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SOBOLEV GRADIENTS IN UNIFORMLY CONVEX SPACES

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1. Introduction. The main idea of this paper is to show how the Beurling-Deny theorem presented in [**11**] can be extended to find a function from the uniformly convex Sobolev space $H^{1,p}[0,1]$ to the space $L_p[0,1]$, $p > 2$. We also look at the possibility of using that function to establish a relationship between the ordinary gradient $\nabla\varphi$ associated with the Euclidean norm in R^{n+1} and the p-gradient $\nabla_p\varphi$ of a C^1 function φ defined on the uniformly convex Banach space R^{n+1} with the p-norm

$$\|h\| = \left(\sum_{i=1}^{n} \left(\left| \frac{h_i - h_{i-1}}{\delta} \right|^p + \left| \frac{h_i + h_{i-1}}{2} \right|^p \right) \right)^{1/p}, \quad h = (h_0, h_1, \dots, h_n) \in R^{n+1}, \quad \delta = \frac{1}{n}, \tag{1}$$

which is a finite-dimensional emulation of the Sobolev norm

$$\|f\| = \left(\int_0^1 |f|^p + |f'|^p \right)^{1/p}, \quad f \in H^{1,p}[0,1], \tag{2}$$

in the Sobolev space $H^{1,p}[0,1]$.

In a previous work [**16**, page 4], we had

$$(\nabla\varphi)(x) = D^t Q(D(\nabla_p\varphi)(x)), \tag{3}$$

where D_0, D_1 are functions from R^{n+1} to R^n such that

$$D_0 h = \begin{pmatrix} h_1 + h_0/2 \\ h_2 + h_1/2 \\ \vdots \\ h_n + h_{n-1}/2 \end{pmatrix}, \quad D_1 h = \begin{pmatrix} h_1 - h_0/\delta \\ h_2 - h_1/\delta \\ \vdots \\ h_n - h_{n-1}/\delta \end{pmatrix}.$$

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D is a function from R^{n+1} to $R^n \times R^n$ such that

$$Dh = \begin{pmatrix} D_0 h \\ D_1 h \end{pmatrix}, \quad \text{for all} \quad h \in R^{n+1}.$$

D^t is the adjoint of D as defined in [**13**], and

$$Q(t) = \operatorname{diag}\left(pt_1 \left|t_1\right|^{p-2}, pt_2 \left|t_2\right|^{p-2}, \dots, pt_{2n} \left|t_{2n}\right|^{p-2}\right), \\ \text{for all} \quad t = (t_1, t_2, \dots, t_{2n}) \in R^{2n}.$$

The relationship (3) between the two gradients generalizes the following one found in [**12**, page 24]:

$$(\nabla\varphi)(x) = (D^t D)(\nabla_2 \varphi)(x), \quad \text{for all} \quad x \in R^{n+1},$$

where $p = 2$ and R^{n+1} is then a Hilbert space. $(\nabla_2\varphi)(x)$ is called the Sobolev gradient of φ at x.

The paper also shows with a detailed proof that the dual space $H^{1,q}[0,1]^*$ of the space $H^{1,q}[0,1]$, $q \neq 2$, is isomorphic to the space $H^{1,p}[0,1]$, where $1/p + 1/q = 1$.

2. Duals of Sobolev spaces. In this section, we present a useful characterization of the dual space of the space $H^{1,p}[0,1]$ with the Sobolev norm (2). Some other characterizations can be found in [**1**].

Since the dual space of the Hilbert space $H^{1,2}[0,1]$ is the dual of the space $H^{1,2}[0,1]$ itself, we will be interested in working with the space $H^{1,p}[0,1]$, $p \neq 2$. The fact that the space $L_p[0,1]$ is isomorphic to the dual space of $L_q[0,1]$, where $1/p+1/q = 1$ has given us some motivation to show that the space $H^{1,p}[0,1]$ is isomorphic to the dual space of the space $H^{1,q}[0,1]$ with $1/p + 1/q = 1$.

Theorem 1. *The dual space* $(H^{1,q}[0,1])^*$ *of the space* $H^{1,q}[0,1]$, $q \neq 2$, *is isomorphic to the space* $H^{1,p}[0,1]$, *where* $1/p + 1/q = 1$.

Proof. Suppose $q < 2$. Define the function $F: H^{1,p}[0,1] \to (H^{1,q}[0,1])^*$ as follows: for every f in $H^{1,p}[0,1]$,

$$F(f)(g) = \int_0^1 fg + f'g', \quad \text{for all} \quad g \in H^{1,q}[0,1],$$

and denote $F(f)$ by F_f. F is clearly linear. We intend to show that F is a well defined, one-to-one, and onto function.

$$\begin{aligned}|F_f(g)| &= \left|\int_0^1 fg + f'g'\right| \leq \left|\int_0^1 fg\right| + \left|\int_0^1 f'g'\right| \\ &\leq \|f\|_{L^p[0,1]} \cdot \|g\|_{L^q[0,1]} + \|f'\|_{L^p[0,1]} \cdot \|g'\|_{L^q[0,1]} \\ &\leq \left(\|f\|^p_{L^p[0,1]} + \|f'\|^p_{L^p[0,1]}\right)^{1/p} \left(\|g\|^q_{L^q[0,1]} + \|g'\|^q_{L^q[0,1]}\right)^{1/q} \\ &= \|f\|_{H^{1,p}[0,1]} \|g\|_{H^{1,q}[0,1]}\,.\end{aligned}$$

Hence,

$$|F_f| \leq \|f\|_{H^{1,p}[0,1]}\,.$$

Therefore,

$$F_f \in \left(H^{1,q}[0,1]\right)^*$$

and consequently F is well defined.

Now to show that F is one-to-one we need to show that if $F_f = 0$, then $f = 0$. Suppose $f \in H^{1,p}[0,1]$ so that $F_f = 0$

$$0 = |F_f| \geq \frac{|F_f(m)|}{\|m\|_{H^{1,q}[0,1]}} = \frac{\left|\int_0^1 fm + f'm'\right|}{\|m\|_{H^{1,q}[0,1]}}$$
$$\text{for all} \quad m \in H^{1,q}[0,1], m \neq 0.$$

Hence,

$$\int_0^1 fm + f'm' = 0 \quad \text{for all} \quad m \in H^{1,q}[0,1].$$

Let $g = |f|^{p/q}(\operatorname{sgn} f)$. We intend to show that g is a member of the space $H^{1,q}[0,1]$. $f \in H^{1,p}[0,1]$ implies that

$$\int_0^1 |g|^q = \int_0^1 |f|^p < \infty.$$

Now

$$\begin{aligned}g' &= \frac{p}{q} f\,|f|^{(p/q)-2}\,(\operatorname{sgn} f) f' \\ &= \frac{p}{q}\,|f|^{(p/q)-1}\,f'.\end{aligned}$$

Recall that if α and β are two nonnegative real numbers and $0 < \lambda < 1$, then $\alpha^\lambda \beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$, see [**15**, page 112].

Suppose $q < 2$. Let $\lambda = q/p$, $\alpha = |f'|^p$, and $\beta = |f|^p$. Since $q < 2$, then $\lambda < 1$ and

$$\left(|f'|^p\right)^{q/p} \left(|f|^p\right)^{1-(q/p)} \le \frac{q}{p}\,|f'|^p + \frac{p-q}{p}\,|f|^p .$$

Hence,

$$|f'|^q\,|f|^{p-q} \le \frac{q}{p}\,|f'|^p + \frac{p-q}{p}\,|f|^p .$$

So

$$\begin{aligned}
\int_0^1 |g'|^q &= \left(\frac{p}{q}\right)^q \int_0^1 |f'|^q\,|f|^{p-q} \\
&\le \left(\frac{p}{q}\right)^q \int_0^1 \left(\frac{q}{p}\,|f'|^p + \frac{p-q}{p}\,|f|^p\right) \\
&\le \left(\frac{p}{q}\right)^q \max\left(\frac{q}{p}, \frac{p-q}{p}\right) \int_0^1 |f'|^p + |f|^p \\
&= \left(\frac{p}{q}\right)^q \max\left(\frac{q}{p}, \frac{p-q}{p}\right) \|f\|^p_{H^{1,p}[0,1]} \\
&< \infty.
\end{aligned}$$

Therefore, $g \in H^{1,q}[0,1]$. Now

$$0 = \int_0^1 fg + f'g' = \int_0^1 \left(|f|^p + \frac{p}{q}\,|f|^{(p/q)-1}\,f'^2\right) \ge \int_0^1 |f|^p .$$

Hence, $\int_0^1 |f|^p \le 0$. Therefore, $\|f\|_{L^p[0,1]} = 0$ and $f = 0$.

Now to show that F is onto let us suppose that φ is in $(H^{1,q}[0,1])^*$. We need to find $f \in H^{1,p}[0,1]$ such that $\varphi = F_f$.

Let β be the extension of φ to $L_q[0,1]$. Then there is a $g \in L_p[0,1]$ such that $\beta(v) = \int_0^1 gv$ for all $v \in L_q[0,1]$. Now for all $v \in H^{1,q}[0,1]$, we have

$$\varphi(v) = F_f(v) = \int_0^1 fv + f'v'$$

and

$$\varphi(v) = \beta(v) = \int_0^1 gv.$$

Hence,

$$\int_0^1 fv + f'v' = \int_0^1 gv \quad \text{for all} \quad v \in H^{1,q}[0,1].$$

This implies that

$$\int_0^1 (f-g)v + \int_0^1 f'v' = 0.$$

Define the function $h(t) = \int_0^t (f-g)$, so that

$$\int_0^1 h'v + \int_0^1 f'v' = 0.$$

If we integrate by parts, we get

$$v(1)h(1) - v(0)h(0) - \int_0^1 hv' + \int_0^1 f'v' = 0,$$

but $h(0) = 0$. So

$$v(1)h(1) + \int_0^1 v'(f' - h) = 0, \quad \text{for all} \quad v \in H^{1,q}[0,1].$$

If we choose v to be a nonzero constant function, we get $v(1)h(1) = 0$, and hence $h(1) = 0$. Therefore,

$$\int_0^1 (f' - h)v' = 0, \quad \text{for all} \quad v \in H^{1,q}[0,1].$$

Thus, $f' - h = 0$. So we have the following system of equations

$$\begin{pmatrix} f' \\ h' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

with the boundary condition

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ h(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(1) \\ h(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

whose solution $\binom{f}{h}$ is given by

$$\begin{aligned}(\,f\mathbf{C}h\,) = &\begin{pmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{pmatrix}\begin{pmatrix}f(0)\\ h(0)\end{pmatrix}\\ &+\begin{pmatrix}\int_0^t \cosh(t-s) & \int_0^t \sinh(t-s)\\ \int_0^t \sinh(t-s) & \int_0^t \cosh(t-s)\end{pmatrix}\begin{pmatrix}0\\ -g\end{pmatrix}\,ds.\end{aligned}$$

Since $h(0)=0$,

$$\begin{aligned}f(t) &= \cosh(t)f(0) - \int_0^t \sinh(t-s)g(s)\,ds,\\ h(t) &= \sinh(t)f(0) - \int_0^t \cosh(t-s)g(s)\,ds, \quad 0 \le t \le 1.\end{aligned}$$

Since $h(1)=0$,

$$\sinh(1)f(0) - \int_0^1 \cosh(1-s)g(s)\,ds = 0.$$

This implies that

$$f(0) = \frac{\int_0^1 \cosh(1-s)g(s)\,ds}{\sinh(1)}.$$

Hence,

$$f(t) = \frac{\cosh(t)}{\sinh(1)}\int_0^1 \cosh(1-s)g(s)\,ds - \int_0^t \sinh(t-s)g(s)\,ds,$$

and

$$\begin{aligned}|f(t)| &\le \left(\frac{e+1}{2\sinh(1)}\right)\left(\frac{e+1}{2}\right)\int_0^1 |g|\\ &\quad + \frac{e+1}{2}\int_0^1 |g|\\ &= \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]\int_0^1 |g|\\ &\le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]\|g\|_{L^p[0,1]}.\end{aligned}$$

This gives

$$|f(t)|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|g\|^p_{L^p[0,1]}.$$

Hence,

$$\int_0^1 |f|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|g\|^p_{L^p[0,1]}.$$

Also

$$h(t) = \frac{\sinh(t)}{\sinh(1)} \int_0^1 \cosh(1-s)g(s)\,ds - \int_0^t \cosh(t-s)g(s)\,ds,$$

and

$$\begin{aligned} |h(t)| &\le \frac{(e+1)^2}{4\sinh(1)} \int_0^1 |g| + \frac{e+1}{2} \int_0^1 |g| \\ &= \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right] \int_0^1 |g| \\ &\le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right] \|g\|_{L^p[0,1]}. \end{aligned}$$

This gives

$$|h(t)|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|g\|^p_{L^p[0,1]}.$$

Hence,

$$\int_0^1 |f'|^p = \int_0^1 |h|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|g\|^p_{L^p[0,1]}.$$

Therefore, $f \in H^{1,p}[0,1]$ and consequently F is onto and $(H^{1,q}[0,1])^*$ is isomorphic to $H^{1,p}[0,1]$. Now if $q > 2$, then $p < 2$ and $(H^{1,p}[0,1])^*$ is isomorphic to $H^{1,q}[0,1]$. Hence, $((H^{1,p}[0,1])^*)^*$ is isomorphic to $(H^{1,q}[0,1])^*$. Therefore, $H^{1,p}[0,1]$ is isomorphic to $(H^{1,q}[0,1])^*$. The proof of the theorem is now complete. □

The above argument can be generalized to show that $(H^{m,p}[0,1])^*$ is isomorphic to $H^{m,q}[0,1]$, where m is a nonnegative positive integer.

3. Gradients. In this section, we first present some facts from [**11**] where the Beurling-Deny theorem was used in the Hilbert space setting to establish a relationship between the ordinary gradient and the Sobolev gradient. Then we show how that theorem can be extended to find a function from $H^{1,p}[0,1]$ to $L_p[0,1]$, $p > 2$ using Theorem 1.

Theorem 2 [**11**]. *Suppose that each of H and J is a Hilbert space so that the points of J form a dense subset of H. Suppose also that $\|x\|_J \geq \|x\|_H$ for all $x \in J$. Then there is an $M \in L(H,J)$, the set of all continuous linear operators from H to J, so that*

(i) *$R(M)$ is a dense subset of J, where $R(M)$ is the range of M.*

(ii) $|M|_{L(H,J)} \leq 1$.

(iii) *M^{-1} exists.*

A proof of that theorem can be found in [**11**]. It may be useful to note how the function M is constructed.

Suppose $x \in H$. Let f be an element of H^* (the dual space of H) so that $f(z) = \langle z, x\rangle_H$, for all $z \in H$. Let g be the restriction of f to J. If $z \in J$,

$$|g(z)| = |f(z)| = |\langle z, x\rangle_H| \leq \|z\|_H \|x\|_H \leq \|z\|_J \|x\|_H .$$

Hence $g \in J^*$. So there is a unique y in J so that $g(z) = \langle z, y\rangle_J$ for all $z \in J$. Denote y by Mx. M is clearly a linear function from H to J. M^{-1} is called the Laplacian for the pair H, J.

Now we recall some facts concerning use of the function M to establish a relationship between the ordinary and the Sobolev gradients in the Hilbert space setting.

For the discrete case, we consider the two Hilbert spaces $H = R^{n+1}$ with the Euclidean norm and $J = R^{n+1}$ with the p-norm (1), where $p = 2$.

For every $z \in J$, $\langle z, x\rangle_H = \langle z, Mx\rangle_J = \langle Dz, DMx\rangle_{R^{2n}} = \langle z, D^tDMx\rangle_H$, see [**12**, page 24], where D is the function defined in the introduction. Therefore, $x = D^tDMx$ and $M^{-1} = D^tD$.

If φ is a real-valued C^1 function on J, then

$$\begin{aligned}\varphi'(y)h &= \langle h, (\nabla_2\varphi)(y)\rangle_J = \langle Dh, D(\nabla_2\varphi)(y)\rangle_{R^{2n}} \\ &= \left\langle h, (D^tD)(\nabla_2\varphi)(y)\right\rangle_H .\end{aligned}$$

$\nabla_2\varphi$ is the Sobolev gradient as we mentioned in the introduction. Hence the ordinary gradient $(\nabla\varphi)(y) = (D^tD)\nabla_2\varphi(y)$. So we have the following relationship between the ordinary and the Sobolev gradients using the function M.

$$(\nabla\varphi)(x) = M^{-1}(\nabla\varphi_2)(x), \quad \text{for all} \quad x \in R^{n+1}. \tag{4}$$

For the continuous case where $H = L_2[0,1]$ and $J = H^{1,2}[0,1]$, the following argument shows that we get the same results. Let z be an element of J. Then $\langle z, x\rangle_H = \langle z, Mx\rangle_J$. Let $Mx = y$. Hence,

$$\int_0^1 zy + z'y' = \int_0^1 zx, \quad \text{for all} \quad z \in J.$$

So

$$\int_0^1 zy + \int_0^1 z'y' = \int_0^1 zx.$$

This implies that

$$\int_0^1 zy + [y'z]_0^1 - \int_0^1 zy'' = \int_0^1 zx.$$

Thus,

$$\int_0^1 z(y - y'' - x) = y'(0)z(0) - y'(1)z(1).$$

Hence,

$$\int_0^1 z(y - y'' - x) = 0, \quad \text{for all} \quad z \in J \ni z(0) = z(1) = 0.$$

We claim that $y - y'' - x = 0$. Suppose not. Then, without loss of generality, take a subinterval of $[0,1]$ over which the function $y - y'' - x$ is positive, and then define a function z which is positive over the subinterval and vanishes outside the subinterval. Thus, $\int_0^1 z(y-y''-x)$ would be positive which is a contradiction. Therefore, $y - y'' - x = 0$ and consequently $\int_0^1 z(y - y'' - x) = 0$, for all $z \in J$. Hence, $y'(0)z(0) - y'(1)z(1) = 0$, for all $z \in J$. So if we choose a function z so that $z(0) = 0$ and $z(1) = 1$, we get $y'(1) = 0$. Next we choose a function z so that $z(0) = 1$ and $z(1) = 0$, and we get $y'(0) = 0$. So finally the initial value problem $y - y'' = x$, $y'(0) = 0 = y'(1)$ has a unique solution $y = Mx$. This implies that $(I - \Delta)Mx = x$, where I is the identity function, and $I - \Delta = M^{-1}$.

Now consider the linear transformation $D_1 : H^{1,2}[0,1] \to L_2[0,1]$ such that

$$D_1 f = f', \quad \text{for all} \quad f \in H^{1,2}[0,1].$$

D_1 is a closed densely defined operator. Then

$$D_1^t f = -f' \quad \text{for all} \quad f \in H^{1,2}[0,1] \ni f(0) = 0 = f(1),$$

where D_1^t is the adjoint of D_1. Consider also the linear transformation

$$D : H^{1,2}[0,1] \longrightarrow L_2[0,1] \times L_2[0,1]$$

such that

$$D(f) = \begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} If \\ D_1 f \end{pmatrix}, \quad \text{for all} \quad f \in H^{1,2}[0,1].$$

D is a closed densely defined operator. Then

$$D^t \begin{pmatrix} u \\ v \end{pmatrix} = u + D_1^t v,$$
$$\text{for all} \quad u \in L_2[0,1], \quad v \in H^{1,2}[0,1] \ni v(0) = 0 = v(1),$$

where D^t is the adjoint of D. Hence,

$$D^t D f = f + D_1^t f' = f - f'',$$
$$\text{for all} \quad f \in H^{2,2}[0,1] \ni f'(0) = 0 = f'(1).$$

Therefore, $D^t D = I - \Delta$ and consequently $D^t D = M^{-1}$.

Suppose φ is a C^1 function on $H^{1,2}[0,1]$.

$$\varphi'(x)(y) = \langle y, (\nabla_2 \varphi)(x) \rangle_{H^{1,2}[0,1]} = \langle Dy, D(\nabla_2 \varphi)(x) \rangle_{L_2[0,1] \times L_2[0,1]}.$$

If the $L_2[0,1]$ gradient $(\nabla \varphi)(x)$ exists, then

$$\varphi'(x)(y) = \langle y, (\nabla \varphi)(x) \rangle_{L_2[0,1]}.$$

Therefore,

$$\langle Dy, D(\nabla_2\varphi)(x)\rangle_{L_2[0,1]\times L_2[0,1]} = \langle y, (\nabla\varphi)(x)\rangle_{L_2[0,1]}.$$

Hence, $D(\nabla_2\varphi)(x)$ is in the domain of D^t and $D^tD(\nabla_2\varphi)(x) = (\nabla\varphi)(x)$. Therefore, $(\nabla_2\varphi)(x) = (D^tD)^{-1}(\nabla\varphi)(x)$ or $(\nabla_2\varphi)(x) = M(\nabla\varphi)(x)$, for all $x \in H^{1,2}[0,1]$.

When we look for critical points of the C^1 function

$$\varphi(x) = \frac{1}{2}\int_0^1 (x'-x)^2, \quad x \in H^{1,2}[0,1]$$

by solving $(\nabla_2\varphi)(x) = 0$ or $M(\nabla\varphi)(x) = 0$, see [**11**, page 80], we look at

$$\begin{aligned}
\varphi'(x)h &= \int_0^1 (x'-x)(h'-h) \\
&= \int_0^1 (x'-x)h' - \int_0^1 (x'-x)h) \\
&= [(x'-x)h]_0^1 - \int_0^1 (x''-x')h + (x'-x)h \\
&= \int_0^1 (x-x'')h = 0,
\end{aligned}$$

with the boundary conditions $x'(0) = x(0)$ and $x'(1) = x(1)$. This implies that $(\nabla\varphi)(x) = x - x'' = 0$ which requires that $x \in H^{2,2}[0,1]$. Since $\nabla_2\varphi$ is continuous, $M(\nabla\varphi)$ is continuous and hence we can extend it to the whole space $H^{1,2}[0,1]$.

Definition 3 [**7**, page 155]**.** Suppose X is a Banach space and X^* is the dual space of X. Given sets $S \subset X$ and $S^* \subset X^*$, the sets

$$\begin{aligned}
S^{\perp} &= \{f^* \in X^* : \langle f, f^*\rangle = 0, \quad \text{for all} \quad f \in S\} \\
S^{*\perp} &= \{f \in X : \langle f, f^*\rangle = 0, \quad \text{for all} \quad f^* \in S^*\}
\end{aligned}$$

are known as the orthogonal complements of S and S^*, respectively, where the coupling $\langle f, f^*\rangle = f^*(f)$.

Definition 4 [**7**, page 161]**.** Suppose X and Y are two Banach spaces and T is a bounded linear operator from X to Y. The adjoint of T, denoted by T^*, is the mapping from Y^* to X^* defined by

$$T^*(f^*)j = f^*(T(j)), \quad f^* \in Y^*, \quad j \in X.$$

Theorem 5 [**7**, page 164]. *Let X and Y be two Banach spaces, and suppose that T is a linear operator from X to Y. Then $\overline{R(T)} = N(T^*)^\perp$, where $R(T)$ is the range of T and $N(T^*) = \{f^* \in Y^*: T^*(f^*) = 0\}$.*

The following theorem shows that the Beurling-Deny theorem can be extended to the two uniformly convex spaces $H = L_p[0,1]$ and $J = H^{1,p}[0,1]$, $p > 2$.

Theorem 6. *There is an M in $L(L_p[0,1], H^{1,p}[0,1])$, $2 < p < \infty$, such that*

(i) *$R(M)$ is dense in $H^{1,p}[0,1]$.*

(ii) *M^{-1} exists.*

Proof. Suppose that $f \in L_p[0,1]$. Then there is a bounded linear function α on $L_q[0,1]$, $1/p + 1/q = 1$, such that $\alpha(g) = \int_0^1 fg$, for all $g \in L_q[0,1]$. Let $\beta = \alpha_{|H^{1,q}[0,1]}$ be the restriction of α to $H^{1,q}[0,1]$. For every $h \in H^{1,q}[0,1]$, we have

$$|\beta(h)| = \left|\int_0^1 hf\right| \le \int_0^1 |hf| \le \|h\|_{L_q[0,1]} \|f\|_{L_p[0,1]}.$$

Hence, $|\beta| \le \|f\|_{L_p[0,1]}$. Therefore, β is a member of $(H^{1,q}[0,1])^*$. Since $(H^{1,q}[0,1])^*$ is ismorphic to $H^{1,p}[0,1]$, there is a unique k in $H^{1,p}[0,1]$ such that

$$\beta(h) = \int_0^1 kh + k'h', \quad \text{for all} \quad h \in H^{1,q}[0,1].$$

Define $M: L_p[0,1] \to H^{1,p}[0,1]$ so that $Mf = k$. M is clearly linear. We intend to show that M is continuous and $\overline{R(M)} = H^{1,p}[0,1]$. Since

$\beta(h) = \alpha(h)$, for all h$\in H^{1,q}[0,1]$,

$$\int_0^1 kh + k'h' = \int_0^1 fh.$$

But $k = Mf$, so

$$\int_0^1 (Mf)h + (Mf)'h' = \int_0^1 fh$$

which implies that

$$\int_0^1 (Mf - f)h + (Mf)'h' = 0, \quad \text{for all} \quad h \in H^{1,q}[0,1].$$

Let $u(t) = \int_0^t (Mf - f)$. Then

$$\int_0^1 u'h + (Mf)'h' = 0.$$

So

$$[hu]_0^1 - \int_0^1 uh' + \int_0^1 (Mf)'h' = 0.$$

Therefore,

$$h(1)u(1) - h(0)u(0) + \int_0^1 ((Mf)' - u)\, h' = 0, \quad \text{for all} \quad h \in H^{1,q}[0,1],$$

which yields $u(1) = 0$. Thus,

$$\int_0^1 ((Mf)' - u)\, h' = 0, \quad \text{for all} \quad h \in H^{1,q}[0,1].$$

Hence $(Mf)' - u = 0$. So $(Mf)'' = u'$ and $(Mf)'' = Mf - f$. Therefore, we have the following differential equation $(Mf)'' - Mf = -f$ with $(Mf)'(0) = 0 = (Mf)'(1)$ whose solution is given by

$$(Mf)(t) = \cosh(t)(Mf)(0) - \int_0^t \sinh(t-s)f(s)\, ds,$$

and

$$(Mf)'(t) = \sinh(t)(Mf)(0) - \int_0^t \cosh(t-s)f(s)\,ds, \quad 0 \le t \le 1.$$

Hence,

$$0 = (Mf)'(1) = \sinh(1)(Mf)(0) - \int_0^1 \cosh(1-s)f(s)\,ds.$$

This implies that

$$(Mf)(0) = \frac{\int_0^1 \cosh(1-s)f(s)\,ds}{\sinh(1)}.$$

Hence,

$$(Mf)(t) = \frac{\cosh(t)}{\sinh(1)}\int_0^1 \cosh(1-s)f(s)\,ds - \int_0^t \sinh(t-s)f(s)\,ds.$$

So

$$\begin{aligned}|(Mf)(t)| &\le \frac{(e+1)^2}{4\sinh(1)}\int_0^1 |f| + \frac{e+1}{2}\int_0^1 |f| \\ &= \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]\int_0^1 |f| \\ &\le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]\|f\|_{L_p[0,1]}.\end{aligned}$$

Then

$$|(Mf)(t)|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|f\|^p_{L_p[0,1]}.$$

Hence,

$$\int_0^1 |Mf|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|f\|^p_{L_p[0,1]}.$$

Similarly, we get

$$\int_0^1 |(Mf)'|^p \le \left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|f\|^p_{L_p[0,1]}.$$

Therefore,

$$\|Mf\|_{H^{1,p}}^p \le 2\left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right]^p \|f\|_{L_p[0,1]}^p .$$

So

$$\|Mf\|_{H^{1,p}} \le 2^{1/p}\left[\frac{(e+1)^2}{4\sinh(1)} + \frac{e+1}{2}\right] \|f\|_{L_p[0,1]} .$$

Hence, M is continuous.

Suppose $M^*: (H^{1,p}[0,1])^* \to L_p[0,1]^*$ is the adjoint of the function M. Let $N(M^*) = \{g* \in (H^{1,p}[0,1])^* : M^*(g^*) = 0\}$. Suppose g^* is a member of $N(M^*)$. For every h in $L_p[0,1]$, we have $(M^*(g^*))h = g^*(Mh)$. Therefore $g^*(Mh) = 0$.

Now by Theorem 1, there is an m in $H^{1,q}[0,1]$ so that $g^*(k) = \int_0^1 mk + m'k'$, for all $k \in H^{1,p}[0,1]$. Hence,

$$0 = g^*(Mh) = \int_0^1 m(Mh) + m'(Mh)' = \int_0^1 mh, \quad \text{for all} \quad h \in L_p[0,1].$$

So $m = 0$ and $g^* = 0$. Therefore, $N(M^*) = \{0\}$ and consequently $\overline{R(M)} = N(M^*)^\perp = H^{1,p}[0,1]$.

Now to show (ii), we let $h \in L_p[0,1]$ so that $Mh = 0$. Hence,

$$\int_0^1 m(Mh) + m'(Mh)' = 0, \quad \text{for all} \quad m \in H^{1,q}[0,1].$$

Therefore, $\int_0^1 mh = 0$, for all $m \in H^{1,q}[0,1]$. So $h = 0$ and consequently M^{-1} exists. The proof of the theorem is now complete. □

Unlike the case when $p = 2$, we cannot use the function M^{-1} instead of $D^tQ(D(.))$ in (3) to establish a relationship between the ordinary and the p-gradient as we did in (4) simply because the function M^{-1} is linear but $D^tQ(D(.))$ is not.

For the discrete case, if we consider $H = R^{n+1}$ with the Euclidean norm and $J = R^{n+1}$ with the p-norm (1), for $r \in H$ there exists a linear function f_r on J so that $f_r(s) = \langle r, s\rangle_H$ for every $s \in J$. Hence, there exists a unique $h \in J$ such that $f_r(h)$ is maximum subject to $|f_r|_{J^*} = \|h\|_J$. So we have a function T from H to J so that $h = Tr$. Let $\beta(h) = \|h\|^p - |f_r|^p$. Using Lagrange multipliers, we get $\nabla f_r(h) = \nabla\beta(h)$ but $\nabla\beta(h) = D^t Q(D(h))$, see [**16**, page 1542]. Therefore, $r = D^t Q(D(Tr))$ and $T^{-1} = D^t Q(D(.))$. The relationship (3) between the two gradients implies then $(\nabla\varphi)(x) = T^{-1}(\nabla_p\varphi)(x)$, where φ is a C^1 function on R^{n+1}.

Note that the two functions T and M are equal if $p = 2$.

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