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COMPARISON OF CUSP FORMS ON $GL(3,\mathbf{Z})$

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ABSTRACT. We give an estimate for the number of Fourier coefficients needed to determine uniquely a cusp form on $GL(3,\mathbf{Z})\backslash PGL(3,\mathbf{R})/O(3)$. This leads to an estimate on the multiplicity of the space of eigenfunctions with fixed infinity type. More precisely we show the multiplicity of the space of eigenfunctions with fixed infinity type, $M(\lambda)=O(\lambda^2)$ where λ is the eigenvalue of the Laplacian.

1. Introduction. Since the early work of Jacobi on θ functions to study the representation of integers as sums of squares, automorphic forms in various guises have played an important role in number theory. Notable among these is Ramanujan's discriminant function whose reciprocal is intimately connected with the study of the partition function in number theory. The automorphic properties of this function, which include periodicity with period one, give rise to a Fourier expansion whose study has been a central theme in number theory for a century. The estimate of the size of the Fourier coefficients is a key element in these applications. Hecke formalized the concept of automorphic forms in the 1920's. He assumes an analytic function on the upper half plane that has only polynomial growth at infinity, and with respect to some discrete subgroup Γ of $SL(2,\mathbf{R})$, the group of symmetries of the upper half plane satisfies

$$f(\gamma z)=(cz+d)^k f(z)$$

where γ, an element of Γ, equals $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. In particular, we have that f is periodic with respect to x, so we have a Fourier expansion.

In the 1940's, Maass generalized this notion by considering functions that are not holomorphic but are eigenfunctions of the Laplace-Beltrami operators. These functions, in particular when Γ is $SL(2,\mathbf{Z})$, have been of tremendous usefulness in the study of the Riemann zeta function, and in various aspects of the study of Kloosterman sums.

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An interesting question we have is how many Fourier coefficients are needed to determine a cusp form. We study this question for $SL(3,\mathbf{Z})$.

The structure of the argument we use here is a mild variation of the Siegel-Maass method. This method for $SL(2,\mathbf{Z})$ cusp forms can be found in Terras [**20**]. The basic idea of the method is that for a cusp form ϕ, one derives an inequality of

$$\phi \le k\phi,$$

where k is a constant, and shows that k is less than one. This contradiction forces ϕ to be identically zero.

The use of this method is possible because of the Fourier expansion of a cusp form with Whittaker functions as kernels, which can be found in Bump [**1**]. Whittaker functions can be expressed as an integral of two K-Bessel functions, which is due to Vinogradov and Takhtadzhyan. When the variable of a K-Bessel function is larger than the order of the parameter, we can use Olver's error estimate of K-Bessel functions as a numerical tool for our purpose. That is, we overestimate K-Bessel functions by $e^{-k(\lambda^{1/2}+x)}$ where k is a positive constant, λ is the eigenvalue of the Laplacian, and $x > 0$. We also use an asymptotic expansion of Whittaker functions obtained by Bump and Huntley to underestimate the Whittaker functions at a certain value, [**2**]. For the rest of the paper, k denotes various constants suitable for our purpose.

2. Statement of theorem.

Theorem 1. *If Fourier coefficients a_{n_1,n_2} of a cusp form ϕ on*

$$\mathbf{X} = GL(3,\mathbf{Z})\backslash PGL(3,\mathbf{R})/O(3)$$

are zero when $\max(n_1, n_2^{1/3}) = O(\lambda^{1/2})$*, then ϕ is identically zero.*

Corollary 2. *Let ϕ and ψ be cusp forms on $\mathbf{X}$, and let a_{n_1,n_2} and b_{n_1,n_2} be their Fourier coefficients respectively. If $a_{n_1,n_2} = b_{n_1,n_2}$ when* $\max(n_1, n_2^{1/3}) = O(\lambda^{1/2})$*, then $\phi = \psi$.*

Notice that Theorem 1 implies that the multiplicity of an eigenvalue λ is $O(\lambda^2)$, which is an implication of Theorem 1.1 of [**4**] which says:

Let $\mathbf{X} = SL(3,\mathbf{Z})\backslash SL(3,\mathbf{R})/SO(3,\mathbf{R})$, *and* $N(\lambda)$ *denote the dimension of the space of cusp forms on* $\mathbf{X}$ *with Laplace eigenvalue less than* λ. *Then*

$$N(\lambda) = C\lambda^{5/2} + O(\lambda^2)$$

with

$$C = \frac{\mathrm{Vol}\,(\mathbf{X})}{(4\pi)^{(5/2)}\Gamma(7/2)}.$$

However, the theorem in [**4**] doesn't show which Fourier coefficients of the cusp form are nonzero, and Theorem 1 doesn't imply the actual existence of cusp forms for a certain interval of λ.

3. Proof of theorem. We first describe the theorem in [**1**, page 65] and the action of Γ_1^2 on y_1 and y_2 and introduce parameters we will use in the paper. Let Γ_∞ be the group of 3×3 upper triangular unipotent matrices with integer coefficients. Also let

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbf{Z},\ ad-bc = \pm 1 \right\},$$

and

$$\Gamma_\infty^2 = \Gamma^2 \cap \Gamma_\infty.$$

Denote the subgroup of index 2 in Γ^2 as Γ_1^2, which is the group of those elements of determinant one. The parameters ν_1, ν_2 and ν are defined as follows

$$\lambda = 3\left(\nu_1^2 + \nu_1\nu_2 + \nu_2^2 - \nu_1 - \nu_2\right)$$
$$\nu = \frac{3\nu_1 + 3\nu_2 - 2}{2}.$$

Notice that the order of ν is $k\lambda^{1/2}$ for a constant k. We define the type of cusp forms to be the pair (ν_1, ν_2).

The theorem in [**1**] says: *Let* ϕ *be a cusp form of type* (ν_1,ν_2). *There exist coefficients* a_{n_1,n_2} *for positive integers* n_1 *and* n_2 *such that*:

$$\phi(\tau) = \sum_{g\in\Gamma_\infty^2\backslash\Gamma^2} \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} n_1^{-1} n_2^{-1} a_{n_1,n_2} \times W_{1,1}^{(\nu_1,\nu_2)}\left(\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g\tau\right). \tag{3.1}$$

Here $\tau = \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$.

Now

$$\begin{bmatrix} n_1n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1y_2 & y_1x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} y_1'y_2' & y_1'x_2' & x_3' \\ 0 & y_1' & x_1' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (cx_2+d)\,\omega^{-1} & -cy_2\omega^{-1} & 0 \\ cy_2\omega^{-1} & (cx_2+d)\,\omega^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\omega = |c(x_2+iy_2)+d|$. For our purpose we may ignore x_j because they do not contribute to the magnitude of the Whittaker function. Solving equations for y_1' and y_2', we find the action of Γ_1^2 on y_1 and y_2. That is, $y_1' = n_1\omega y_1$ and $y_2' = n_2y_2/\omega^2$.

Since ϕ is a cusp form, there exists τ° such that $|\phi(\tau^\circ)|$ is the maximum. By

$$W_{1,1}^{(\nu_1,\nu_2)}(\tau, w_1) = W_{\nu_1,\nu_2}(y_1,y_2)e^{2\pi i(x_1+x_2)} \qquad [\mathbf{1},\ (3.46)],$$

we have

$$|\phi(\tau^\circ)| \le \sum_{g\in\Gamma_\infty^2\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left| n_1^{-1}n_2^{-1}a_{n_1,n_2}W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2y_2^\circ}{\omega^2}\right)\right|$$

where y_1° and y_2° are corresponding y_1 and y_2 to τ°.

$$\phi_{n_1,n_2}(\tau) = \int_0^1\int_0^1\int_0^1 \phi\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}\tau\right) \times e^{2\pi i(-n_1\xi_1-n_2\xi_2)}\,d\xi_1\,d\xi_2\,d\xi_3 \quad [\mathbf{1},\ (4.9)],$$

whose absolute value is less than $|\phi(\tau^\circ)|$, and by putting $(\operatorname{diag}(n_1n_2, n_1, 1))^{-1}\tau$ instead of τ, we have

$$|\phi(\tau^\circ)| \ge \left|\phi_{n_1,n_2}\left(\begin{bmatrix} n_1n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\tau\right)\right|.$$

$$\phi_{n_1,n_2}(\tau) = a_{n_1,n_2}\,|n_1n_2|^{-1} \times W_{1,1}^{(\nu_1,\nu_2)}\left(\begin{bmatrix} n_1n_2 & 0 & 0\\ 0 & n_1 & 0\\ 0 & 0 & 1\end{bmatrix}\tau\right) \quad [\mathbf{1},\ (4.12)]$$

implies

$$|\phi(\tau^\circ)| \geq \left| a_{n_1,n_2}\,|n_1n_2|^{-1} \times W_{\nu_1,\nu_2}\left(\begin{bmatrix} n_1n_2 & 0 & 0\\ 0 & n_1 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix} n_1n_2 & 0 & 0\\ 0 & n_1 & 0\\ 0 & 0 & 1\end{bmatrix}\tau\right)\right|.$$

So we have

$$\left|a_{n_1,n_2}n_1^{-1}n_2^{-1}\right| \leq \frac{|\phi(\tau^\circ)|}{|W_{\nu_1,\nu_2}(y_1,y_2)|}$$

for any fixed nonzero y_1 and y_2. We will show $W_{\nu_1,\nu_2}(y_1,y_2)$ does not decay faster than

$$e^{-1/4k|\nu|},$$

when $|\nu_1|$ or $|\nu_2| \to \infty$, where k is a constant. Since a_{n_1,n_2} is a Fourier coefficient and $|\phi(\tau^\circ)|$ is the maximum, the above inequality holds for any y_1 and y_2. Thus, we may choose them as an after fact. That is, we first choose ν_1 and ν_2 with sufficiently large absolute values, and then we choose a pair of numbers for y_1 and y_2 that makes $|W_{\nu_1,\nu_2}(y_1,y_2)|$ large enough. A pair of numbers which serves our purpose is $y_1 = y_2 = 1$. We will show

Lemma 1.

$$|W_{\nu_1,\nu_2}(1,1)| \geq \lambda^{-3}$$

where $\lambda = 3(\nu_1^2+\nu_1\nu_2+\nu_2^2-\nu_1-\nu_2)$ *is the eigenvalue of the Laplacian.*

Using Lemma 1 to overestimate $n_1^{-1}n_2^{-1}a_{n_1,n_2}$ in (3.1) by $|\phi(\tau^\circ)|/|W_{\nu_1,\nu_2}(1,1)|$, we have

$$(3.2)\quad |\phi(\tau^\circ)| \leq \sum_{g\in\Gamma_\infty^2\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\lambda^3\,|\phi(\tau^\circ)|\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ,\ \frac{n_2y_2^\circ}{\omega^2}\right)\right|.$$

Assume $a_{n_1,n_2} = 0$ for all n_1 and n_2 such that $\max(n_1, n_2^{1/3}) \le k|\nu|$, where $\nu = (3\nu_1 + 3\nu_2 - 2)/2$ and k is a fixed but large enough constant. Here the number of coefficients a_{n_1,n_2} which equal zero is $k^4|\nu||\nu|^3 = O(\lambda^2)$. With this assumption, we will show

$$\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \lambda^3 \left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right|$$

is less than some positive power of $e^{-|\nu|}$, which is a contradiction unless ϕ is identically zero. This proves Theorem 1 and Corollary 2. Notice that λ^3 factor is immaterial, and we will omit it.

For nonzero coefficients, we have the following three cases: $n_1 \ge n_2^{1/3}$ or $n_2^{1/3} > n_1$ and $n_2/\omega^2 \le k\,|\nu|$, or $n_2^{1/3} > n_1$ and $n_2/\omega^2 > k|\nu|$. In the second case, we may assume $n_2^{1/3} > k|\nu|$, and since $n_2 \le k|\nu|\omega^2$, we have $\omega \ge k|\nu|$. In the third case, $\omega < (n_2/k|\nu|)^{1/2}$. We thus have

$$\begin{aligned}
&\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \\
&\quad \le \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1\ge k|\nu|}^{\infty} \sum_{n_2=1}^{n_1^3} \left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \\
&\qquad + \sum_{\omega\ge k|\nu|}^{\infty} \sum_{n_1=1}^{n_2^{1/3}} \sum_{n_2=(k|\nu|)^3}^{k|\nu|\omega^2} \left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \\
&\qquad + \sum_{\omega<(n_2/k|\nu|)^{1/2}} \sum_{n_1=1}^{n_2^{1/3}} \sum_{n_2=(k|\nu|)^3}^{\infty} \left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right|.
\end{aligned}$$

We will show the following lemmas.

Lemma 2. *If* $\max(n_1, n_2^{1/3}) \ge k|\nu|$ *and* $n_1 \ge n_2^{1/3}$ *where* k *is a sufficiently large constant and* $\nu = (3\nu_1 + 3\nu_2 - 2)/2$, *then*

$$\left| W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \le e^{-1/4 n_1\omega y_1^\circ} \le e^{-1/4k|\nu|}.$$

Lemma 3. *If* $\max(n_1, n_2^{1/3}) \geq k|\nu|$, $n_2^{1/3} \geq n_1$ *and* $n_2/\omega^2 \leq k|\nu|$, *then*

$$\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \leq e^{-1/4n_1\omega y_1^\circ} \leq e^{-1/4k|\nu|}.$$

Lemma 4. *If* $\max(n_1, n_2^{1/3}) \geq k|\nu|$, $n_2^{1/3} \geq n_1$ *and* $n_2/\omega^2 \geq k|\nu|$, *then*

$$\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \leq e^{-1/2(n_2 y_2^\circ/\omega^2)} \leq e^{-1/4k|\nu|}.$$

Given these lemmas, we have

$$\begin{aligned}
\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}&\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\right| \\
&\leq 4\sum_{d=0}^{0}\sum_{c=1}^{\infty}\sum_{n_1\geq k|\nu|}^{\infty}\sum_{n_2=1}^{n_1^3}\left|e^{-1/4n_1\omega y_1^\circ}\right| \\
&\quad + 4\sum_{d=1}^{\infty}\sum_{c=0}^{\infty}\sum_{n_1\geq k|\nu|}^{\infty}\sum_{n_2=1}^{n_1^3}\left|e^{-1/4n_1\omega y_1^\circ}\right| \\
&\quad + \sum_{\omega\geq k|\nu|}^{\infty}\sum_{n_1=1}^{n_2^{1/3}}\sum_{n_2=(k|\nu|)^3}^{k|\nu|\omega^2}\left|e^{-1/4n_1\omega y_1^\circ}\right| \\
&\quad + \sum_{\omega<(n_2/k|\nu|)^{1/2}}\sum_{n_1=1}^{n_2^{1/3}}\sum_{n_2=(k|\nu|)^3}^{\infty}\left|e^{-1/2(n_2 y_2^\circ/\omega^2)}\right|.
\end{aligned}$$

Here we are using the fact that g is a matrix, $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1\end{pmatrix}$ with determinant one, a, b, c and d are integers, $(c,d) = 1$ and only one pair of representatives of a and b for each pair of c and d needs to be considered. The first two summations on the right side are multiplied by four to account for the cases with negative c or d. But for our purpose, a big O estimate, a factor of any finite constant is irrelevant. These are quadruple geometric series. By changing the order of summations if necessary, we overestimate the right side of the above inequality by $(e^{-k|\nu|})^{\varepsilon}$, and we have $|\phi(\tau^\circ)| \le |\phi(\tau^\circ)|(e^{-k|\nu|})^{\varepsilon}$ where the actual size of ε is immaterial as long as it is a positive constant. Thus, we only need to prove the above four lemmas.

Proof of Lemma 1. We have

$$\begin{aligned} W_{\nu_1,\nu_2}(y_1,y_2) \\ &= \pi^{-2} y_1^{1+(\nu_1-\nu_2)/2} y_2^{1+(\nu_2-\nu_1)/2} \\ &\times \int_0^\infty K_\nu\left(y_1\sqrt{1+x}\right) K_\nu\left(y_2\sqrt{\frac{1+x}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx. \end{aligned}$$

We also have

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t-1)^{\nu-1/2}(t+1)^{\nu-1/2}\,dt$$

for $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(\nu) > -1/2$ [**15**, (5.10.24)]. We thus have

$$\begin{aligned} W_{\nu_1,\nu_2}(y_1,y_2) &= \frac{y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}}{\pi 4^\nu\left(\Gamma(\nu+(1/2))\right)^2} \\ &\times \int_0^\infty \int_1^\infty e^{-ty_1\sqrt{1+x}}(t-1)^{\nu-1/2}(t+1)^{\nu-1/2}\,dt \\ &\times \int_1^\infty e^{-ty_2\sqrt{(x+1)/x}}(t-1)^{\nu-1/2}(t+1)^{\nu-1/2}\,dt \\ &\times \frac{(1+x)^{(3\nu_1+3\nu_2-2)/2}}{x^{(3\nu_2-1)/2}}\,dx. \end{aligned}$$

Then

$$\begin{aligned} W_{\nu_1,\nu_2}(\lambda,\lambda) &= \frac{\lambda^{3\nu_1+3\nu_2}}{\pi 4^\nu\left(\Gamma\left(\nu+(1/2)\right)\right)^2} \\ &\times \int_0^\infty \int_1^\infty e^{-t\lambda\sqrt{1+x}}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt \\ &\times \int_1^\infty e^{-t\lambda\sqrt{(x+1)/x}}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt \\ &\times \frac{(1+x)^{(3\nu_1+3\nu_2-2)/2}}{x^{(3\nu_2-1)/2}}\,dx. \end{aligned}$$

Since there is no complex value involved in

$$e^{t(\lambda-1)\sqrt{1+x}} \quad \text{and} \quad e^{t(\lambda-1)\sqrt{(x+1)/x}},$$

and

$$(\lambda-1)\left(\sqrt{1+x}+\sqrt{\frac{x+1}{x}}\right) \geq 2\sqrt{2}\,(\lambda-1)\,,$$

we also have

$$\begin{aligned}|W_{\nu_1,\nu_2}(1,1)| \geq \left|\frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^{3\nu_1+3\nu_2}}\right| \\ \times \left|\begin{array}{c}(\lambda^{3\nu_1+3\nu_2})/(\pi 4^{\nu}\,(\Gamma\,(\nu+(1/2)))^2)\int_0^\infty \\ \times\int_1^\infty e^{-t\lambda\sqrt{1+x}}\,(t-1)^{\nu-(1/2)}\,(t+1)^{\nu-(1/2)}\,dt \\ \times\int_1^\infty e^{-t\lambda\sqrt{(x+1)/x}}\,(t-1)^{\nu-(1/2)}\,(t+1)^{\nu-(1/2)}\,dt \\ \times((1+x)^{(3\nu_1+3\nu_2-2)/2})(x^{(3\nu_2-1)/2})\,dx\end{array}\right| \\ \geq \frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^3}\,|W_{\nu_1,\nu_2}(\lambda,\lambda)|\,.\end{aligned}$$

Theorem 1 in [**2**] says:

$$\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}\,(y_1,y_2) \sim \sum_{n=0}^{\infty} W^n_{\nu_1,\nu_2}\,(y_1,y_2)\,,$$

where for certain constants $c(j,\nu_1,\nu_2)$, we have

$$\begin{aligned}W^n_{\nu_1,\nu_2}\,(y_1,y_2) = y_1^{1/3}y_2^{1/3}\left[\sum_{-n}^{n} c\,(j,\nu_1,\nu_2)\,y_1^{(2j)/3}y_2^{-(2j)/3}\right] \\ \times\left(y_1^{2/3}+y_2^{2/3}\right)^{-(6n+1)/4} e^{-\left(y_1^{2/3}+y_2^{2/3}\right)^{3/2}}\end{aligned}$$

This asymptotic expansion is valid when $y_1 \to \infty$ and $y_1^{-\alpha}y_2$ is kept equal to a positive constant, or when $y_2 \to \infty$ and $y_1 y_2^{-\alpha}$ is kept equal to a positive constant, for any $-1/2 < \alpha < \infty$. In particular, it is valid if y_1 and y_2 both are large.

Furthermore, they computed $c(j,\nu_1,\nu_2)$ by a recursive method which is described in the last part of the article. In particular, $c(j,\nu_1,\nu_2) = 1$

for $n = 0$. We also know that the highest power of λ in $c(j, \nu_1, \nu_2)$ is increased by 1 as n increases for each term in the asymptotic expansion of a Whittaker function, and the exponents of $(y_1^{2/3} + y_2^{2/3})$ are decreased by $3/2$. This implies that the asymptotic expansion of a Whittaker function can be overestimated by a convergent geometric series times $e^{-(y_1^{2/3}+y_2^{2/3})^{3/2}}$ when $|y_1|$ and $|y_2|$ are larger than λ. This asymptotic expansion can be used to underestimate the absolute value of a Whittaker function by a constant factor times the zeroth term because the sum of the rest of terms can be overestimated by a convergent geometric series times $e^{-(y_1^{2/3}+y_2^{2/3})^{3/2}}$.

By asymptotic expansion of [**2**], we have

$$W_{\nu_1,\nu_2}(\lambda, \lambda) = k\sqrt{\frac{2}{3\pi}}\frac{\lambda^{1/2}}{2^{1/4}}e^{-2\sqrt{2}\lambda}.$$

We thus have

$$|W_{\nu_1,\nu_2}(1,1)| = \frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^3}k\sqrt{\frac{2}{3\pi}}\frac{\lambda^{1/2}}{2^{1/4}}e^{-2\sqrt{2}\lambda} \geq \frac{1}{\lambda^3}. \qquad \square$$

Proof of Lemma 2. In this case, we have $n_1 \geq n_2^{1/3}$ since $\max(n_1, n_2) \geq k|\nu|$, $n_1 \geq k|\nu|$. Thus, $n_1\omega y_1^\circ \geq k|\nu|$. By breaking the interval of integration, we have

$$\begin{aligned}
&W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_2^\circ}{\omega^2}\right)\\
&= \pi^{-2}(n_1\omega y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\omega^2}\right)^{1+(\nu_2-\nu_1)/2}\\
&\times \int_0^\infty K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx\\
&= \pi^{-2}(n_1\omega y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\omega^2}\right)^{1+(\nu_2-\nu_1)/2}\\
&\times \left(\int_0^{x^\circ} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx\right.\\
&\quad \left. + \int_{x^\circ}^\infty K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^\circ}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx\right)
\end{aligned}$$

where

$$x^\circ = \frac{n_2^2 y_2^{\circ 2}}{\omega^4 - n_2^2 y_2^{\circ 2}}.$$

The polynomial part of the outside of the parenthesis is immaterial in terms of the magnitude of $W_{\nu_1,\nu_2}(n_1\omega y_1^\circ, (n_2 y_2^\circ)/\omega^2)$.

Notice that x° is where $(n_2 y_2^\circ/\omega^2)\sqrt{(x+1)/x} = 1$. Thus, when $0 \le x \le x^\circ$,

$$\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}} \ge 1,$$

and when $x > x^\circ$,

$$\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}} < 1,$$

then

$$\begin{aligned}\left|\int_0^{x^\circ} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \\ < \int_0^{x^\circ} \left|K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)\right| \left|K_\nu(1)\right| \left|x^{(3\nu_1-3\nu_2-4)/4}\right|\,dx\end{aligned}$$

because of the following.

We know $-1/2 < \mathrm{Re}\,(\nu) < 1/2$. By [**15**, (5.10.24)], for $\mathrm{Re}\,(z) > 0$ and $\mathrm{Re}\,(\nu) > -1/2$,

$$\begin{aligned}K_\nu(z) &= \frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt \\ &= \frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu z^\nu \int_1^\infty e^{-(z-1)t} \\ &\qquad \times e^{-t}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt\end{aligned}$$

By inspecting this, we know $|K_\nu(z)| \le |K_\nu(1)|$ when $z > 1$.

Furthermore,

$$\begin{aligned}|K_\nu(1)| &= \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right.\\ &\qquad \left.\times\int_1^\infty e^{-t}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt\right|\\ &\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right|\\ &\qquad \times\left(\left|\int_1^2 e^{-t}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt\right|\right.\\ &\qquad\quad \left.+\left|\int_2^\infty e^{-t}(t-1)^{\nu-(1/2)}(t+1)^{\nu-(1/2)}\,dt\right|\right).\end{aligned}$$

Since $\mathrm{Re}\,(\nu-(1/2))<0$, we have

$$\begin{aligned}|K_\nu(1)| &\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right|\\ &\qquad \times\left(\left|\int_1^2 e^{-t}(t-1)^{\nu-(1/2)}\,dt\right|+\left|\int_2^\infty e^{-t}\,dt\right|\right)\\ &= \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\left|\int_0^1 e^{-t}t^{\nu-(1/2)}\,dt\right|+e^{-1}\right)\\ &\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right|\\ &\qquad \times e^{-1}\left(\left|\frac{e^{-t}t^{\nu+(1/2)}}{\nu+(1/2)}\right|_0^1+\left|\int_0^1\frac{e^{-t}t^{\nu+(1/2)}}{\nu+(1/2)}\,dt\right|+e^{-1}\right).\end{aligned}$$

Since $\mathrm{Re}\,(\nu+(1/2))>0$ and so $|t^{\nu+(1/2)}|\le 1$, and

$$\begin{aligned}|K_\nu(1)| &\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right|\\ &\qquad \times e^{-1}\left(\left|\frac{e^{-1}}{\nu+(1/2)}\right|+\left|\int_0^1\frac{e^{-t}}{\nu+(1/2)}\,dt\right|+e^{-1}\right)\\ &= \left|\frac{\sqrt{\pi}}{\Gamma(\nu+(1/2))}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\frac{e^{-1}}{\nu+(1/2)}+e^{-1}\right).\end{aligned}$$

But $|\Gamma(\nu+(1/2))|\sim e^{-\pi|\nu+(1/2)|/2}$ by Stirling's formula. So

$$|K_\nu(1)| = O\left(e^{\pi|\nu+(1/2)|/2}\right).$$

By Olver [**17**, page 269], we have

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}(1+\gamma_1),$$

where

$$|\gamma_1| \le e^{\left|\left(\nu^2-(1/4)\right)z^{-1}\right|}\left(\left(\nu^2-\frac{1}{4}\right)z^{-1}\right) \quad \text{for} \quad \mathrm{Re}\,(z) \ge 0.$$

Thus when z is a real number as in our case,

$$|K_\nu(z)| \le \Bigg|\left(\frac{\pi}{2z}\right)^{1/2} \times e^{-z}\left(1+e^{\left|(\nu-(1/2))(\nu+(1/2))z^{-1}\right|}\left(\left(\nu-\frac{1}{2}\right)\left(\nu+\frac{1}{2}\right)z^{-1}\right)\right)\Bigg|.$$

If $z > 4\pi|\nu+(1/2)|$, then

$$|K_\nu(z)| \le \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left|\left(1+e^{|(\nu-(1/2))|}\left(\nu-\frac{1}{2}\right)\right)\right|,$$

as $|\nu|$ is large enough.

Then, by letting $n_1 > 4\pi|\nu+(1/2)|$ and overestimating the first K-Bessel function, we obtain

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \le \int_0^{x^\circ} e^{-3/4 n_1\omega y_1^\circ\sqrt{1+x}}\,|K_\nu(1)|\,x^{(3\nu_1-3\nu_2-4)/4}\,dx.$$

In the above inequalities, 4π is not fundamental. Any large enough constant serves for our purpose. We have already shown $|K_\nu(1)| = O(e^{\pi/2|\nu+(1/2)|})$ which can be absorbed by $e^{-3/4n_1\omega y_1^\circ\sqrt{1+x}}$. So

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \le e^{-3/4 n_1\omega y_1^\circ\sqrt{1+x}}.$$

For

$$\int_{x^\circ}^{\infty} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx,$$

we use Lebedev [**15**, (5.7.1)]

$$I_\nu\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(z/2\right)^{\nu+2n}}{\Gamma\left(n+1\right)\Gamma\left(n+\nu+1\right)},$$

and [**15**, (5.7.2)]

$$K_\nu\left(z\right)=\frac{\pi}{2}\frac{I_{-\nu}\left(z\right)-I_\nu\left(z\right)}{\sin\nu z}$$

where $|z|<\infty$, $\nu\neq 0,\pm1,\pm2,\dots$ and $\arg(z)<\pi$. Notice that when x is within the limits of the integral,

$$\frac{n_2y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}<\infty.$$

So we can use these equalities for the second K-Bessel function.

Then

$$\begin{aligned}&\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx\right|\\ &\leq\int_{x^\circ}^{\infty} e^{-3/4n_1\omega y_1^\circ\sqrt{1+x}}\left|\frac{\pi}{2\sin\nu\pi}\right|\\ &\qquad\times\left\{\sum_{n=0}^{\infty}\left|\frac{\left(n_2y_2^\circ/2\omega^2\sqrt{(x+1)/x}\right)^{-\nu+2n}}{\Gamma\left(n+1\right)\Gamma\left(n-\nu+1\right)}\right|\right.\\ &\qquad\qquad\left.-\left|\frac{\left(n_2y_2^\circ/2\omega^2\sqrt{(x+1)/x}\right)^{\nu+2n}}{\Gamma\left(n+1\right)\Gamma\left(n+\nu+1\right)}\right|\right\}x^{(3\nu_1-3\nu_2-4)/4}\,dx.\end{aligned}$$

Here, $|\sin\nu\pi|$ is not small. In fact, $\sin\nu\pi=(e^{-i\nu\pi}-e^{i\nu\pi})/2i$ and the imaginary part of ν has a large absolute value, so one of the exponential expressions is large. The absolute values of Γ functions in the denominators are also sufficiently large. That is,

$$\left|\Gamma\left(n\pm\nu+1\right)\right|\geq\left(n\pm\mathrm{Re}\,\left(\nu\right)\right)!e^{-\pi|\nu|}$$

which will be shown shortly. Since $((n_2y_2^\circ/2\omega^2)\sqrt{(x+1)/x})^2 \leq 1/4$, when x is within the limits of integration we have

$$\begin{aligned}
&\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \\
&\leq \int_{x^\circ}^{\infty} e^{-(3/4)n_1\omega y_1^\circ\sqrt{1+x}}\Bigg|\Bigg\{\left|\Gamma\left(-\nu+1\right)\right|^{-1} \\
&\quad\times\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{-\nu}\sum_{n=0}^{\infty}\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{2n} \\
&\quad-\left|\Gamma\left(\nu+1\right)\right|^{-1}\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{\nu} \\
&\qquad\qquad+\sum_{n=0}^{\infty}\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{2n}\Bigg\}\Bigg|x^{(3\nu_1-3\nu_2-4)/4}\,dx \\
&\leq \int_{x^\circ}^{\infty} e^{-(3/4)n_1\omega y_1^\circ\sqrt{1+x}}e^{\pi|\nu+1|} \\
&\qquad\times\Bigg|\Bigg\{\left(\frac{2\omega^2}{n_2y_2^\circ}\sqrt{\frac{x}{x+1}}\right)^{\nu}\sum_{n=0}^{\infty}\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{2n} \\
&\quad-\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{\nu}\sum_{n=0}^{\infty}\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{2n}\Bigg\}\Bigg|x^{(3\nu_1-3\nu_2-4)/4}\,dx \\
&\leq \int_{x^\circ}^{\infty} e^{-(3/4)n_1\omega y_1^\circ\sqrt{1+x}}e^{\pi|\nu+1|}\frac{1}{1-((n_2y_2^\circ/2\omega^2)\sqrt{(x+1)/x})^2} \\
&\qquad\quad\times\Bigg|\Bigg\{\left(\frac{2\omega^2}{n_2y_2^\circ}\sqrt{\frac{x}{x+1}}\right)^{\nu} \\
&\qquad\qquad-\left(\frac{n_2y_2^\circ}{2\omega^2}\sqrt{\frac{x+1}{x}}\right)^{\nu}\Bigg\}\Bigg|x^{(3\nu_1-3\nu_2-4)/4}\,dx \\
&\leq e^{-(1/2)n_1\omega y_1^\circ}.
\end{aligned}$$

Thus we can conclude, if we show that the Γ functions in the denominators are sufficiently large,

$$\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2}{\omega^2}\right)\right| \leq e^{-(\pi/2)n_1\omega y_1^\circ}$$

when $n_1 \geq n_2^{1/3}$.

By abuse of notation, we can write $\Gamma(n \pm \nu + 1) = (n \pm \nu)!\Gamma(\pm\nu)$. Then we only need to show $|\Gamma(\pm\nu)| \geq e^{-\pi|\nu|}$. Stirling's formula [**15**, (1.4.23)] says

$$\Gamma(z) = e^{(z-(1/2))\ln z - z + (1/2)\ln 2\pi}\left[1 + O\left(|z|^{-1}\right)\right].$$

Let $\nu = x + iy$. Then we know $-1/2 < x < 1/2$, and $y \to \pm\infty$. Thus $1 > x + 1/2 > 0$. If $y > 0$,

$$\begin{aligned}\Gamma(-\nu) &= \Gamma(-x-iy)\\ &= e^{(-x-iy-(1/2))\ln(-x-iy)+x+iy+(1/2)\ln 2\pi}\left[1+O\left(|-x-iy|^{-1}\right)\right]\\ &= e^{(-(x+(1/2))-iy)(\ln(|-x-iy|)+i(-(\pi/2)+2J\pi))+x+iy+(1/2)\ln 2\pi}\\ &\quad\times\left[1+O\left(|-x-iy|^{-1}\right)\right]\\ &= e^{-(x+(1/2))\ln(|-x-iy|)-iy\ln(|-x-iy|)-i(x+(1/2))}\\ &\quad\times e^{(-(\pi/2)+2J\pi)+y(-(\pi/2)+2J\pi)+x+iy+(1/2)\ln 2\pi}\\ &\quad\times\left[1+O\left(|-x-iy|^{-1}\right)\right],\end{aligned}$$

and

$$|\Gamma(-\nu)| \geq e^{-\ln(x+y)-y(\pi/2)} \geq e^{-\pi|\nu|}.$$

If $y < 0$,

$$\begin{aligned}\Gamma(-\nu) &= \Gamma(-x-iy)\\ &= e^{-(x+(1/2))\ln(|-x-iy|)-iy\ln(|-x-iy|)-i(x+(1/2))}\\ &\quad\times e^{(-(\pi/2)+2J\pi)+y((\pi/2)+2J\pi)+x+iy+(1/2)\ln 2\pi}\\ &\quad\times\left[1+O\left(|-x-iy|^{-1}\right)\right],\end{aligned}$$

and

$$|\Gamma(-\nu)| \geq e^{-\ln(x+y)+y(\pi/2)} \geq e^{-\pi|\nu|}.$$

Likewise, when $y \to \pm\infty$,

$$\begin{aligned}\Gamma(\nu) &= e^{(x-(1/2)+iy)(\ln(|x+iy|)+i(\pm(\pi/2)+2J\pi))-x-iy+(1/2)\ln 2\pi}\\ &\quad\times\left[1+O\left(|-x-iy|^{-1}\right)\right]\\ &= e^{(x-(1/2))\ln(|x+iy|)+iy\ln(|x+iy|)+i(x-(1/2))(\pm(\pi/2)+2J\pi)}\\ &\quad\times e^{\mp y((\pi/2)+2J\pi)-x-iy+(1/2)\ln 2\pi}\left[1+O\left(|-x-iy|^{-1}\right)\right]\end{aligned}$$

Since $-1 < x - 1/2 < 0$,

$$|\Gamma(\nu)| \geq e^{-\ln(x+y) \mp y(\pi/2)} \geq e^{-\pi|\nu|}. \qquad \square$$

Proof of Lemma 3. In this case, we have $\omega \geq k|\nu|$ because $n_2 \geq (k|\nu|)^3$ and $\omega^2 \geq n_2/k|\nu|$. Then $n_1\omega y_1^\circ\sqrt{1+x} \geq k|\nu|$ as in Lemma 1. By the same argument in the proof of Lemma 1, we have

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \leq e^{-(3/4)n_1\omega y_1^\circ\sqrt{1+x}},$$

and

$$\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}\,dx\right| \leq e^{-(1/2)n_1\omega y_1^\circ}.$$

Thus, we conclude

$$\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2}{\omega^2}\right)\right| \leq e^{-(1/2)n_1\omega y_1^\circ}$$

when

$$n_2^{1/3} \geq n_1 \quad \text{and} \quad \frac{n_2}{\omega^2} \leq k\,|\nu|\,. \qquad \square$$

Proof of Lemma 4. In this case, we have $n_2/\omega^2 \geq k|\nu|$. Thus, we overestimate the second K Bessel function in a similar way to overestimating the first K Bessel function in the proof of Lemma 1. That is,

$$K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right) \leq e^{-(1/2)(n_2 y_2^\circ/\omega^2)}\sqrt{(x+1)/x}.$$

The first K Bessel function here does not cause any trouble because, unlike the variable in the second K Bessel function, the variable in the first K Bessel function, $n_1\omega y_1^\circ\sqrt{1+x}$ is never too small. Better yet, when $x \geq (k|\nu|)^2$, we have

$$\left|K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)\right| \leq e^{-(3/4)n_1\omega y_1^\circ\sqrt{1+x}}.$$

Thus,

$$\begin{aligned}
&\left|W_{\nu_1,\nu_2}\left(n_1\omega y_1^\circ, \frac{n_2 y_{2^\circ}}{\omega^2}\right)\right| \\
&\leq \pi^{-2}\left|(n_1\omega y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\omega^2}\right)^{1+(\nu_2-\nu_1)/2}\right| \\
&\quad\times\int_0^\infty \left|K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)\right| \\
&\qquad\qquad\times\left|K_\nu\left(\frac{n_2 y_2^\circ}{\omega^2}\sqrt{\frac{x+1}{x}}\right)\right|\left|x^{(3\nu_1-3\nu_2-4)/4}\right|\,dx \\
&\leq \pi^{-2}\left|(n_1\omega y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\omega^2}\right)^{1+(\nu_2-\nu_1)/2}\right| \\
&\quad\times\int_0^\infty \left|K_\nu\left(n_1\omega y_1^\circ\sqrt{1+x}\right)\right| e^{-(3\pi/4)n_1\omega y_1^\circ\sqrt{1+x}}\left|x^{(3\nu_1-3\nu_2-4)/4}\right|\,dx \\
&\leq e^{-(1/2)(n_2 y_2^\circ/\omega^2)}. \qquad \square
\end{aligned}$$

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REFERENCES

1. D. Bump, *Automorphic forms on $GL(3,\mathbf{R})$*, Springer Lecture Notes Math. **1083**, 1984.

2. D. Bump and J. Huntley, *Unramified Whittaker functions for $GL(3,\mathbf{R})$*, J. D'Analyse Math. **65** (1995), 19–44.

3. S. Catto, J. Huntley, J. Jorgenson and D. Tepper, *On an analogue of Selberg's eigenvalue conjecture for $SL_3(\mathbf{Z})$*, Proc. Amer. Math. Soc. **126**, 3455–3459.

4. S. Catto, J. Huntley, N. Moh and D. Tepper, *Weyl's law with error estimate*, Contemp. Math. **251** (1999), 107–116.

5. J. Huntley and D. Tepper, *A local Weyl's law–The angular distribution and multiplicity of cusp forms on product spaces*, Trans. Amer. Math. Soc. **330**, 97–110.

6. N. Lebedev, *Special functions and their applications*, Dover Books, New York, 1972.

7. N. Moh, *Analysis of the spectrum of* $SL(3, \mathbf{Z})\backslash SL(3, \mathbf{R})/SO(3, \mathbf{R})$, Ph.D. thesis, City University of New York Graduate Center, 2001.

8. F. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.

9. E. Stade, *On explicit formulas for* $GL(n, \mathbf{R})$*-Whittaker functions*, Duke Math. J. **60** (1989), 313–362.

10. A. Terras, *Harmonic analysis on symmetric spaces and applications*, Springer-Verlag, New York, 1985.

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