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ON THE HENSTOCK-KURZWEIL-DUNFORD AND KURZWEIL-HENSTOCK-PETTIS INTEGRALS

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ABSTRACT. In this paper, we discuss the Kurzweil-Henstock-Dunford integral and Kurzweil-Henstock-Pettis integral of the functions mapping a compact interval into a Banach space. We firstly show that the Pettis and Dunford integrability for measurable functions are equivalent if and only if the Banach space contains no copy of c_0. Then we prove that the Kurzweil-Henstock-Pettis and Kurzweil-Henstock-Dunford integrability for measurable functions are equivalent if and only if the Banach space is weakly sequentially complete. The equivalence results on the Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrability are also discussed in Schur spaces.

1. Introduction. It is well known that the Kurzweil-Henstock integral of real-valued functions is a kind of nonabsolute integral that contains the Lebesgue integral and equals the Perron integral. The Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrals are generalizations of the Kurzweil-Henstock integral of the real functions to Banach space-valued functions, see [**6**]. The relationships between the Kurzweil-Henstock-Pettis integral and Pettis, Kurzweil-Henstock-Dunford and Dunford integrals for Banach-space-valued functions were discussed in [**6**]. It can be seen from the corresponding definitions that a Kurzweil-Henstock integrable function is Kurzweil-Henstock-Pettis integrable and a Kurzweil-Henstock-Pettis integrable function is Kurzweil-Henstock-Dunford integrable, but the reverse does not hold. An example shows that the Kurzweil-Henstock-Dunford integrability of Banach-valued functions cannot imply Kurzweil-Henstock-Pettis integrability. We would like to know what is the relationship between the Kurzweil-Henstock-Pettis and Kurzweil-Henstock-Dunford integrability in Banach spaces? In this paper we study this problem

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and give some interesting results. Firstly, we prove that the Pettis and Dunford integrability for measurable functions are equivalent if and only if the Banach space contains no copy of c_0, and then prove that the Kurzweil-Henstock-Pettis and Kurzweil-Henstock-Dunford integrability for measurable functions are equivalent if and only if the Banach space is weakly sequentially complete. Moreover, in Schur spaces equivalence results on the Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrability for measurable functions are discussed.

2. Basic definitions.

Throughout this paper, X denotes a real Banach space with norm $\|\cdot\|$ and X^* its dual. $B(X^*) = \{x^* \in X^*;\ \|x^*\| \le 1\}$ is the closed unit ball in X^*. Let $I_0 = [a, b]$ be a compact interval in $\mathbf{R}^1$ and $E \subset \mathbf{R}^1$ a measurable subset of I_0. $\mu(E)$ stands for the Lebesgue measure. The Lebesgue integral of a function f over a set E will be denoted by $(L)\int_E f$.

We say that intervals I and J are nonoverlapping if $\operatorname{int}(I) \cap \operatorname{int}(J) = \varnothing$. By $\operatorname{int}(J)$ the interior of J is denoted.

A *partial M-partition* D in I_0 is a finite collection of interval-point pairs (I, ξ) with nonoverlapping intervals $I \subset I_0$, $\xi \in I_0$ being the associated point of I. Requiring $\xi \in I$ for the associated point of I, we get the concept of a *partial K-partition* D in I_0. We write $D = \{(I, \xi)\}$.

A partial M-partition $D = \{(I, \xi)\}$ in I_0 is an *M-partition* of I_0 if the union of all the intervals I equals I_0 and similarly for a K-partition.

Let δ be a positive function defined on the interval I_0. A partial M-partition (K-partition) $D = \{(I, \xi)\}$ is said to be δ-fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$, where $B(\xi, \delta(\xi)) = (\xi - \delta(\xi), \xi + \delta(\xi))$.

Definition 2.1. An X-valued function f is said to be *McShane integrable* on I_0 if there exists an $S_f \in X$ such that for every $\varepsilon > 0$ there exists a $\delta(\xi) > 0$ such that for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0, we have

$$\left\| \sum_D f(\xi)\mu(I) - S_f \right\| < \varepsilon.$$

We write $(M)\int_{I_0} f = S_f$ and call S_f the *McShane integral* of f over I_0.

f is McShane integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E$ is McShane integrable on I_0, where χ_E denotes the characteristic function of E. We write $(M)\int_E f = (M)\int_{I_0} f\chi_E = F(E)$ for the McShane integral of f on E.

Denote the set of all McShane integrable functions $f : I_0 \mapsto X$ by $\mathcal{M}$.

Replacing the term "M-partition" by "K-partition" in the definition above, we obtain *Kurzweil-Henstock integrability* and the definition of the *Kurzweil-Henstock integral* $(KH)\int_{I_0} f$.

It is clear that if $f : I_0 \mapsto X$ is McShane integrable, then it is also Kurzweil-Henstock integrable because every K-partition is an M-partition.

It is known that linearity, integrability on subintervals, additivity of intervals of McShane and Kurzweil-Henstock integrals hold. For details, see [**2, 3, 5**–**10**].

Definition 2.2. (a) A function $f : I_0 \to X$ is *Kurzweil-Henstock-Dunford integrable* if for each x^* in X^* the function x^*f is Kurzweil-Henstock integrable on I_0 and for each interval I in I_0 there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. We write $x_{I_0}^{**} = (KHD)\int_{I_0} f = F(I_0)$, and F is the primitive of f on I_0.

(b) A function $f : I_0 \to X$ is *Kurzweil-Henstock-Pettis integrable* on I_0 if f is Kurzweil-Henstock-Dunford integrable on I_0 and $x_I^{**} \in X$ for every interval I in I_0. We write $x_{I_0}^{**} = (KHP)\int_{I_0} f = F(I_0)$.

For simplicity, the letters M, KH, KHD and KHP stand for McShane, Kurzweil-Henstock, Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis, respectively, and we denote the sets of all McShane, Kurzweil-Henstock, Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrable functions $f : I_0 \mapsto X$ by $\mathcal{M}$, $\mathcal{KH}$, $\mathcal{KHD}$, $\mathcal{KHP}$, respectively.

From the corresponding definitions of different integrals, we have

$$\mathcal{M} \subset \mathcal{KH} \subset \mathcal{KHP} \subset \mathcal{KHD}.$$

For further discussion of the McShane and Kurzweil-Henstock integrals, see [**2, 3, 5**–**10**].

Remark. A function f is scalarly Kurzweil-Henstock integrable on I_0 if for each x^* in X^* the function x^*f is Kurzweil-Henstock integrable on I_0. It is well known from [**6**, Theorem 8.2.26] that a function f is scalarly Kurzweil-Henstock integrable on I_0 then there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. Therefore, by Definition 2.2 f is Kurzweil-Henstock-Dunford integrable on I_0 if f is scalarly Kurzweil-Henstock integrable on I_0. This means that a function f is Kurzweil-Henstock-Dunford integrable on I_0 if and only if f is scalarly Kurzweil-Henstock integrable on I_0.

3. The main results.

In this section the main theorems are Theorem 3.3–Theorem 3.5. We first show an equivalence result of the Dunford integral and the Pettis integral.

Theorem 3.1. *Suppose that a function $f : I_0 \to X$ is measurable. Then the Dunford and Pettis integrability of f are equivalent if and only if X contains no copy of c_0.*

Proof. Suppose that X contains no copy of c_0 and f is measurable on I_0. Then by [**6**, Proposition 1.1.9], there exists a bounded measurable $g : I_0 \to X$ and a measurable $h : I_0 \to X$ with

$$h(t) = \sum_{k=1}^{\infty} x_k \chi_{E_k}(t), \quad x_k \in X,\ k \in \mathbf{N},\ t \in I_0,$$

where $E_k \subset I_0$, $k \in \mathbf{N}$, are pairwise disjoint measurable sets such that $f = g + h$. Obviously, g is Bochner integrable on I_0 and therefore g is Pettis integrable on I_0.

If f is Dunford integrable, then $h = f - g$ is Dunford integrable and

$$(\text{Dunford}) \int_{I_0} h = (\text{Dunford}) \int_{I_0} f - (\text{Pettis}) \int_{I_0} g.$$

Hence, for each $x^* \in X^*$, x^*h is Lebesgue integrable on I_0. It follows that

$$(L) \int_{I_0} |x^*h| = \sum_{k=1}^{\infty} |x^*(x_k)| \mu(E_k) < \infty.$$

Moreover, for every measurable set $E \subset I_0$,

$$(L)\int_E |x^*h| = \sum_{k=1}^{\infty} |x^*(x_k)|\mu(E\cap E_k) < \infty.$$

This means that $\sum_{n=1}^{\infty} x^*(x_n)\mu(E_n \cap E)$ is absolutely convergent. Since X contains no copy of c_0, by the Bessaga-Pelczynski theorem [**1**, page 22], $\sum_{n=1}^{\infty} x_n\mu(E_n\cap E)$ is unconditionally convergent. Consequently, there exists an $x_E \in X$ such that $x_E = \sum_{k=1}^{\infty} x_k\mu(E\cap E_k)$ and

$$(L)\int_E x^*h = \sum_{k=1}^{\infty} x^*(x_k)\mu(E\cap E_k) = x^*(x_E).$$

We obtain

$$(\text{Dunford})\int_E h = \sum_{n=1}^{\infty}(x_n)\mu(E_n\cap E) = x_E \in X.$$

Hence, h is Pettis integrable on I_0. It follows from $f = g + h$ that f is Pettis integrable on I_0.

Conversely, the following example shows that if the Dunford and Pettis integrability for measurable functions f are equivalent, then X contains no copy of c_0.

Define $f : [0,1] \to c_0$ by

$$f(t) = (\chi_{[0,1]}(t), 2\chi_{[0,1/2]}(t), \dots, n\chi_{[0,1/n]}(t), \dots).$$

For every $x^* \in c_0^* = l^1$, let $x^* = g = (g_1, g_2, \dots, g_n, \dots)$. Then $x^*f(t) = \sum_{n=1}^{\infty} ng_n\chi_{[0,1/n]}(t)$ and $\int_0^1 |x^*f(t)| = \sum_{n=1}^{\infty}|g_n| < \infty$. This means that f is Dunford integrable on $[0,1]$, but $\int_0^1 f = (1,1,\dots,1,\dots) \notin c_0$. So f is not Pettis integrable on $[0,1]$. This leads to a contradiction. Hence, X contains no copy of c_0. □

Now we would like to know under what conditions are the Kurzweil-Henstock-Pettis and Kurzweil-Henstock-Dunford integrability equivalent? Therefore, we need the following two results.

Theorem 3.2. *A function $f : I_0 \to X$ is Kurzweil-Henstock-Dunford integrable if and only if for each closed set $E \subset I_0$ there exists a portion $P = E \cap I$ of E on which f is Dunford integrable.*

Proof. Since f is Kurzweil-Henstock-Dunford integrable on I_0, for each $x^* \in X^*$, x^*f is Kurzweil-Henstock integrable and therefore the primitive $x^*F(t) = \int_{[a,t]} x^*f$ is ACG^*. By [**6**, Theorems 8.2.8, 8.2.9 and 7.2.3], for each closed set $E \subset I_0$ there exists a portion $P = E \cap I$ of E such that for each $x^* \in X^*$ x^*f is Lebesgue integrable on P. Hence, f is Dunford integrable on P. The reverse process is also valid. □

Theorem 3.3. *Suppose that X is weakly sequentially complete. Then a function $f : I_0 \to X$ is Kurzweil-Henstock-Pettis integrable if and only if for each closed set $E \subset I_0$ there exists a portion $P = E \cap I$ of E on which f is Pettis integrable.*

Proof. (*Necessity*). Since f is Kurzweil-Henstock-Pettis integrable on I_0, f is Kurzweil-Henstock-Dunford integrable on I_0. By Theorem 3.2, for each closed set $E \subset I_0$ there exists a portion $P = E \cap I$ of E on which f is Dunford integrable. In what follows, we prove f being Pettis integrable on P.

By Kurzweil-Henstock-Pettis integrability on subintervals, for each interval J in I_0 we have $(KHP)\int_J \in X$.

Let G be an open set in I_0, x^*f Kurzweil-Henstock integrable on G and x^*f McShane integrable on $I_0 \setminus G$ for each $x^* \in X^*$. Then G can be expressed as the union of nonoverlapping open intervals J_n and

$$(KH)\int_G x^*f = \sum_{n=1}^{\infty}(KH)\int_{J_n} x^*f = \sum_{n=1}^{\infty} x^*(KHP)\int_{J_n} f, \quad x^* \in X^*.$$

Since X is weakly sequentially complete, $\sum_{n=1}^{\infty}(KHP)\int_{J_n} f$ exists in X. Denote $x_G = \sum_{n=1}^{\infty}(KHP)\int_{J_n} f$; then $x_G \in X$.

Now suppose that A is any measurable subset of P. It follows from the Dunford integrability of f on P that f is Dunford integrable on $A \subset P$ and therefore x^*f is McShane integrable on A for each $x^* \in X^*$.

If A is an open subset of P, then from the above discussion there exists an $x_A \in X$ such that $(M)\int_A x^*f = x^*(x_A)$.

If A is a closed subset of P, then $G = I_0 \setminus A$ is open. There is an $x_G = x_{I_0 \setminus A} \in X$ such that $(M)\int_A x^*f = (KH)\int_{I_0} x^*f - (KH)\int_G x^*f = x^*((KHP)\int_{I_0} f - x_G)$. It follows that there exists an $x_A = (KHP)\int_{I_0} f - x_G$ such that $(M)\int_A x^*f = x^*(x_A)$.

If A is any measurable subset of P, then a family of closed sets F_n exists such that

$$F_n \subset F_{n+1}, \quad \bigcup_{n=1}^{\infty} F_n = A,$$

and $x_{F_n} \in X$ such that $(M)\int_{F_n} x^*f = x^*(x_{F_n})$. Since

$$\lim_{n\to\infty} (M)\int_{F_n} x^*f = \lim_{n\to\infty} x^*(x_{F_n}) = (M)\int_A x^*f$$

and X is weakly sequentially complete, there exists an $x_A \in X$ such that $(M)\int_A x^*f = x^*(x_A)$. Hence, we obtain that f is Pettis integrable on P.

(*Sufficiency*). Suppose f is not Kurzweil-Henstock-Pettis integrable on I_0. Denote by Δ the family of the closed intervals $I \subset I_0$ such that f is Kurzweil-Henstock-Pettis integrable on I. Since the Kurzweil-Henstock-Pettis integral has integrability on the subintervals and the additivity for intervals, with no loss of generality, suppose that the I's are pairwise nonoverlapping intervals. By using the Cauchy extension property, we further suppose that $I \cap J = \varnothing$ when $I, J \in \Delta$. That is, if $I = [c,d] \in \Delta$, then for every $\eta > 0$, $I_1 = [c-\eta, d] \notin \Delta$ and $I_2 = [c, d+\eta] \notin \Delta$. Obviously, Δ is not empty and at most countable. We write $\Delta = \{I_n\}_{n=1}^{\infty}$.

Let $E = I_0 \setminus \cup_{I_n \in \Delta} I_n^0$ and I_n^0 be the interior of I_n. Then E is a closed set and $I_0 = E \cup (\cup_n I_n^0)$.

We will prove that E contains only two endpoints of I_0. Otherwise, suppose that E contains an inner point of I_0. Since E is a closed set, there exists an open interval K with endpoints in E such that $E \cap K \neq 0$ and f is Pettis integrable on $E \cap K$. By the property of the Pettis integral, for every interval $\tilde{I} \subseteq K$, f is Pettis integrable on $E \cap \tilde{I}$ and

$$(P)\int_{\tilde{I}} f\chi_E = (P)\int_{E\cap\tilde{I}} f \in X.$$

Since f is Kurzweil-Henstock-Pettis integrable on each I_n, for each $J \subseteq I_n$, f is Kurzweil-Henstock-Pettis integrable on J and

$$(KH)\int_J x^* f = x^*(KHP)\int_J f, \quad (KHP)\int_J f \in X.$$

Especially, $(KHP)\int_{\tilde{I}\cap I_n} f \in X$ for each $n \in \mathbf{N}$.

Note that $\tilde{I} = (\tilde{I} \cap E) \cup (\cup_n(\tilde{I} \cap I_n^0))$ and $(\tilde{I} \cap I_n)$ are pairwise nonoverlapping intervals. Let $G = \cup_n(\tilde{I} \cap I_n^0) = \tilde{I} \setminus (\tilde{I} \cap E)$. Then f is Kurzweil-Henstock-Pettis integrable on G and, for each $x^* \in X^*$, $(KH)\int_G x^* f = \sum_{n=1}^{\infty}(KH)\int_{\tilde{I}\cap I_n} x^* f = \sum_{n=1}^{\infty} x^*(KHP)\int_{\tilde{I}\cap I_n} f$.

Since X is weakly sequentially complete, $\sum_{n=1}^{\infty}\int_{\tilde{I}\cap I_n} f$ exists in X.

Moreover,

$$\begin{aligned}(KH)\int_{\tilde{I}} x^* f &= (L)\int_{\tilde{I}\cap E} x^* f + \sum_{n=1}^{\infty}(KH)\int_{\tilde{I}\cap I_n} x^* f \\ &= x^*(P)\int_{\tilde{I}\cap E} f + x^*\sum_{n=1}^{\infty}(KHP)\int_{\tilde{I}\cap I_n} f \\ &= x^*((P)\int_{\tilde{I}\cap E} f + \sum_{n=1}^{\infty}(KHP)\int_{\tilde{I}\cap I_n} f).\end{aligned}$$

By

$$(P)\int_{\tilde{I}\cap E} f + \sum_{n=1}^{\infty}(KHP)\int_{\tilde{I}\cap I_n} f \in X$$

and the randomness of $\tilde{I} \subseteq K$, we obtain that f is Kurzweil-Henstock-Pettis integrable on K and

$$(KHP)\int_K f = (P)\int_{K\cap E} f + \sum_{n=1}^{\infty}(KHP)\int_{K\cap I_n} f.$$

So there is an $I_{n_0} \in \Delta$ such that $K = I_{n_0}^0$. On one hand, by the hypothesis of $E = I_0 \setminus \cup_{I_n\in\Delta} I_n^0$, $K \cap E = I_{n_0}^0 \cap E = \varnothing$. On the other hand, $K \cap E = I_{n_0}^0 \cap E \neq \varnothing$. This is a contradiction. So f is Kurzweil-Henstock-Pettis integrable on I_0. □

Theorem 3.2 and Theorem 3.3 may serve as alternative definitions of the Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrals.

Theorem 3.4. *Suppose that X is weakly sequentially complete and $f : I_0 \to X$ is a measurable function. If f is Kurzweil-Henstock-Dunford integrable on I_0, then f is Kurzweil-Henstock-Pettis integrable on I_0.*

Proof. Since f is Kurzweil-Henstock-Dunford integrable on I_0, by Theorem 3.2, for each closed set $E \subset I_0$, there exists a portion $P = E \cap I$ on which f is Dunford integrable. Since X is weakly sequentially complete, therefore X contains no copy of c_0. It follows from Theorem 3.1 that f is Pettis integrable on $P = E \cap I$. By Theorem 3.3, f is Kurzweil-Henstock-Pettis integrable on I_0. □

Theorem 3.5. *The Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrability for measurable functions on I_0 are equivalent if and only if X is weakly sequentially complete.*

Proof. The sufficiency follows from Theorem 3.4.

Conversely, if the Kurzweil-Henstock-Dunford and Kurzweil-Henstock-Pettis integrability for the measurable functions on I_0 are equivalent, we prove that X is weakly sequentially complete.

Suppose X is not weakly sequentially complete. Then there exists a series $\sum_{n=1}^{\infty} x_n$ in X such that the series $\sum_{n=1}^{\infty} x^*(x_n)$ converges for each x^* in X^* but the series $\sum_{n=1}^{\infty} x_n$ converges *weak** to $x_0^{**} \in X^{**} - X$. For each positive integer n, let $I_n = ((1/n+1), 1/n]$.

Define $f : [0,1] \to X$ by

$$f(t) = \sum_{n=1}^{\infty} \frac{x_n}{\mu(I_n)} \chi_{I_n}(t), f(0) = 0.$$

Then the function f is measurable. For every $x^* \in X^*$, $x^* f$ is McShane integrable on $[a, 1]$ for every $a \in (0, 1)$.

Especially, for $a \in ((1/N+1), 1/N)$,

$$\int_a^1 x^* f = \int_a^{1/N} x^* f + \int_{1/N}^1 x^* f = \frac{x^*(x_N)}{\mu(I_N)}(1/N - a) + \sum_{n=1}^{N-1} x^*(x_n).$$

Since $\sum_{n=1}^{\infty} x^*(x_n)$ converges,

$$\lim_{a\to 0+} \int_a^1 x^* f = \sum_{n=1}^{\infty} x^*(x_n).$$

Hence, $x^* f$ is Kurzweil-Henstock integrable on $[0,1]$ and

$$(KH) \int_0^1 x^* f = \sum_{n=1}^{\infty} x^*(x_n) = x_0^{**}(x^*).$$

This means that f is Kurzweil-Henstock-Dunford integrable on $[0,1]$, but not Kurzweil-Henstock-Pettis integrable on $[0,1]$. This leads to a contradiction. □

Corollary 3.1. *Assume that X is weakly sequentially complete. If X is separable and function f is Kurzweil-Henstock-Dunford integrable on I_0, then f is Kurzweil-Henstock-Pettis integrable on I_0.*

Proof. If f is Kurzweil-Henstock-Dunford integrable on I_0, then f is weakly measurable. Since X is separable, f is measurable. By Theorem 3.5, f is Kurzweil-Henstock-Pettis integrable on I_0. □

Recall that a Banach space X is a Schur space if weakly convergent sequences in X are norm convergent.

Corollary 3.2. *Assume that X is a Schur space and $f : I_0 \to X$ is measurable. If f is Kurzweil-Henstock-Dunford integrable on I_0, then f is Kurzweil-Henstock-Pettis integrable on I_0.*

Proof. Since a Schur space is weakly sequentially complete, by Theorem 3.5, f is Kurzweil-Henstock-Pettis integrable on I_0. □

Theorem 3.6. *If $f : I_0 \to X$ is weakly continuous and bounded, then f is Kurzweil-Henstock-Pettis integrable on I_0.*

Proof. Since f is weakly continuous and bounded, f is measurable and Dunford integrable. It follows that f is Pettis integrable. Hence, f is Kurzweil-Henstock-Pettis integrable on I_0. □

Theorem 3.7. *If $F : I_0 \to X$ is weakly differentiable with the weak derivative f, then f is Kurzweil-Henstock-Pettis integrable on I_0 and the Newton-Leibniz formula holds, that is, $(KHP)\int_a^t f = F(t) - F(a)$.*

Proof. Since F is weakly differentiable with the weak derivative f, for each $x^* \in X^*$, $(x^*F)'(t) = x^*f(t)$ for $t \in I_0$. So, x^*f is Kurzweil-Henstock integrable on I_0 and $(KH)\int_a^t x^*f = x^*(F(t) - F(a))$. By $F(t) - F(a) \in X$, we obtain that f is Kurzweil-Henstock-Pettis integrable on I_0 and $(KHP)\int_a^t f = F(t) - F(a)$. □

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