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DEGREE GRAPHS OF SIMPLE GROUPS

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ABSTRACT. Let G be a finite group, and let $\mathrm{cd}\,(G)$ be the set of irreducible character degrees of G. The degree graph $\Delta(G)$ is the graph whose set of vertices is the set of primes that divide degrees in $\mathrm{cd}\,(G)$, with an edge between p and q if pq divides a for some degree $a \in \mathrm{cd}\,(G)$. We compile here the graphs $\Delta(G)$ for all finite simple groups G.

1. Introduction. The theory of characters is an important tool in the study of finite groups. The irreducible complex characters of a finite group G encode much information about the structure of G. For example, using these characters it is possible to determine the normal subgroups of G and therefore whether G is simple. It is also possible to determine if G is abelian, nilpotent, or solvable.

Somewhat surprisingly, it is also possible to obtain some information just from the set of degrees of the characters, that is, their (integer) values on the identity element of G. There has recently been much interest in studying the connections between the structure of a finite group G and the structure of its set of character degrees. Of particular interest is the connection between the structure of G and common divisors among character degrees. A useful tool for studying this connection is the character degree graph.

Let G be a finite group, and let $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G. Denote the set of irreducible character degrees of G by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, and denote by $\rho(G)$ the set of primes that divide degrees in $\mathrm{cd}(G)$. The *character degree graph* $\Delta(G)$ of G is the graph whose set of vertices is $\rho(G)$, with primes p, q in $\rho(G)$ joined by an edge if pq divides a for some character degree $a \in \mathrm{cd}(G)$.

These graphs have been studied for some time, primarily for solvable groups initially but more recently for nonsolvable groups as well. Some of the earliest results on character degree graphs were obtained by

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Manz, Staszewski, and Willems [**13**] on the number of connected components of the graph and by Manz, Willems, and Wolf [**14**] on the diameter of the graph for solvable groups. More recently, Lewis and the author have been able to classify nonsolvable groups G for which $\Delta(G)$ is disconnected [**10**] and to bound the diameter of $\Delta(G)$ when G is a nonsolvable group [**11, 12**]. See the article [**9**] by Lewis for an overview of results concerning $\Delta(G)$ and related graphs.

To each finite group G corresponds a unique collection of simple groups called the composition factors of G. In this sense, the finite simple groups are the building blocks for finite groups. The composition factors of a solvable group are all cyclic groups of prime order. A nonsolvable group, however, has at least one nonabelian composition factor.

The basic approach for studying degree graphs and attacking other character degree problems for nonsolvable groups is to first obtain as much information as possible about the situation for the nonabelian finite simple groups. The results are then extended to general nonsolvable groups using the information about their composition factors.

Of course, more complete results for the simple groups lead to better results for the nonsolvable groups. Using the partial results on degree graphs of simple groups available at the time, Lewis and the author proved in [**11**] that, for a nonsolvable group G, the diameter of $\Delta(G)$ is at most 4. Later, using the complete results in [**1, 18, 19, 20**], we were able to lower this bound to 3 in [**12**].

Our purpose in this survey is to describe the degree graphs for all nonabelian finite simple groups. By the classification of finite simple groups (see [**7**]) a nonabelian finite simple group must be an alternating group $\mathrm{Alt}(n)$ with $n \geqslant 5$, a simple group of Lie type, or one of 26 sporadic simple groups. The groups of Lie type may be further divided into the categories of groups of exceptional type and groups of classical type.

The irreducible characters of the 26 sporadic groups are available in the Atlas [**3**]. The degree graphs for these groups are therefore known implicitly, but the graphs and proofs have not appeared in print previously. The situation is similar for the alternating groups $\mathrm{Alt}(n)$ for $5 \leqslant n \leqslant 14$. The results of Barry and Ward in [**1**] determine $\Delta(\mathrm{Alt}(n))$ for $n \geqslant 15$. The graphs for the groups of Lie type were completely determined by the author in [**18, 19, 20**].

We explicitly describe here the graphs for all nonabelian finite simple groups and list the degrees used in the proofs. We give proofs in those cases for which proofs have not previously appeared in print. For the other cases, we outline the proofs and give references to the complete published proofs. The sporadic groups are considered in Section 2, the alternating groups in Section 3, exceptional groups of Lie type in Section 4 and classical groups of Lie type in Section 5. Finally, in Section 6, we summarize the results on connectedness and diameters of the degree graphs of the simple groups.

We say that a graph is a *complete* graph if there is an edge between every pair of vertices. Thus, $\Delta(G)$ is a complete graph if whenever p and q are any distinct primes dividing character degrees of G, there is some $\chi \in \mathrm{Irr}(G)$ such that $pq \mid \chi(1)$. The graph $\Delta(G)$ tends to be a complete graph for simple groups G. In fact, $\Delta(G)$ fails to be complete only for certain "small" simple groups.

Moreover, many groups satisfy the stronger condition that there is a character $\chi \in \mathrm{Irr}(G)$ such that every prime in $\rho(G)$ divides $\chi(1)$. In this case, we say G has *covering number* 1. (The covering number is the smallest number of irreducible character degrees required to "cover" the primes in $\rho(G)$.) It is shown in [**1**] that $\mathrm{Alt}\,(n)$ has covering number 1 for all $n \geqslant 15$.

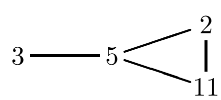
Notice that if $\Delta(G)$ is a complete graph but G does not have covering number 1, then a proof that the graph is complete requires at least three character degrees. In our proofs, particularly for the sporadic and small alternating groups, we attempt to use the fewest possible degrees in each case. Of course, if $\Delta(G)$ is *not* complete, all degrees must be used in order to determine the graph.

Our notation is mostly standard. Notation for the sporadic simple groups is as in the Atlas [**3**], as are the labels for the characters of the small alternating groups. We refer the reader to [**2**] for notation and results for the simple groups of Lie type. We will use $\pi(n)$ to denote the set of prime divisors of a positive integer n. If G is a nonabelian finite simple group, then by the Itô-Michler theorem (see [**15**, Remarks 13.13]), $\rho(G) = \pi(|G|)$. We will also denote by $\Phi_k = \Phi_k(q)$ the value of the kth cyclotomic polynomial evaluated at q.

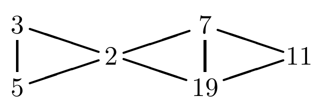
2. Sporadic simple groups. We first describe the graph $\Delta(G)$ for a sporadic simple group G. As noted in the introduction, the explicit character tables of the sporadic groups are known, so it is straightforward to determine the graphs. The graphs were described partially in [**11**, Lemma 2.3], but the graphs and proofs have not appeared in print elsewhere. It was shown in [**11**] that the sporadic simple group J_1 is the only simple group whose degree graph has diameter greater than 2, and the diameter of $\Delta(J_1)$ is 3.

Theorem 2.1. *If G is a sporadic simple group other than M_{11}, J_1 or M_{23}, then $\Delta(G)$ is a complete graph. The graphs of M_{11}, J_1, and M_{23} are as follows.*

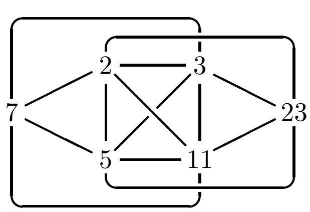
1. $\Delta(M_{11})$ *is*



2. $\Delta(J_1)$ *is*



3. $\Delta(M_{23})$ *is*



Proof. The character tables of the sporadic groups are found in the Atlas [**3**].

We have

$$\mathrm{cd}\,(M_{11}) = \{1,\ 2\cdot 5,\ 11,\ 2^4,\ 2^2\cdot 11,\ 3^2\cdot 5,\ 5\cdot 11\}.$$

Hence, $\rho(M_{11}) = \{2,3,5,11\}$ and in $\Delta(G)$, the primes 2, 5, and 11 are all adjacent to each other, but 3 is adjacent only to 5, so the graph is as claimed.

Next,

$$\mathrm{cd}\,(J_1) = \{1,\ 2^3\cdot 7,\ 2^2\cdot 19,\ 7\cdot 11,\ 2^3\cdot 3\cdot 5,\ 7\cdot 19,\ 11\cdot 19\}$$

and $\rho(J_1) = \{2,3,5,7,11,19\}$. The character degrees show that the primes 3 and 5 are adjacent in $\Delta(J_1)$, as are all of 7, 11, and 19, but neither of 3, 5 is adjacent to any of 7, 11, or 19. Also, 2 is adjacent to all of 3, 5, 7, and 19, and the graph is as claimed.

We have

$$\begin{aligned}\mathrm{cd}\,(M_{23}) = \{1,\ 2\cdot 11,\ 3^2\cdot 5,\ 2\cdot 5\cdot 23,\ 3\cdot 7\cdot 11,\ 11\cdot 23,\ 2\cdot 5\cdot 7\cdot 11,\\ 2^7\cdot 7,\ 2\cdot 3^2\cdot 5\cdot 11,\ 3^2\cdot 5\cdot 23,\ 2^3\cdot 11\cdot 23\}\end{aligned}$$

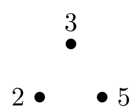
and $\rho(M_{23}) = \{2,3,5,7,11,23\}$. It is easily verified that the complete graph on $\{2,3,5,7,11\}$ is a subgraph of $\Delta(M_{23})$ and that 23 is adjacent to each of 2, 3, 5, and 11, but not to 7.

The orders of the remaining sporadic groups are listed in Table 1. It is easily verified that $\Delta(G)$ is a complete graph by considering the character degrees in Table 2, Table 3 and Table 4. The groups listed in Table 4 are those for which a single irreducible character degree is divisible by all primes in $\pi(|G|)$, that is, the sporadic groups with covering number 1. □

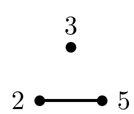
3. Alternating groups. We next consider the simple alternating groups $\mathrm{Alt}\,(n)$ for $n \geqslant 5$. The graphs for these groups are complete except when $n = 5$, 6 or 8, and in fact, $\mathrm{Alt}\,(n)$ has covering number 1 for all $n \geqslant 15$ by [**1**]. The exceptions in this case are also special cases for the linear groups (subsection 5.1), because $\mathrm{Alt}\,(5) \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$, $\mathrm{Alt}\,(6) \cong \mathrm{PSL}_2(9)$, and $\mathrm{Alt}\,(8) \cong \mathrm{PSL}_4(2)$.

Theorem 3.1. *Let* $G = \mathrm{Alt}\,(n)$, *the alternating group of degree* n, *with* $n \geqslant 5$. *If* n *is not* 5, 6 *or* 8, *then the graph* $\Delta(G)$ *is complete. The graphs for* $\mathrm{Alt}\,(5)$, $\mathrm{Alt}\,(6)$, *and* $\mathrm{Alt}\,(8)$ *are as follows.*

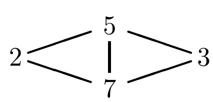
1. $\Delta(\mathrm{Alt}\,(5))$ *is*



2. $\Delta(\mathrm{Alt}\,(6))$ *is*



3. $\Delta(\mathrm{Alt}\,(8))$ *is*



Proof. For the graphs of Alt (5), Alt (6), and Alt (8), we have

$$\mathrm{cd}\,(\mathrm{Alt}\,(5)) = \{1, 3, 2^2, 5\},$$
$$\mathrm{cd}\,(\mathrm{Alt}\,(6)) = \{1, 5, 2^3, 3^2, 2 \cdot 5\},$$

and

$$\mathrm{cd}\,(\mathrm{Alt}\,(8)) = \{1, 7, 2 \cdot 7, 2^2 \cdot 5, 3 \cdot 7, 2^2 \cdot 7, 5 \cdot 7, 3^2 \cdot 5, 2^3 \cdot 7, 2^6, 2 \cdot 5 \cdot 7\}$$

by the Atlas [**3**] character tables. Therefore, the graphs are as claimed. We also have

$$\mathrm{cd}\,(\mathrm{Alt}\,(7)) = \{1, 2\cdot 3, 2\cdot 5, 2\cdot 7, 3\cdot 5, 3\cdot 7, 5\cdot 7\},$$

so $\Delta(\mathrm{Alt}\,(7))$ is complete.

The character degrees needed to prove that the graphs of $\mathrm{Alt}\,(n)$ are complete for $9 \leqslant n \leqslant 14$ are given in Table 5. The degrees for $n \leqslant 13$ may be found in the Atlas [**3**], and the degrees for $\mathrm{Alt}\,(14)$ were computed using GAP [**5**].

It is shown by Barry and Ward in [**1**] that for $n \geqslant 15$, $\mathrm{Alt}\,(n)$ has an irreducible character whose degree is divisible by all primes dividing the order of $\mathrm{Alt}\,(n)$. Therefore, $\Delta(\mathrm{Alt}\,(n))$ is complete for all $n \geqslant 15$. □

4. Exceptional groups of Lie type. In this section, G is a simple group of exceptional Lie type. Thus, G will be one of the following:

$$\begin{gathered} G_2(q) \quad \text{for } q \neq 2, \\ F_4(q),\ E_6(q),\ E_7(q),\ E_8(q),\ {}^2E_6(q^2),\ {}^3D_4(q^3) \text{ for all } q, \\ {}^2B_2(q^2) \text{ and } {}^2F_4(q^2) \quad \text{for } q^2 = 2^{2m+1},\ q^2 \neq 2, \\ {}^2F_4(2)', \\ {}^2G_2(q^2) \text{ for } q^2 = 3^{2m+1},\ q^2 \neq 3. \end{gathered}$$

(See [**2**] or [**3**] for notation.) Note that certain values of q are excluded for some types because the corresponding groups are not simple. The group ${}^2B_2(2)$ is solvable. The derived groups of $G_2(2)$ and ${}^2G_2(3)$ are simple but are isomorphic to the simple classical groups ${}^2A_2(3^2)$ and $A_1(8)$, respectively, which will be considered in Section 5. The derived group of ${}^2F_4(2)$ is simple and does not appear elsewhere in the classification of finite simple groups, so will be considered here.

Theorem 4.1 [**18**, Theorem 3.3]. *Let $G \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. The set of primes $\rho(G)$ can be partitioned as*

$$\rho(G) = \{2\} \cup \pi(q^2 - 1) \cup \pi(q^4 + 1).$$

The subgraph of $\Delta(G)$ on $\rho(G) - \{2\}$ is complete and 2 is adjacent in $\Delta(G)$ to precisely the primes in $\pi(q^2 - 1)$.

Proof. We have $|G| = q^4\Phi_1\Phi_2\Phi_8$ and

$$\mathrm{cd}\,(\mathrm{G}) = \left\{1,\ \mathrm{q}^4,\ \frac{1}{\sqrt{2}}\mathrm{q}\Phi_1\Phi_2,\ \Phi_8,\ \Phi_1\Phi_2\Phi_8',\ \Phi_1\Phi_2\Phi_8''\right\},$$

where $\Phi_8' = q^2 + \sqrt{2}q + 1$ and $\Phi_8'' = q^2 - \sqrt{2}q + 1$. (Thus, Φ_8' and Φ_8'' are integers with $\Phi_8'\Phi_8'' = \Phi_8$.) See [**18**] for details. □

Theorem 4.2 [**18**, Theorem 3.4]. *If G is a simple group of exceptional Lie type, other than type ${}^2B_2$, then the graph $\Delta(G)$ is complete.*

Proof. If $G \cong F_4(2)$, then $|G| = 2^{24}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13\cdot 17$. By the Atlas [**3**], G has character degree $\chi_{46}(1) = 1299480 = 2^3\cdot 3\cdot 5\cdot 7^2\cdot 13\cdot 17$, so the graph is complete.

If $G \cong {}^2F_4(2)$, then $G' \cong {}^2F_4(2)'$ is simple and $|{}^2F_4(2)'| = 2^{11}\cdot 3^3\cdot 5^2\cdot 13$. By the Atlas [**3**], ${}^2F_4(2)'$ has character degrees $\chi_6(1) = 78 = 2\cdot 3\cdot 13$, $\chi_7(1) = 300 = 2^2\cdot 3\cdot 5^2$ and $\chi_8(1) = 325 = 5^2\cdot 13$, and these degrees show that the graph is complete.

For the other cases, the group orders are given in Table 6 and the degrees used to prove that the graphs are complete are given in Tables 7 and 8. See [**2**] for the notation for the character labels, and see [**18**] for details of the proof. □

5. Classical groups of Lie type. The classical groups of Lie type are the classical simple linear, unitary, symplectic, and orthogonal groups. For the classical groups, it is generally easier to compute the degrees of the so-called adjoint group, which contains the simple group of the given type as a normal subgroup. For the purpose of finding the character degree graphs, we will use the degrees for the adjoint groups along with the following well-known lemma (see [**8**, Corollary 11.29]).

Lemma 5.1. *If $N \trianglelefteq G$ with $|G : N| = d$, $\chi \in \mathrm{Irr}(G)$, and μ is a constituent of the restriction of χ to N, then $\chi(1)/d$ divides $\mu(1)$.*

Various notations are used for these groups. In the terminology of groups of Lie type, the linear groups are of type A_ℓ, the unitary groups are of type ${}^2A_\ell$, the various orthogonal groups are of types B_ℓ, D_ℓ and

${}^2D_\ell$, and the symplectic groups are of type C_ℓ. The table below gives the classical matrix group notation and the notation used in the Atlas [**3**] for these groups. Also listed are the adjoint group and the index of the simple group in the adjoint group for each type. The notation for the adjoint groups is as in [**2**].

Lie Type	**Simple Group Classical**	**Atlas**	**Adjoint Group**	**Index**
A_ℓ	$\mathrm{PSL}_{\ell+1}(q)$	$\mathrm{L}_{\ell+1}(q)$	$\mathrm{PGL}_{\ell+1}(q)$	$(\ell+1, q-1)$
${}^2A_\ell$	$\mathrm{PSU}_{\ell+1}(q^2)$	$\mathrm{U}_{\ell+1}(q)$	$\mathrm{PU}_{\ell+1}(q)$	$(\ell+1, q+1)$
B_ℓ	$\Omega_{2\ell+1}(q)$	$\mathrm{O}_{2\ell+1}(q)$	$\mathrm{SO}_{2\ell+1}(q)$	$(2, q-1)$
C_ℓ	$\mathrm{PSp}_{2\ell}(q)$	$\mathrm{S}_{2\ell}(q)$	$\mathrm{PCSp}_{2\ell}(q)$	2
D_ℓ	$\mathrm{P}\Omega^+_{2\ell}(q)$	$\mathrm{O}^+_{2\ell}(q)$	$\mathrm{P}(\mathrm{CO}_{2\ell}(q)^0)$	$(4, q^\ell-1)$
${}^2D_\ell$	$\mathrm{P}\Omega^-_{2\ell}(q)$	$\mathrm{O}^-_{2\ell}(q)$	$\mathrm{P}(\mathrm{CO}^-_{2\ell}(q)^0)$	$(4, q^\ell+1)$

5.1 Type A_ℓ: Linear groups. In this section, G is the projective special linear group $\mathrm{PSL}_{\ell+1}(q)$ of type A_ℓ, where q is a power of a prime p and $\ell \geqslant 1$. If $\ell = 1$, then we take $q \geqslant 4$, as $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$ are not simple. Note also that $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4) \cong \mathrm{Alt}\,(5)$.

Theorem 5.2 [**19**, Theorem 3.1]. *Let $G \cong \mathrm{PSL}_2(q)$, where $q \geqslant 4$ is a power of a prime p.*

1. *If q is even, then $\Delta(G)$ has three connected components, $\{2\}$, $\pi(q-1)$ and $\pi(q+1)$, and each component is a complete graph.*

2. *If $q > 5$ is odd, then $\Delta(G)$ has two connected components, $\{p\}$ and $\pi((q-1)(q+1))$.*

(a) *The connected component $\pi((q-1)(q+1))$ is a complete graph if and only if $q-1$ or $q+1$ is a power of 2.*

(b) *If neither of $q-1$ or $q+1$ is a power of 2, then $\pi((q-1)(q+1))$ can be partitioned as $\{2\} \cup M \cup P$, where $M = \pi(q-1) - \{2\}$ and $P = \pi(q+1) - \{2\}$ are both nonempty sets. The subgraph of $\Delta(G)$ corresponding to each of the subsets M, P is complete, all primes are adjacent to 2, and no prime in M is adjacent to any prime in P.*

Proof. If $q = 2^n$, $n \geqslant 2$, then

$$\mathrm{cd}(G) = \{1, 2^n - 1, 2^n, 2^n + 1\}.$$

If $q = p^n > 5$ is odd, then

$$\mathrm{cd}(G) = \{1, q-1, q, q+1, (q+\varepsilon)/2\},$$

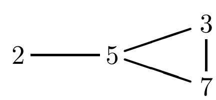
where $\varepsilon = (-1)^{(q-1)/2}$. See [**19**] for details. □

We next describe the degree graph for $\mathrm{PSL}_3(q)$. Note that $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, so if $q = 2$, then the graph is described in Theorem 5.2. We will therefore take $q > 2$.

Theorem 5.3 [**19**, Theorem 3.2]. *Let $G \cong \mathrm{PSL}_3(q)$, where $q > 2$ is a power of a prime p.*

1. *The graph $\Delta(G)$ is complete if and only if q is odd and $q-1 = 2^i 3^j$ for some $i \geqslant 1$, $j \geqslant 0$.*

2. (a) *If $q = 4$, then $G \cong \mathrm{PSL}_3(4)$ and $\Delta(G)$ is*



(b) *If $q \neq 4$, then $\rho(G) = \{p\} \cup \pi((q-1)(q+1)(q^2+q+1))$. The subgraph of $\Delta(G)$ corresponding to $\pi((q-1)(q+1)(q^2+q+1))$ is complete and p is adjacent to precisely those primes dividing $q+1$ or q^2+q+1.*

Proof. If $q = 3$, then $G \cong \mathrm{PSL}_3(3)$ and by the Atlas [**3**] character table, we have

$$\mathrm{cd}\,(\mathrm{PSL}_3(3)) = \{1, 2^2 \cdot 3, 13, 2^4, 2 \cdot 13, 3^3, 3 \cdot 13\}.$$

If $q = 4$, then $G \cong \mathrm{PSL}_3(4)$ and again by the character table in the Atlas [**3**], we have

$$\mathrm{cd}\,(\mathrm{PSL}_3(4)) = \{1, 2^2 \cdot 5, 5 \cdot 7, 3^2 \cdot 5, 3^2 \cdot 7, 2^6\}.$$

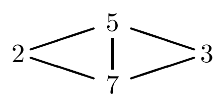
If $q > 4$, then by the character table of G in either [**16**] or the CHEVIE system [**6**], every character degree of G divides one of the degrees in the subset

$$\{q^3, q(q+1), (q-1)(q^2+q+1), q(q^2+q+1), (q+1)(q^2+q+1), (q-1)^2(q+1)\}$$

of $\mathrm{cd}(G)$. The theorem follows from these lists of degrees. See [**19**] for details. □

We now consider the simple groups of type A_ℓ for all $\ell \geqslant 3$. The one exceptional case is $\mathrm{PSL}_4(2)$, which is isomorphic to the alternating group $\mathrm{Alt}\,(8)$.

Theorem 5.4 [**19**, Theorem 3.3]. *Let $G \cong \mathrm{PSL}_{\ell+1}(q)$, where $\ell \geqslant 3$ and q is a power of a prime p. The graph $\Delta(G)$ is complete unless $\ell = 3$ and $q = 2$. If $\ell = 3$ and $q = 2$, then $G \cong \mathrm{PSL}_4(2) \cong \mathrm{Alt}\,(8)$ and $\Delta(G)$ is*



Proof. The order of $G \cong \mathrm{PSL}_{\ell+1}(q)$ is given by

$$|G| = \frac{1}{d}\, q^{\ell(\ell+1)/2}(q^2-1)(q^3-1)\cdots(q^\ell-1)(q^{\ell+1}-1),$$

where $d = (\ell+1, q-1) = [\mathrm{PGL}_{\ell+1}(q) : \mathrm{PSL}_{\ell+1}(q)]$. Therefore, $\rho(G)$ is precisely the set of primes dividing q or some Φ_k for $1 \leqslant k \leqslant \ell+1$.

For $\ell \geqslant 4$, the graph $\Delta(\mathrm{PSL}_{\ell+1}(q))$ is seen to be complete using Lemma 5.1 and the following degrees of $\mathrm{PGL}_{\ell+1}(q)$:

$$\begin{aligned}
\chi^{(1,1,\ell-1)}(1) &= q^3\,\frac{(q^{\ell-1}-1)(q^\ell-1)}{(q-1)(q^2-1)}\\
\chi_1(1) &= (q-1)(q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^\ell-1)\\
\chi_2(1) &= (q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^{\ell+1}-1)\\
\chi_3(1) &= q\,\frac{(q^2-1)(q^3-1)\cdots(q^{\ell+1}-1)}{(q^2-1)(q^{\ell-1}-1)}.
\end{aligned}$$

For $\ell = 3$, so $G \cong \mathrm{PSL}_4(q)$, the primes in $\rho(G)$ are those primes dividing one of q, Φ_1, Φ_2, Φ_3, or Φ_4. In this case, $\mathrm{PGL}_4(q)$ has the degrees

$$\chi_1(1) = \Phi_1^3\Phi_2\Phi_3$$
$$\chi_2(1) = \Phi_1^2\Phi_2^2\Phi_4$$
$$\chi_3(1) = q\Phi_1\Phi_3\Phi_4.$$

For $q = 3$, these are sufficient to prove that the graph is complete. For $q > 3$, $\mathrm{PGL}_4(q)$ also has the degree $\chi(1) = q\Phi_2\Phi_3\Phi_4$, and the graph is shown to be complete using this degree along with those listed above and Lemma 5.1. Finally, if $q = 2$, then $G \cong \mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)$ and

$$\mathrm{cd}(G) = \{1, 7, 2\cdot 7, 2^2\cdot 5, 3\cdot 7, 2^2\cdot 7, 5\cdot 7, 3^2\cdot 5, 2^3\cdot 7, 2^6, 2\cdot 5\cdot 7\},$$

as noted in Theorem 3.1. The graph follows in this case. See [**19**] for details. □

5.2. Type $^2A_\ell$: Unitary groups. In this section, G is the projective special unitary group $\mathrm{PSU}_{\ell+1}(q^2)$ of type $^2A_\ell$, where q is a power of a prime p. Note that this case is very similar to the case where G is of type A_ℓ. Roughly speaking, replacing q with $-q$ in the results for type A_ℓ yields the analogous results for type $^2A_\ell$.

Because $\mathrm{PSU}_2(q^2) \cong \mathrm{PSL}_2(q)$, we take $\ell \geqslant 2$. If $\ell = 2$, then we take $q > 2$, as $\mathrm{PSU}_3(2^2)$ is not simple.

Theorem 5.5 [**19**, Theorem 3.4]. *Let $G \cong \mathrm{PSU}_3(q^2)$, where $q > 2$ is a power of a prime p.*

1. *The graph $\Delta(G)$ is complete if and only if q satisfies $q + 1 = 2^i3^j$ for some $i \geqslant 0$, $j \geqslant 0$.*

2. *If $q > 2$, then $\rho(G) = \{p\} \cup \pi((q-1)(q+1)(q^2-q+1))$. The subgraph of $\Delta(G)$ corresponding to $\pi((q-1)(q+1)(q^2-q+1))$ is complete, and p is adjacent to precisely those primes dividing $q-1$ or q^2-q+1.*

Proof. Since $q > 2$, the character table of G in either [**16**] or the CHEVIE system [**6**] shows that every character degree of G divides one of the degrees in the subset

$$\{q^3, q(q-1), (q-1)(q^2-q+1), q(q^2-q+1), (q+1)(q^2-q+1), (q-1)(q+1)^2\}$$

of $\mathrm{cd}(G)$. The graph follows from this list of degrees. See [**19**] for details. □

We next determine the graph $\Delta(G)$ for the simple groups of type ${}^2A_\ell$ for all $\ell \geqslant 3$.

Theorem 5.6 [**19**, Theorem 3.5]. *If $G \cong \mathrm{PSU}_{\ell+1}(q^2)$, where $\ell \geqslant 3$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. The order of $G \cong \mathrm{PSU}_{\ell+1}(q^2)$ is given by

$$|G| = \frac{1}{d}\, q^{\ell(\ell+1)/2}(q^2-1)(q^3+1)\cdots(q^\ell-(-1)^\ell)(q^{\ell+1}-(-1)^{\ell+1}),$$

where $d = (\ell+1, q+1) = [\mathrm{PU}_{\ell+1}(q^2) : \mathrm{PSU}_{\ell+1}(q^2)]$. Therefore, $\rho(G)$ is precisely the set of primes dividing q or some $q^k - (-1)^k$ for $2 \leqslant k \leqslant \ell+1$.

If (ℓ, q) is in $\{(3,2),(3,3),(4,2)\}$, then in each case there is an irreducible character whose degree is divisible by all primes in $\rho(G)$. Using Atlas [**3**] notation for the characters, we have $\rho(\mathrm{PSU}_4(2^2)) = \{2,3,5\}$ and $\chi_{11}(1) = 30 = 2\cdot 3\cdot 5$, $\rho(\mathrm{PSU}_4(3^2)) = \{2,3,5,7\}$ and $\chi_8(1) = 210 = 2\cdot 3\cdot 5\cdot 7$, and $\rho(\mathrm{PSU}_5(2^2)) = \{2,3,5,11\}$ and $\chi_{26}(1) = 330 = 2\cdot 3\cdot 5\cdot 11$. Hence, $\Delta(G)$ is complete in these cases.

If $(\ell, q) \notin \{(3,2),(3,3),(4,2)\}$, then $\mathrm{PU}_{\ell+1}(q^2)$ has the character degrees

$$\begin{aligned}
\chi^{(1,1,\ell-1)}(1) &= q^3\frac{(q^{\ell-1}-(-1)^{\ell-1})(q^\ell-(-1)^\ell)}{(q+1)(q^2-1)}\\
\chi_1(1) &= (q+1)(q^2-1)(q^3+1)\cdots(q^{\ell-1}-(-1)^{\ell-1})(q^\ell-(-1)^\ell)\\
\chi_2(1) &= (q^2-1)(q^3+1)\cdots(q^{\ell-1}-(-1)^{\ell-1})(q^{\ell+1}-(-1)^{\ell+1})\\
\chi_3(1) &= q\frac{(q^2-1)(q^3+1)\cdots(q^{\ell+1}-(-1)^{\ell+1})}{(q^2-1)(q^{\ell-1}-(-1)^{\ell-1})}.
\end{aligned}$$

For $\ell \geqslant 4$, these degrees, along with Lemma 5.1, are sufficient to prove the graph is complete.

If $\ell = 3$, so $G \cong \mathrm{PSU}_4(q^2)$, then the primes in $\rho(G)$ are those primes dividing one of q, Φ_1, Φ_2, Φ_4, or Φ_6. In this case, since $q > 3$, $\mathrm{PU}_4(q^2)$

has degrees

$$\chi_1(1) = \Phi_1\Phi_2^3\Phi_6$$
$$\chi_2(1) = \Phi_1^2\Phi_2^2\Phi_4$$
$$\chi_3(1) = q\Phi_2\Phi_4\Phi_6$$
$$\chi(1) = q\Phi_1^2\Phi_4.$$

The graph is shown to be complete for $\ell = 3$ and $q > 3$ using these degrees and Lemma 5.1. See [**19**] for details. □

5.3. Type B_ℓ: Odd dimensional orthogonal groups. In this section, G is the simple odd dimensional orthogonal group $\Omega_{2\ell+1}(q)$ (or $\mathrm{O}_{2\ell+1}(q)$ in Atlas [**3**] notation) of type B_ℓ, where q is a power of a prime p. Since $B_1(q) \cong A_1(q)$, we assume $\ell \geqslant 2$, and since $B_2(2)$ is isomorphic to the symplectic group $\mathrm{Sp}_4(2)$, which is not simple, we assume $q > 2$ when $\ell = 2$.

Theorem 5.7 [**20**, Theorem 1.1]. *If $G \cong \Omega_{2\ell+1}(q)$ is a simple group of type B_ℓ, where $\ell \geqslant 2$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. The order of G is given by

$$|G| = \frac{1}{d}q^{\ell^2}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1),$$

where $d = (2, q-1) = [\mathrm{SO}_{2\ell+1}(q) : \Omega_{2\ell+1}(q)]$. It follows that $\rho(G)$ consists of p and the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell$.

If $\ell \geqslant 3$, then $\mathrm{SO}_{2\ell+1}(q)$ has character degrees

$$\chi^\alpha(1) = \frac{1}{2}q^4\frac{(q^{\ell-2}-1)(q^{\ell-1}-1)(q^{\ell-1}+1)(q^\ell+1)}{(q^2-1)^2}$$

$$\chi_c(1) = \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^\ell+1}$$

$$= (q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1)$$

$$\chi_1(1) = q\frac{(q^4-1)(q^6-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell-1}+1}$$

(see [**20**, Lemma 2.2]). These degrees, along with Lemma 5.1, are sufficient to show $\Delta(G)$ is complete for $\ell \geqslant 3$.

Now let $\ell = 2$ so $G \cong B_2(q) \cong C_2(q)$, hence $G \cong \mathrm{PSp}_4(q)$. As noted above, G is not simple if $q = 2$. If $q = 3$, then $|G| = 2^6 \cdot 3^4 \cdot 5$ and, by the Atlas [**3**] character table, G has the character degree $\chi_{11}(1) = 30 = 2 \cdot 3 \cdot 5$, so the graph is complete. Hence, we may assume $q > 3$. The order of G is

$$|G| = |\mathrm{PSp}_4(q)| = \frac{1}{(2,\Phi_1)} q^4 \Phi_1^2 \Phi_2^2 \Phi_4,$$

and the character tables in [**4, 17**] show that for $q > 3$, G has the character degrees

$$\chi_1(1) = \Phi_1^2\Phi_2^2$$
$$\chi_2(1) = q\Phi_1\Phi_4$$
$$\chi_3(1) = q\Phi_2\Phi_4.$$

These degrees show that the graph $\Delta(G)$ is complete for $\ell = 2$. See [**20**] for details. □

5.4. Type C_ℓ: Symplectic groups. In this section, G is the simple symplectic group $\mathrm{PSp}_{2\ell}(q)$ of type C_ℓ, where q is a power of a prime p. Since $C_2(q) \cong B_2(q)$ for all q, we assume $\ell \geqslant 3$.

Theorem 5.8 [**20**, Theorem 1.1]. *If $G \cong \mathrm{PSp}_{2\ell}(q)$ is a simple group of type C_ℓ, where $\ell \geqslant 3$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. Since $C_\ell(q) \cong B_\ell(q)$ for all ℓ if q is even, by Theorem 5.7 we may assume that q is odd. Since q is odd, the order of $G \cong \mathrm{PSp}_{2\ell}(q)$ is

$$|G| = \frac{1}{2} q^{\ell^2}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1).$$

It follows that $\rho(G)$ consists of p and the primes dividing Φ_j or Φ_{2j} for $j = 1, \dots, \ell$.

If $\ell \geqslant 4$, then the projective conformal symplectic group $\mathrm{PCSp}_{2\ell}(q)$ has character degrees

$$\begin{aligned}
\chi^{\alpha}(1) &= q^3 \frac{(q^{2(\ell-2)}-1)(q^{2\ell}-1)}{(q^2-1)^2} \\
\chi_c(1) &= \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell}+1} \\
&= (q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{\ell}-1) \\
\chi_1(1) &= q^2 \frac{(q^2+1)(q^6-1)(q^8-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell-2}+1}
\end{aligned}$$

(see [**20**, Lemma 2.3]), and we have $[\mathrm{PCSp}_{2\ell}(q) : \mathrm{PSp}_{2\ell}(q)] = 2$. Using Lemma 5.1, these degrees are sufficient to show that $\Delta(G)$ is complete for $\ell \geqslant 4$.

Now let $\ell = 3$ so that $G \cong \mathrm{PSp}_6(q)$. The order of G is

$$|G| = |\mathrm{PSp}_6(q)| = \frac{1}{2} q^9 \Phi_1^3 \Phi_2^3 \Phi_3 \Phi_4 \Phi_6.$$

Since q is odd, the character table of the conformal symplectic group $\mathrm{CSp}_6(q)$ is available in the CHEVIE system [**6**]. This table shows that $\mathrm{PCSp}_6(q)$ has a character of degree

$$\chi_{82}(1,1,q-2)(1) = q\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6,$$

and since $[\mathrm{PCSp}_6(q) : \mathrm{PSp}_6(q)] = 2$, this degree, along with Lemma 5.1, shows that $\Delta(G)$ is complete. See [**20**] for details. □

5.5. Types D_ℓ and ${}^2D_\ell$: Even dimensional orthogonal groups. In this section, G is a simple even dimensional orthogonal group and so is either $\mathrm{P}\Omega_{2\ell}^{+}(q)$ of type D_ℓ or $\mathrm{P}\Omega_{2\ell}^{-}(q)$ of type ${}^2D_\ell$ (that is, $\mathrm{O}_{2\ell}^{+}(q)$ or $\mathrm{O}_{2\ell}^{-}(q)$, respectively, in Atlas [**3**] notation), where q is a power of a prime p. Since $D_\ell \cong A_\ell$ and ${}^2D_\ell \cong {}^2A_\ell$ if ℓ is less than 4, we assume $\ell \geqslant 4$.

Theorem 5.9 [**20**, Theorem 1.1]. *If $G \cong \mathrm{P}\Omega_{2\ell}^{+}(q)$ is a simple group of type D_ℓ, where $\ell \geqslant 4$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. The order of G is given by

$$|G| = \frac{1}{d} q^{\ell(\ell-1)}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1),$$

where $d = (4, q^\ell - 1) = [\mathrm{P}(\mathrm{CO}_{2\ell}(q)^0) : \mathrm{P}\Omega^+_{2\ell}(q)]$. (See [**2**, subsection 1.19] for notation.) It follows that $\rho(G)$ consists of p, the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell - 1$, and the primes dividing Φ_ℓ.

If $\ell \geq 6$, then $\mathrm{P}(\mathrm{CO}_{2\ell}(q)^0)$ has character degrees

$$\chi^\alpha(1) = q^6 \frac{(q^{\ell-4}+1)(q^{2(\ell-3)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q^2-1)^2(q^4-1)}$$

$$\chi_c(1) = \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-1}+1)}$$

$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-3}+1)}$$

(see [**20**, Lemma 2.4]). Using Lemma 5.1, these degrees are sufficient to show $\Delta(G)$ is complete when $\ell \geqslant 6$.

If $\ell = 5$, then

$$|G| = \frac{1}{d} q^{20}\Phi_1^5\Phi_2^4\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8$$

and $\mathrm{P}(\mathrm{CO}_{10}(q)^0)$ has character degree

$$\chi_1(1) = q^2\Phi_1^3\Phi_2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8.$$

Since $[\mathrm{P}(\mathrm{CO}_{10}(q)^0) : G] = (4, q^5 - 1) = d$, it follows using Lemma 5.1 that $\Delta(G)$ is complete for $\ell = 5$.

Finally, let $\ell = 4$. If $q = 2$, then $|G| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ and by the character table of G in the Atlas [**3**], G has an irreducible character of degree $\chi_{11}(1) = 210 = 2 \cdot 3 \cdot 5 \cdot 7$. If $q = 3$, then $|G| = 2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ and again by the Atlas character table, G has the degree $\chi_{17}(1) = 5460 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. Hence, $\Delta(G)$ is a complete graph if $q = 2$ or $q = 3$.

If $\ell = 4$ and $q > 3$, then

$$|G| = \frac{1}{d} q^{12}\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_6$$

and $\mathrm{P}(\mathrm{CO}_8(q)^0)$ has character degrees

$$\chi_1(1) = q^2\Phi_1^2\Phi_3\Phi_4^2\Phi_6$$
$$\chi_c(1) = \Phi_1^4\Phi_2^2\Phi_3\Phi_4^2$$
$$\chi^\beta(1) = \frac{1}{2}q^3\Phi_2^4\Phi_6.$$

Since $[\mathrm{P}(\mathrm{CO}_8(q)^0) : G] = (4, q^4 - 1) = d$, these degrees, along with Lemma 5.1, show that $\Delta(G)$ is complete in this case as well. See [**20**] for details. □

Theorem 5.10 [**20**, Theorem 1.1]. *If $G \cong \mathrm{P}\Omega_{2\ell}^-(q)$ is a simple group of type $^2D_\ell$, where $\ell \geqslant 4$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. The order of G is given by

$$|G| = \frac{1}{d}q^{\ell(\ell-1)}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell+1),$$

where $d = (4, q^\ell + 1) = [\mathrm{P}(\mathrm{CO}_{2\ell}^-(q)^0) : \mathrm{P}\Omega_{2\ell}^-(q)]$. (See [**2**, subsection 1.19] for notation.) It follows that $\rho(G)$ consists of p, the primes dividing Φ_j or Φ_{2j} for $j = 1, \dots, \ell - 1$, and the primes dividing $\Phi_{2\ell}$.

If $\ell \geqslant 5$, then $\mathrm{P}(\mathrm{CO}_{2\ell}^-(q)^0)$ has character degrees

$$\chi^\alpha(1) = \frac{1}{2}q^3\frac{(q^{\ell-3}-1)(q^{\ell-2}+1)(q^{\ell-1}-1)(q^\ell+1)}{(q^2+1)(q-1)^2}$$
$$\chi_c(1) = (q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)$$
$$\chi_1(1) = q^2\frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell+1)}{q^{\ell-2}+1}$$

(see [**20**, Lemma 2.5]). Using Lemma 5.1, these degrees are sufficient to show $\Delta(G)$ is complete when $\ell \geqslant 5$.

If $\ell = 4$, then

$$|G| = \frac{1}{d} q^{12} \Phi_1^3 \Phi_2^3 \Phi_3 \Phi_4 \Phi_6 \Phi_8$$

and $\mathrm{P}(\mathrm{CO}_8^-(q)^0)$ has character degrees

$$\begin{aligned} \chi_c(1) &= \Phi_1^3 \Phi_2^3 \Phi_3 \Phi_4 \Phi_6, \\ \chi_1(1) &= q^2 \Phi_1 \Phi_2 \Phi_3 \Phi_6 \Phi_8, \\ \chi_2(1) &= q^3 \Phi_1 \Phi_3 \Phi_4 \Phi_8. \end{aligned}$$

Since $[\mathrm{P}(\mathrm{CO}_8^-(q)^0) : G] = (4, q^4 + 1) = d$, these degrees, along with Lemma 5.1, show that $\Delta(G)$ is complete for $\ell = 4$. See [**20**] for details. □

6. Diameters of degree graphs. Combining the results above, we have the following theorem summarizing results on connectedness and diameters of degree graphs of simple groups. This result is used in [**12**] to improve the bound on the diameter of the degree graph for nonsolvable groups found in [**11**].

Theorem 6.1 [**20**, Corollary 1.2]. *Let G be a finite simple group. The graph $\Delta(G)$ is disconnected if and only if $G \cong \mathrm{PSL}_2(q)$ for some prime power q. If $\Delta(G)$ is connected, then the diameter of $\Delta(G)$ is at most 3 and $\Delta(G)$ is a complete graph except in the following cases*:

1. *The diameter of $\Delta(G)$ is 3 if and only if $G \cong J_1$.*

2. *The diameter of $\Delta(G)$ is 2 if and only if G is isomorphic to one of*

(a) *the sporadic Mathieu group M_{11} or M_{23},*

(b) *the alternating group A_8,*

(c) *the Suzuki group ${}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geqslant 1$,*

(d) *the linear group $\mathrm{PSL}_3(q)$, where $q > 2$ is even or q is odd and $q - 1$ is divisible by a prime other than 2 or 3, or*

(e) *the unitary group $\mathrm{PSU}_3(q^2)$, where $q > 2$ and $q + 1$ is divisible by a prime other than 2 or 3.*

APPENDIX

A. Tables of orders and degrees.

TABLE 1. Orders of sporadic groups.

Group	Order
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

TABLE 2. Degrees of sporadic groups.

Group	Char	Degree
M_{12}	χ_8	$55 = 5 \cdot 11$
	χ_{11}	$66 = 2 \cdot 3 \cdot 11$
	χ_{13}	$120 = 2^3 \cdot 3 \cdot 5$
M_{22}	χ_5	$55 = 5 \cdot 11$
	χ_6	$99 = 3^2 \cdot 11$
	χ_7	$154 = 2 \cdot 7 \cdot 11$
	χ_8	$210 = 2 \cdot 3 \cdot 5 \cdot 7$
J_2	χ_7	$63 = 3^2 \cdot 7$
	χ_8	$70 = 2 \cdot 5 \cdot 7$
	χ_{10}	$90 = 2 \cdot 3^2 \cdot 5$
HS	χ_4	$154 = 2 \cdot 7 \cdot 11$
	χ_{13}	$825 = 3 \cdot 5^2 \cdot 11$
	χ_{22}	$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$
J_3	χ_9	$816 = 2^4 \cdot 3 \cdot 17$
	χ_{10}	$1140 = 2^2 \cdot 3 \cdot 5 \cdot 19$
	χ_{13}	$1615 = 5 \cdot 17 \cdot 19$
M_{24}	χ_{10}	$770 = 2 \cdot 5 \cdot 7 \cdot 11$
	χ_{14}	$1035 = 3^2 \cdot 5 \cdot 23$
	χ_{21}	$3312 = 2^4 \cdot 3^2 \cdot 23$
	χ_{23}	$5313 = 3 \cdot 7 \cdot 11 \cdot 23$
M^cL	χ_3	$231 = 3 \cdot 7 \cdot 11$
	χ_5	$770 = 2 \cdot 5 \cdot 7 \cdot 11$
	χ_{12}	$4500 = 2^2 \cdot 3^2 \cdot 5^3$

TABLE 3. Degrees of sporadic groups (continued).

Group	Char	Degree
He	χ_{12}	$1920 = 2^7 \cdot 3 \cdot 5$
	χ_{17}	$7497 = 3^2 \cdot 7^2 \cdot 17$
	χ_{25}	$11900 = 2^2 \cdot 5^2 \cdot 7 \cdot 17$
Ru	χ_{11}	$27000 = 2^3 \cdot 3^3 \cdot 5^3$
	χ_{21}	$52780 = 2^2 \cdot 5 \cdot 7 \cdot 13 \cdot 29$
	χ_{24}	$71253 = 3^3 \cdot 7 \cdot 13 \cdot 29$
Suz	χ_3	$364 = 2^2 \cdot 7 \cdot 13$
	χ_9	$5940 = 2^2 \cdot 3^3 \cdot 5 \cdot 11$
	χ_{13}	$15015 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
$O'N$	χ_{12}	$58311 = 3^2 \cdot 11 \cdot 19 \cdot 31$
	χ_{25}	$175770 = 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 31$
	χ_{29}	$234080 = 2^5 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
Co_3	χ_{13}	$5544 = 2^3 \cdot 3^2 \cdot 7 \cdot 11$
	χ_{22}	$31625 = 5^3 \cdot 11 \cdot 23$
	χ_{27}	$57960 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23$
Ly	χ_9	$381766 = 2 \cdot 7 \cdot 11 \cdot 37 \cdot 67$
	χ_{10}	$1152735 = 3 \cdot 5 \cdot 31 \cdot 37 \cdot 67$
	χ_{33}	$28787220 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 31 \cdot 67$
J_4	χ_{19}	$35411145 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 43$
	χ_{25}	$300364890 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
	χ_{42}	$1182518964 = 2^2 \cdot 3^2 \cdot 11^3 \cdot 23 \cdot 29 \cdot 37$

TABLE 4. More degrees of sporadic groups.

Group	Char	Degree
Co_2	χ_{28}	$212520 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	χ_{24}	$150150 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
HN	χ_{29}	$1053360 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
Th	χ_{39}	$40199250 = 2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 31$
Fi_{23}	χ_{43}	$35225190 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Co_1	χ_{36}	$9669660 = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Fi'_{24}	χ_{34}	$7150713570 = 2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
B	χ_{65}	$40955835260340 = 2^2 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
M	χ_{188}	$1982039000444238454944825600 =$
		$2^7 \cdot 3^2 \cdot 5 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

TABLE 5. Orders and degrees of alternating groups.

Group	Order	Char	Degree
Alt(9)	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	χ_8	$35 = 5 \cdot 7$
		χ_9	$42 = 2 \cdot 3 \cdot 7$
		χ_{14}	$120 = 2^3 \cdot 3 \cdot 5$
Alt(10)	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	χ_{11}	$210 = 2 \cdot 3 \cdot 5 \cdot 7$
Alt(11)	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	χ_{31}	$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
Alt(12)	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	χ_{10}	$320 = 2^6 \cdot 5$
		χ_{12}	$462 = 2 \cdot 3 \cdot 7 \cdot 11$
		χ_{19}	$1155 = 3 \cdot 5 \cdot 7 \cdot 11$
Alt(13)	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	χ_6	$220 = 2^2 \cdot 5 \cdot 11$
		χ_{36}	$6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$
		χ_{52}	$15015 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
Alt(14)	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	χ_{10}	$715 = 5 \cdot 11 \cdot 13$
		χ_{24}	$6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$
		χ_{36}	$13650 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13$

TABLE 6. Orders of exceptional groups.

Group	Order
$G_2(q)$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$
$F_4(q)$	$q^{24}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
$E_6(q)$	$q^{36}\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
$E_7(q)$	$q^{63}\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
$E_8(q)$	$q^{120}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
${}^2B_2(q^2)$	$q^4\Phi_1\Phi_2\Phi_8$
${}^3D_4(q^3)$	$q^{12}\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$
${}^2G_2(q^2)$	$q^6\Phi_1\Phi_2\Phi_4\Phi_{12}$
${}^2F_4(q^2)$	$q^{24}\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8^2\Phi_{12}\Phi_{24}$
${}^2E_6(q^2)$	$q^{36}\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$

TABLE 7. Character degrees of exceptional groups.

Group	Label	Degree
$G_2(q)$	$\phi_{2,2}$	$\frac{1}{2}q\Phi_2^2\Phi_6$
	$G_2[-1]$	$\frac{1}{2}q\Phi_1^2\Phi_3$
	X_a, χ_{10}, χ_6	$\Phi_1\Phi_2\Phi_3\Phi_6$
$F_4(q)$		
$q \neq 2$	$\phi'_{8,3}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$
	B_2, r	$\frac{1}{4}q^4\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8$
	χ_s	$\Phi_1^4\Phi_2^2\Phi_3^2\Phi_4^2\Phi_6\Phi_8\Phi_{12}$
$E_6(q)$	$\phi_{64,4}$	$q^4\Phi_2^3\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
	$D_4, 1$	$\frac{1}{2}q^3\Phi_1^4\Phi_3^2\Phi_5\Phi_9$
	χ_{s_1}	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_{12}$
	χ_{s_2}	$\Phi_1^6\Phi_2^4\Phi_3^2\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9$
	$\phi_{60,5}$	$q^5\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$

TABLE 7. Character degrees of exceptional groups (continued).

Group	Label	Degree
$E_7(q)$	$\phi_{512,12}$	$\frac{1}{2}q^{11}\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
	$E_7[\xi]$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}$
	χ_s	$\Phi_1^7\Phi_2^6\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$
	D_4, σ_2	$\frac{1}{2}q^{10}\Phi_1^4\Phi_3^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{18}$
	$\phi_{512,11}$	$\frac{1}{2}q^{11}\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
$E_8(q)$	$\phi_{4096,12}$	$\frac{1}{2}q^{11}\Phi_2^7\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
	$E_7[\xi], 1$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$
	$\phi_{1344,19}$	$\frac{1}{4}q^{16}\Phi_2^4\Phi_4^2\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
	$D_4, \phi_{16,5}$	$\frac{1}{4}q^{16}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_5^2\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$
	$D_4, \phi'_{2,4}$	$\frac{1}{2}q^4\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{30}$

TABLE 8. Character degrees of twisted exceptional groups.

Group	Label	Degree
${}^3D_4(q^3)$	${}^3D_4[-1]$	$\frac{1}{2}q^3\Phi_1^2\Phi_3^2$
	$\phi_{2,1}$	$\frac{1}{2}q^3\Phi_2^2\Phi_6^2$
	$\phi'_{1,3}$	$q\Phi_{12}$
	χ_8	$\Phi_1\Phi_2\Phi_3\Phi_6^2\Phi_{12}$
${}^2G_2(q^2)$	cuspidal	$\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$
	cuspidal	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi'_{12}$
	cuspidal	$\frac{1}{2\sqrt{3}}(q\Phi_1\Phi_2\Phi''_{12}$
	ξ_2	Φ_{12}
	η_i^-	$\Phi_1\Phi_2\Phi_4\Phi''_{12}$
	η_i^+	$\Phi_1\Phi_2\Phi_4\Phi'_{12}$

TABLE 8. Character degrees of twisted exceptional groups (continued).

Group	Label	Degree
${}^2F_4(q^2)$	${}^2B_2[a],1$	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_{12}$
	ρ_2	$\frac{1}{2}q^4\Phi_8^2\Phi_{24}$
	χ_s	$\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8\Phi_{12}\Phi_{24}$
${}^2E_6(q^2)$	${}^2A_5,1$	$q^4\Phi_1^3\Phi_3^2\Phi_4^2\Phi_8\Phi_{12}$
	$\phi'_{8,3}$	$\frac{1}{2}q^3\Phi_2^4\Phi_6^2\Phi_{10}\Phi_{18}$
	χ_{s_1}	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}$
	χ_{s_2}	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{10}\Phi_{18}$
	$\phi'_{4,7}$	$q^5\Phi_4\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$

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