of genus 13, there are 9 groups of order 48. Each of these nine is either a direct product or a semi-direct product. One is the direct product of Z_2 and $G_{24} = (4,6 \mid 2,2)$ [**4**, page 134]. One is a semi-direct product of Z_{24} by Z_2 with presentation

$$X^{24} = T^2 = 1, TXT = X^{11}.$$

Two groups are semi-direct products of Z_3 by a group of order 16. Let $H_1 = (4,4 \mid 2,2)$ [**4**, page 134]. The group $Z_3 \times_\theta H_1$ has presentation

$$X^3 = R^4 = S^4 = (RS)^2 = (R^{-1}S)^2 = 1,\ XR = RX,\ S^{-1}XS = X^{-1}.$$

Let $H_2 = \langle 2,2,2 \rangle_2$ [**4**, page 134]. The group $Z_3 \times_\psi H_2$ has presentation

$$X^3 = R^2 = S^2 = T^2 = 1, \quad RST = STR = TRS,$$
$$RXR = X^{-1}, \quad XS = SX, \quad TXT = X^{-1}.$$

TABLE 7. Groups of real genus 13.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
$Z_2 \times Q$	16	f	[**13**]	$Z_3 \times H_1$	48	e		$Z_3 \times_\phi S_4$	72	q	
DC_6	24	c	[**11**]	$Z_3 \times QA_4$	48	e		$D_3 \times A_4$	72	c	
$Z_2 \times DC_3$	24	c	[**15**]	$Z_4 \times A_4$	48	e		$Z_2 \times (S_3)^2$	72	q	
$(Z_2)^2 \times Z_6$	24	f	[**13**]	$Z_3 \times_\psi H_2$	48	q		$G(R3.5)$	96	q	[**3**]
$\Gamma_2 b$	32	p	[**17**]	$Z_2 \times G_{24}$	48	q		$G(R5.7)$	96	q	[**3**]
$Z_6 \times D_3$	36	c	[**5**]	$(Z_2)^2 \times A_4$	48	e		$G(R6.5)$	96	q	[**3**]
$Z_3 \times A_4$	36	c		$(Z_2)^3 \times D_3$	48	p		$G(N10.2)$	96	q	[**3**]
$Z_{24} \times_\phi Z_2$	48	c		$Z_3 \times S_4$	72	c		$S_3 \times S_4$	144	q	[**9**]
$Z_3 \times_\theta H_1$	48	c									

TABLE 8. Groups of real genus 14.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
$\langle -2,2,3\rangle$	24	c	[**15**]	$Z_3 \times_\phi D_8$	48	q		$\langle 13,6,10\rangle$	78	c	
$\langle 13,3,3\rangle$	39	c	[**14**]	$\langle 13,4,8\rangle$	52	c					

There are also 4 groups of order 72 with genus 13. In addition to three direct products, there is the semi-direct product of Z_3 by S_4 with presentation

$$X^3 = S^4 = T^2 = (ST)^3 = 1, \quad S^{-1}XS = X^{-1}, \quad TXT = X^{-1}.$$

There are only 5 groups with real genus 14, and three of these involve the prime 13. The group of order 48 is the semi-direct product of Z_3 by D_8 with presentation

$$X^3 = A^2 = B^8 = (AB)^2 = 1, \quad AXA = X^{-1}, \quad B^{-1}XB = X^{-1}.$$

There are 9 groups with real genus 15, including 4 of order 56. Two of these four are direct products of familiar groups. One is the semi-direct product of Z_7 by D_4 with presentation

$$X^7 = A^2 = B^4 = (AB)^2 = 1, \quad AXA = X^{-1}, \quad B^{-1}XB = X^{-1}.$$

The fourth one is a semi-direct product of $(Z_2)^3$ by Z_7. Let A, B and C be a set of generators for $(Z_2)^3$, and let X be a generator for Z_7.

TABLE 9. Groups of real genus 15.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
DC_7	28	c	[**11**]	$Z_4 \times D_7$	56	c	[**5**]	$(Z_2)^3 \times_\psi Z_7$	56	e	
$Z_7 \times D_3$	42	e	Thm. 3	$Z_7 \times D_4$	56	e	[**5**]	$D_3 \times D_7$	84	q	
$Z_3 \times D_7$	42	c	Thm. 3	$Z_7 \times_\phi D_4$	56	c		$G(R6.4)$	112	q	[**3**]

TABLE 10. Groups of real genus 16.

Group	o	cl	Ref	Group	o	cl	Ref	Group	o	cl	Ref
$Z_5 \times Z_5$	25	c		$Z_5 \times D_5$	50	c	Thm. 3	$(Z_3)^2 \times_\phi D_4$	72	q	
$Z_3 \times Z_9$	27	c		$(Z_5)^2 \times_\phi Z_2$	50	q		$G(R6.6)$	100	q	[**3**]
$Z_9 \times_\theta Z_3$	27	c	[**15**]	$(Z_3 \times Z_9) \times_\psi Z_2$	54	q		$G(N5.1)$	120	q	[**3**]
$Z_3 \times DC_3$	36	c		$Z_5 \times A_4$	60	e		$G(N7.1)$	120	q	[**3**]

Then the action is given by

$$X^{-1}AX = B, \quad X^{-1}BX = C, \quad X^{-1}CX = AC.$$

There are 12 groups with real genus 16, including several groups of relatively small order. One of the groups of order 50 is the semi-direct product of $(Z_5)^2$ by Z_2 with presentation

$$X^5 = Y^5 = C^2 = 1, XY = YX, CXC = X^{-1}, CYC = Y^{-1}.$$

The group of order 54 is the semi-direct product of $Z_3 \times Z_9$ by Z_2 with presentation

$$X^3 = Y^9 = C^2 = 1, \quad XY = YX, CXC = X^{-1}, \quad CYC = Y^{-1}.$$

The group of order 72 is the semi-direct product of $Z_3 \times Z_3$ by D_4 with presentation

$$X^3 = Y^3 = A^2 = B^4 = (AB)^2 = 1, \quad XY = YX,$$
$$AXA = X, \quad AYA = Y^{-1},$$
$$B^{-1}XB = X^{-1}, \quad B^{-1}YB = Y.$$

The large group of order 100 is in class (q) with $m = 5$. It is a group of a map of type $\{5, 10\}$, the smallest group with this genus action.

11. The groups of real genus 18, 20, 22 and 24. The same procedure applied to the next four even values of the real genus yields some interesting results.

Theorem 5. *The unique group of real genus* 18 *is the metacyclic group* $\langle 17, 4, 13\rangle$ *of order* 68.

Theorem 6. *There are exactly two groups with real genus* 20; *these groups are the metacyclic group* $G = \langle 19, 3, 7\rangle$ *of order* 57 *and the semi-direct product* $G \times_\phi Z_2$ *with presentation*

$$X^{19} = Y^3 = A^2 = 1,\ Y^{-1}XY = X^7,\ AXA = X^{-1},\ AY = YA.$$

The result for genus 24 is especially interesting. The same procedure handles the classification for this genus. There are no large groups, and each remaining group G has $|G| \leq 6(24-1) = 138$. Theorem 1 applies in case $|G| > 2(24-1) = 46$, and the real genus of each group of smaller order is known.

Theorem 7. *There is no group of real genus* 24.

On the other hand, real genus 22 is more typical. The group $G_{63} \cong Z_3 \times \langle 7, 3, 2\rangle$ of order 63 has presentation

$$C^3 = X^7 = Y^3 = 1,\ Y^{-1}XY = X^2,\ CX = XC,\ CY = YC. \tag{5}$$

The group of order 126 is the semi-direct product of G_{63} by Z_2 with presentation (5) together with

$$A^2 = 1,\ AXA = X^{-1},\ AY = YA,\ ACA = C^{-1}.$$

The group of order 72 is a semi-direct product of $Z_3 \times Z_{12}$ by Z_2 with presentation

$$X^3 = Y^{12} = A^2 = 1,\ XY = YX,\ AXA = X^{-1}, AYA = Y^{-1}.$$

TABLE 11. Groups of real genus 22.

Group	o	cl	Group	o	cl	Group	o	cl
$Z_3 \times Z_{12}$	36	c	$(Z_3 \times Z_{12}) \times_\psi Z_2$	72	q	$G_{63} \times_\phi Z_2$	126	c
$Z_2 \times \langle 7,3,2 \rangle$	42	c	$Z_7 \times A_4$	84	e	$G(N19.1)$	168	q
$Z_3 \times \langle 7,3,2 \rangle$	63	c						

TABLE 12. Number of groups of real genus g.

g	$\nu(g)$	Ref	g	$\nu(g)$	Ref	g	$\nu(g)$	Ref	g	$\nu(g)$	Ref
2	0	[**11**]	7	3	[**7**]	12	0	[**21**]	18	1	Thm. 5
3	2	[**11**]	8	2	[**7**]	13	25	Tbl. 7	20	2	Thm. 6
4	4	[**12**]	9	24	Tbl. 4	14	5	Tbl. 8	22	7	Tbl. 11
5	10	[**15**]	10	10	Tbl. 5	15	9	Tbl. 9	24	0	Thm. 7
6	4	[**7**]	11	10	Tbl. 6	16	12	Tbl. 10			

12. Comments and open problems. Our approach will handle the groups with odd real genus $\rho \leq 23$, but there are quite a few of these groups. In particular, $\rho = 17$ is a big problem (primarily because of the groups of order 64). Extending the classification (with this procedure) to larger values of ρ would require determining the genus of more "small" groups, that is, groups with order at least 48.

One original hope was that this classification would provide enough data to shed some light on the following interesting general problem. For each integer $g \geq 2$, define $\nu(g)$ to be the number of groups with real genus g. Of course, $\nu(g)$ is a finite number for each g.

The known values of $\nu(g)$ are Table 12. These values reflect the correction of the mistake in [**15**, Theorem 10]; here see [**17**, Theorem 3]. In particular, QA_4 is a group of real genus 5, not 7. This group should be in the list of groups of genus 5 [**15**, Theorem 11] but not in the genus 7 list [**7**, page 698]. That genus 7 list also includes $Z_2 \times D_8$, a group of genus one. Further, $\rho(Z_3 \times D_4) = 7$ [**5**, page 152], and there are exactly three groups of genus 7, $Z_4 \times D_3$, $Z_3 \times D_4$ and $D_3 \times D_4$ [**5**, page 155].

Although $\nu(g)$ is positive for all odd $g \geq 3$ [**11**, Theorem 9], we have $\nu(2) = \nu(12) = 0$ and now $\nu(24) = 0$ as well. It may be that this is still only a small genus phenomenon, however. Certainly the genus 2 case can be considered this way. Also, for $g = 12$ and again for $g = 24$, $g-1$ is prime, which is a limiting consideration. Certainly, there seems to be a relative scarcity of groups of genus $p+1$, where p is prime.

In any case, general questions remain. Is $\nu(g) = 0$ for any even integer $g > 24$? Is there a group of real genus g for all $g \geq K$ for some constant K? Is $\nu(g) = 0$ for infinitely many values of g? If there are more zeros of the function ν, we suspect they will be at values of g of the form $p+1$. However, we note that there is at least one group of real genus 48 [**16**].

These questions are especially interesting because the corresponding question for the strong symmetric genus, a related parameter, has recently been settled. If n is a nonnegative integer, then there is at least one group of strong symmetric genus n [**19**, Theorem 1].

We are not at all sure whether a similar result holds for the real genus parameter.

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