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MINIMAL USCO MAPS, DENSELY CONTINUOUS FORMS AND UPPER SEMI-CONTINUOUS FUNCTIONS

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ABSTRACT. New characterizations of minimal USCO maps and densely continuous forms are given. Let X and Y be topological spaces, and let Y be a T_1 regular space. Let $F : X \to Y$ be a set-valued mapping. The following are equivalent: (1) F is a minimal USCO map; (2) There is a quasicontinuous, subcontinuous function $f : X \to Y$ such that the closure of the graph $\overline{\mathrm{Gr}\, f}$ of f in $X \times Y$ is equal to the graph $\mathrm{Gr}\, F$ of F. For $Y = \mathbf{R}$ we also prove some isomorphic results between the class of minimal USCO maps and a certain class of quasicontinuous functions as well as between the class of densely continuous forms and a certain class of densely continuous functions equipped with uniformity of uniform convergence.

1. Introduction. Let X and Y be Hausdorff topological spaces. In our paper we give new characterizations of minimal USCO maps and densely continuous forms from X to Y.

There is a close relation between these two important classes of set-valued mappings. In particular, every minimal USCO map from a Baire space X into a metric space Y is a densely continuous form, and densely continuous forms have a kind of minimality property found in the theory of minimal USCO maps.

Interesting results concerning minimal USCO maps were found by Drewnowski and Labuda in their paper [**5**]. Our paper extends some results from [**5**]. We prove the following result: Let $F : X \to Y$ be a set-valued mapping, and let Y be a T_1 regular space. Then F is a USCO map if and only if there is a quasicontinuous and subcontinuous function $f : X \to Y$ such that the closure of the graph $\overline{\mathrm{Gr}\, f}$ of f is equal to the graph $\mathrm{Gr}\, F$ of F.

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We also study the set of selections of real-valued minimal USCO maps, $M(X)$, as well as the set of selections of locally bounded real-valued densely continuous forms, $D^\star(X)$. Further, we investigate the mapping which assigns to every element $F \in M(X)$ the supremum function f^F defined by $f^F(x) = \sup\{y : y \in F(x)\}$.

We prove that this mapping is a uniform isomorphism between $M(X)$ and a certain class of quasicontinuous functions as well as between $D^\star(X)$ and a certain class of densely continuous functions equipped with the uniformity of uniform convergence.

Continuing research of the papers [**11, 12, 16**] we also study the cardinal invariants of the topology of uniform convergence on $D^\star(X)$.

2. Minimal USCO maps. In what follows, let X and Y be Hausdorff topological spaces. Following [**5**] the term map is reserved for set-valued mappings. Also, for $x \in X$ and $y \in Y$, $\mathcal{U}(x)$ and $\mathcal{V}(y)$ are always used to denote a base of open neighborhoods of x in X and $y \in Y$, respectively. If $F : X \to Y$ is a (set-valued) map, then

$$\operatorname{Gr} F = \{(x, y) \in X \times Y : y \in F(x)\}$$

is the graph of F.

Notice that if $f : X \to Y$ is a single-valued function we will use the symbol $\operatorname{Gr} f$ also for the graph of f and the symbol $\overline{A}$ for closure of the set A in a topological space.

Given two maps $F, G : X \to Y$, we write $G \subset F$ and say that G is contained in F if $G(x) \subset F(x)$ for every $x \in X$, equivalently, if $\operatorname{Gr} G \subset \operatorname{Gr} F$.

A map $F : X \to Y$ is upper semi-continuous at a point $x \in X$ if, for every open set V containing $F(x)$, there exists a $U \in \mathcal{U}(x)$ such that

$$F(U) = \cup\{F(u) : u \in U\} \subset V.$$

F is upper semi-continuous if it is upper semi-continuous at each point of X. Following Christensen [**4**], we say that a map F is USCO if it is upper semi-continuous and takes nonempty compact values. Finally, a map F is said to be minimal USCO if it is a minimal element in the family of all USCO maps (with domain X and range Y); that is, if it is USCO and does not properly contain any other USCO map from X into Y. By an easy application of the Kuratowski-Zorn principle we can guarantee that every USCO map from X to Y contains a minimal USCO map from X to Y, see [**5**].

Minimal USCO maps were studied by Drewnowski and Labuda in [**5**]. In their paper they gave an interesting characterization of minimal USCO maps. In the first part of our paper we extend some results of [**5**].

Minimal multi-functions were studied in [**15**].

Of course, a natural question arises when functions $f : X \to Y$ do have the property that the closures of their graphs $\overline{\mathrm{Gr}\, f}$ in $X \times Y$ are the graphs of minimal USCO maps.

If $f : [0,1] \to [0,1]$ is a function with the property that $\overline{\mathrm{Gr}\, f} = [0,1] \times [0,1]$, then $\overline{\mathrm{Gr}\, f}$ is a USCO map which is not minimal. (It is very easy to define such a function f.)

In the first part of our paper we give a complete answer to the above question. To do this we need the following notions.

We say that a (single-valued) function $f : X \to Y$ is subcontinuous [**8**] at $x \in X$ if, for every net $\{x_\sigma : \sigma \in \Sigma\}$ in X converging to x, there is a convergent subnet of $\{f(x_\sigma) : \sigma \in \Sigma\}$. A function f is subcontinuous if it is subcontinuous at every point of X.

By [**18**, Theorem 2.1], $f : X \to Y$ is subcontinuous at $x \in X$ if and only if for every open cover $\mathcal{H}$ of Y there is a finite subset $\mathcal{F}$ of $\mathcal{H}$ and $U \in \mathcal{U}(x)$ such that $f(U) \subset \cup\mathcal{F}$.

A function $f : X \to Y$ is locally compact at $x \in X$ if there is a compact subset K of Y and $U \in \mathcal{U}(x)$ such that $f(U) \subset K$. A function f is locally compact if it is locally compact at every point of X.

Of course, every locally compact function is subcontinuous, and if the range space is locally compact, then these two notions coincide, see [**18**].

A function $f : X \to Y$ is called quasicontinuous [**17**] at $x \in X$ if, for every $V \in \mathcal{V}(f(x))$ and every $U \in \mathcal{U}(x)$, there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

The notion of quasi-continuity was perhaps used the first time by Baire in [**1**] in the study of points of continuity of separately continuous functions. As Baire indicated in his paper [**1**] the condition of quasi-continuity has been suggested by Vito Volterra.

There is a rich literature concerning the study of quasi-continuity, see for example [**1, 14, 19**], and a survey paper [**17**].

Proposition 2.1. *Let X and Y be topological spaces, and let Y be Hausdorff. Let F be a minimal* USCO *map from X to Y. If f is a selection of F, then* $\operatorname{Gr} F = \overline{\operatorname{Gr} f}$.

Proof. $\operatorname{Gr} F$ is a closed subset of $X \times Y$, thus $\overline{\operatorname{Gr} f} \subset \operatorname{Gr} F$. By [**5**, Proposition 3.2], $\overline{\operatorname{Gr} f}$ is a USCO map. The minimality of F implies that $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$. $\square$

Proposition 2.2. *Let X and Y be topological spaces, and let Y be Hausdorff. Let F be a* USCO *map from X to Y. If, for every selection f of F* $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$, *then F is a minimal* USCO *map.*

Proof. Suppose, by way of contradiction, that F is not a minimal USCO map. Let G be a minimal USCO map which is contained properly in F. Let $(x, y) \in \operatorname{Gr} F \setminus \operatorname{Gr} G$. Let g be a selection of G. Then $\overline{\operatorname{Gr} g} \subset \operatorname{Gr} G$, since $\operatorname{Gr} G$ is a closed set in $X \times Y$. Thus, $(x, y) \notin \overline{\operatorname{Gr} g}$, a contradiction since g is also a selection of F. $\square$

Proposition 2.3. *Let X and Y be topological spaces. Let F be a* USCO *map from X to Y. Then every selection f of F is subcontinuous.*

Proof. Suppose there is a selection f of F which is not subcontinuous. Thus there is a net $\{x_\sigma : \sigma \in \Sigma\} \subset X$ convergent to a point x such that $\{f(x_\sigma) : \sigma \in \Sigma\}$ has no cluster point in Y. The compactness of $F(x)$ implies that there is an open set $O \supset F(x)$ and $\sigma_0 \in \Sigma$ with $f(x_\sigma) \notin O$ for every $\sigma \geq \sigma_0$. The upper semi-continuity of F at x implies that there is a $V \in \mathcal{U}(x)$ with $F(z) \subset O$ for every $z \in V$. There is a $\sigma_1 \geq \sigma_0$ such that $x_\sigma \in V$ for every $\sigma \geq \sigma_1$; i.e., $f(x_\sigma) \in F(x_\sigma) \subset O$ for every $\sigma \geq \sigma_1$, a contradiction. $\square$

The following theorem extends Proposition 4.5 in [**5**].

Theorem 2.4. *Let X and Y be topological spaces. Let F be a* USCO *map from X to Y. Then the following are equivalent.*

(1) *F is minimal*;

(2) *F maps isolated points into singletons, and every selection f of F is quasi-continuous.*

Proof. (1) $\Rightarrow$ (2). Let F be a minimal USCO map from X to Y. It is easy to verify that F maps isolated points into singletons. Let f be a selection of F. Suppose f is not quasi-continuous at x_0; of course, x_0 cannot be an isolated point. Thus, there are open sets O_{x_0} and $O_{f(x_0)}$ in X and Y, respectively, such that $x_0 \in O_{x_0}$, $f(x_0) \in O_{f(x_0)}$ and such that for every nonempty open set $V \subset O_{x_0}$ there is a $z \in V$ with $f(z) \notin O_{f(x_0)}$. By Proposition 2.3, f is subcontinuous. The subcontinuity of f guarantees that, for every $x \in O_{x_0}$, $F(x) \cap (Y \setminus O_{f(x_0)}) \neq \varnothing$. Thus, $G = \operatorname{Gr} F \setminus (O_{x_0} \times O_{f(x_0)})$ is the graph of a USCO map and $G \subset \operatorname{Gr} F$, a contradiction with the minimality of F.

(2) $\Rightarrow$ (1). Suppose F is not minimal. Let $G \subset F$ be a minimal USCO map, and let $(x_0, y_0) \in \operatorname{Gr} F \setminus \operatorname{Gr} G$. Let g be any selection of G. Define function h from X to Y as follows:

$$h(x) = \begin{cases} y_0 & x = x_0, \\ g(x) & x \neq x_0. \end{cases}$$

Then of course h is a selection of F which is not quasi-continuous. (There are open sets U and V in X and Y, respectively, such that $x_0 \in U$, $y_0 \in V$ and $(U \times V) \cap \operatorname{Gr} G = \varnothing$, i.e., $(U \times V) \cap (\operatorname{Gr} h \setminus \{(x_0, h(x_0))\}) = \varnothing$.) $\square$

Theorem 2.5. *Let X and Y be topological spaces, and let Y be a T_1 regular space. Let F be a map from X to Y. Then the following are equivalent*:

(1) *F is a minimal* USCO *map*;

(2) *There exist a quasi-continuous and subcontinuous function f from X to Y such that $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$*;

(3) *Every selection f of F is quasi-continuous, subcontinuous and* $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$.

Proof. (1) ⇒ (3) by Propositions 2.1, 2.3 and Theorem 2.4.

(3) ⇒ (2) is clear.

(2) ⇒ (1). Let f be quasi-continuous, subcontinuous and $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$. First we prove that $F(x)$ is compact for every $x \in X$. Let $x \in X$. It is very easy to verify that $F(x) = \cap\{\overline{f(O)} : O \in \mathcal{U}(x)\}$. Let $\mathcal{G}$ be an open cover of $F(x)$. Let $\mathcal{H}$ be an open cover of $F(x)$ such that the family $\{\overline{U} : U \in \mathcal{H}\}$ is a refinement of $\mathcal{G}$. For every $y \in Y \setminus F(x)$, let O_y be an open set in Y such that $\overline{O_y} \cap F(x) = \varnothing$. Then the family $\mathcal{H} \cup \{O_y : y \in Y \setminus F(x)\}$ is an open cover of Y. The subcontinuity of f at x implies that there is an $O \in \mathcal{U}(x)$ and a finite subfamily $\mathcal{H}^*$ of $\mathcal{H}$ and a finite indexed set I such that $f(O) \subset \cup\{U : U \in \mathcal{H}^*\} \cup \cup\{O_{y_i} : i \in I\}$. Then $F(x) \subset \overline{f(O)} \subset \cup\{\overline{U} : U \in \mathcal{H}^*\} \cup \{\overline{O_{y_i}} : i \in I\}$. Thus, $F(x) \subset \cup\{\overline{U} : U \in \mathcal{H}^*\}$, i.e., there is a finite subfamily of $\mathcal{G}$ which covers $F(x)$; i.e., $F(x)$ is compact.

Now we prove that F is upper semi-continuous. Let $x \in X$. Let V be an open set such that $F(x) \subset V$. The regularity of Y implies that there is an open set H with $F(x) \subset H \subset \overline{H} \subset V$. We claim that there is an $O \in \mathcal{U}(x)$ such that $f(O) \subset H$. Suppose that, for every $O \in \mathcal{U}(x)$, there exists an $x_O \in O$ with $f(x_O) \notin H$. The net $\{x_O;\ O \in \mathcal{U}(x)\}$ converges to x. The subcontinuity of f at x implies that the net $\{f(x_O);\ O \in \mathcal{U}(x)\}$ has a cluster point $y \notin H$, but $(x, y) \in \overline{\operatorname{Gr} f} = \operatorname{Gr} F$, a contradiction. Thus, there is an $O \in \mathcal{U}(x)$ such that $F(O) \subset \overline{f(O)} \subset \overline{H} \subset V$.

Now we prove that F is minimal. Suppose that $\overline{\operatorname{Gr} f}$ is not minimal. Thus, there is a $G \subset \overline{\operatorname{Gr} f}$, $G \neq \overline{\operatorname{Gr} f}$ which is the graph of a minimal USCO map. Let $(x, y) \in \overline{\operatorname{Gr} f} \setminus G$. Thus, there are open sets U and V in X and Y, respectively, such that $x \in U$, $y \in V$ and $(U \times V) \cap G = \varnothing$. Let $O \in \mathcal{V}(y)$ be such that $O \subset \overline{O} \subset V$. Let $(u, f(u)) \in U \times O$. The quasi-continuity of f at u implies that there is a nonempty open set $L \subset U$ such that $f(z) \in O$ for every $z \in L$. Take $s \in L$ and $y \in G(s)$. Then $(s, y) \in L \times (Y \setminus \overline{O})$ but $L \times (Y \setminus \overline{O}) \cap \operatorname{Gr} f = \varnothing$, a contradiction, since $G \subset \overline{\operatorname{Gr} f}$. □

We have the following variant of Theorem 2.5 for locally compact Hausdorff spaces Y.

Proposition 2.6. *Let X and Y be topological spaces, and let Y be a locally compact Hausdorff space. Let F be a map from X to Y. Then the following are equivalent*:

(1) *F is a minimal* USCO *map*;

(2) *There exists a quasi-continuous, locally compact function $f : X \to Y$ such that* $\overline{\mathrm{Gr}\, f} = \mathrm{Gr}\, F$;

(3) *Every selection f of F is quasi-continuous, locally compact and* $\overline{\mathrm{Gr}\, f} = \mathrm{Gr}\, F$.

3. Densely continuous forms. In this part of our paper we continue the study of so-called densely continuous forms introduced by McCoy and Hammer in [**9**] and then studied by Holá, McCoy, Holý and Vadovič in their papers [**10**–**13**].

Let X and Y be Hausdorff topological spaces. Densely continuous forms from X to Y can be considered as maps (set-valued mappings) from X to Y which have a kind of minimality property found in the theory of minimal USCO maps. In particular, every minimal USCO map from a Baire space X into a metric space Y is a densely continuous form.

To define a densely continuous form from X to Y [**9**], denote by $DC(X,Y)$ the set of all functions from X to Y which are continuous at all points of some dense subset of X. We call such functions densely continuous. Of course, $DC(X,Y)$ contains the set $C(X,Y)$ of all continuous functions from X to Y. There are many other interesting subsets in $DC(X,Y)$. For example, if Y is a locally compact second countable space and X is a Baire space, then $DC(X,Y)$ contains all functions from X to Y with closed graphs [**10**].

If Y is the set $\mathbf{R}$ of all real numbers and X is a Baire space, then also all upper and lower semi-continuous functions on X belong to $DC(X,Y)$.

For each function f from X to Y, denote by $C(f) = \{x \in X : f$ is continuous at $x\}$. For every $f \in DC(X,Y)$, $\mathrm{Gr}\,(f\lceil C(f))$ is a subset of $X \times Y$. Denote by $\overline{\mathrm{Gr}\,(f\lceil C(f))}$ the closure of $\mathrm{Gr}\,(f\lceil C(f))$ in $X \times Y$.

We define the set $D(X,Y)$ of densely continuous forms by

$$D(X,Y) = \{\overline{\mathrm{Gr}\,(f\lceil C(f))} : f \in DC(X,Y)\}.$$

The densely continuous forms from X to Y may be considered as maps (set-valued) mappings. For each $x \in X$ and $\Phi \in D(X,Y)$, define $\Phi(x) = \{y \in Y : (x,y) \in \Phi\}$.

Define by $A(X,Y)$ the following set of functions

$$A(X,Y) = \{f : X \to Y : \text{ for every } x \in X \text{ and for every neighborhood } U \text{ of } (x, f(x)) \text{ there exist } y \in C(f) \text{ such that } (y, f(y)) \in U\}.$$

Of course, $A(X,Y) \subset DC(X,Y)$.

It is very easy to verify that every function from $A(X,Y)$ is quasi-continuous. The following example shows that the opposite is not true.

Example 3.1 [**19**]. Let $X = \mathbf{R}$ with the usual Euclidean topology, and let $Y = \mathbf{R}$ with the Sorgenfrey topology. Let $f : X \to Y$ be the identity function. Then f is quasi-continuous, but the set $C(f) = \varnothing$.

However, if X is a Baire space and Y is a metric space, then every quasi-continuous function $f : X \to Y$ has a dense set $C(f)$ of the points of continuity [**17**], i.e., f belongs to $A(X,Y)$.

Clearly, if $f : X \to Y$, then $f \in A(X,Y)$ if and only if $\overline{\mathrm{Gr}\, f} = \overline{\mathrm{Gr}\,(f\lceil C(f))}$.

We have the following characterization of elements of $D(X,Y)$.

Proposition 3.2. *Let X and Y be topological spaces, Y regular and $F : X \to Y$ such that $F(x) \neq \varnothing$ for every $x \in X$. Then the following are equivalent*:

(1) $F \in D(X,Y)$;

(2) *There is a function $f \in A(X,Y)$ such that $\overline{\mathrm{Gr}\, f} = \mathrm{Gr}\, F$;*

(3) *Every selection f of F belongs to $A(X,Y)$ and $\overline{\mathrm{Gr}\, f} = \mathrm{Gr}\, F$.*

Proof. (1) $\Rightarrow$ (3). Let f be a selection of F. There is a $g \in DC(X,Y)$ such that $F = \overline{\mathrm{Gr}\,(g\lceil C(g)}$. Of course, $F(x) = \{g(x)\}$ for every $x \in C(g)$, i.e., $f(x) = g(x)$ for every $x \in C(g)$. It is easy to verify that $C(g) \subset C(f)$. (Let $x \in C(g)$. Suppose $x \notin C(f)$. There is a $V \in \mathcal{V}(f(x))$ such that for every $U \in \mathcal{U}(x)$ there is an $x_U \in U$ with $f(x_U) \notin V$. Let $H \in \mathcal{V}(f(x))$ be such that $\overline{H} \subset V$. The continuity of g at x implies that there is an $O \in \mathcal{U}(x)$ such that $g(O) \subset H$. Then $O \times (Y \setminus \overline{H})$ is a neighborhood of $(x_O, f(x_O))$ which has an empty intersection with the graph $\mathrm{Gr}\,g$ of g, a contradiction, since $\mathrm{Gr}\,f \subset \overline{\mathrm{Gr}\,(g\lceil C(g))}$.)

Thus, the set $C(f)$ of the points of continuity of f is dense in X, i.e., $f \in DC(X,Y)$. Since $\mathrm{Gr}\,f \subset \mathrm{Gr}\,F = \overline{\mathrm{Gr}\,(g\lceil C(g))} \subset \overline{\mathrm{Gr}\,(f\lceil C(f))}$, we have that $f \in A(X,Y)$ and $\mathrm{Gr}\,F = \overline{\mathrm{Gr}\,f}$.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) is also trivial since if $f \in A(X,Y)$, then of course $f \in DC(X,Y)$ and by the above, $\overline{\mathrm{Gr}\,f} = \overline{\mathrm{Gr}\,f\lceil C(f)}$. □

Corollary 3.3. *Let X be a Baire space and Y a metric space. Let $F : X \to Y$ be such that $F(x) \neq \varnothing$ for every $x \in X$. The following are equivalent*:

(1) $F \in D(X,Y)$;

(2) *There is a quasi-continuous function $f : X \to Y$ such that $\overline{\mathrm{Gr}\,f} = \mathrm{Gr}\,F$;*

(3) *Every selection f of F is quasi-continuous and $\overline{\mathrm{Gr}\,f} = \mathrm{Gr}\,F$.*

Denote now by $D(X)$ [**16**] the space of all real-valued densely continuous forms from a topological space X.

If $\Phi : X \to \mathbf{R}$ is a mapping (single-valued or set-valued) and $A \subset X$, we say that Φ is bounded on A, provided that the set

$$\Phi(A) = \bigcup\{\Phi(x) : \ x \in A\}$$

is a bounded subset of $\mathbf{R}$. We say that Φ is locally bounded, provided that each point of X has a neighborhood on which Φ is bounded.

Now define $D^\star(X)$ to be the set of all members of $D(X)$, that are locally bounded.

Remark 3.4. Let $U(X)$ and $M(X)$ be the set of all real-valued USCO maps and minimal USCO maps, respectively. Then $D^\star(X) \subset M(X)$. In fact, if $\Phi \in D^\star(X)$, then for all $x \in X$, $\Phi(x)$ is a nonempty compact set. By a result of Berge [**3**, page 112] any map with a closed graph which has a compact range is upper semi-continuous. Since upper semi-continuity is a local property, every $\Phi \in D^\star(X)$ belongs to $U(X)$. Now by [**5**, Theorem 4.7], Φ is minimal USCO and $D^\star(X) \subset M(X)$.

If X is a Baire space, then $M(X) \subset D^\star(X)$. In fact, if Φ is an upper semi-continuous map with nonempty values, then by [**7**] there is a dense subset E of X such that Φ is lower semi-continuous at each $x \in E$. The minimality of Φ implies that Φ must be single-valued at every point of E. Then any selection of Φ is continuous in each $x \in E$ and by [**5**] $\Phi \in D(X)$. It is easy to show that every USCO map from X to $\mathbf{R}$ is locally bounded. As a result, we have that if X is a Baire space then $M(X) = D^\star(X)$.

We use the notation $DC(X)$ and $A(X)$ for $DC(X, \mathbf{R})$ and $A(X, \mathbf{R})$, respectively, and $DC^\star(X), A^\star(X)$ for locally bounded elements of $DC(X)$ and $A(X)$, respectively. By $UC(X)$ we denote the set of all upper semi-continuous functions.

Remark 3.5. We have an equivalence relation on $DC^\star(X)$ defined by $f \sim g$ if and only if $\overline{\mathrm{Gr}\,(f\lceil C(f))} = \overline{\mathrm{Gr}\,(g\lceil C(g))}$, so that $D^\star(X)$ can be identified with the set of equivalence classes of $DC^\star(X)$ under $\sim$, see [**9**].

Let F be a USCO map from a topological space X to $\mathbf{R}$. Define the function f^F as follows:

$$f^F(x) = \sup\{y : y \in F(x)\}.$$

Then of course f^F is a selection of F and f^F is upper semi-continuous.

If F is a minimal USCO map from a topological space X to $\mathbf{R}$, then by Theorem 2.6, f^F is also quasi-continuous and locally bounded.

In what follows denote by $Q(X)$ the space of all quasi-continuous real-valued functions defined on a topological space X and by $Q^\star(X)$ the set of all locally bounded elements of $Q(X)$.

Define a mapping $\Omega : M(X) \to Q^\star(X) \cap UC(X)$ by $\Omega(F) = f^F$.

Proposition 3.6. *The mapping* $\Omega : M(X) \to Q^\star(X) \cap UC(X)$ *is a bijection and* $\Omega(D^\star(X)) = A^\star(X) \cap UC(X)$.

Proof. To show that Ω is one-to-one, let $F, G \in M(X)$ be such that $F \neq G$, i.e., $\overline{\mathrm{Gr}\, f^F} \neq \overline{\mathrm{Gr}\, f^G}$. Without loss of generality, we can suppose that there is a point $(u,v) \in \overline{\mathrm{Gr}\, f^F}$ such that $(u,v) \notin \overline{\mathrm{Gr}\, f^G}$. Then there exists an open neighborhood U of (u,v) such that $U \cap \overline{\mathrm{Gr}\, f^G} = \varnothing$. There must exist a point $w \in X$ such that $(w, f^F(w)) \in U$ and so $(w, f^F(w)) \notin \overline{\mathrm{Gr}\, f^G}$. Since f^G is a selection of G, we have that $f^F \neq f^G$.

To show that the mapping Ω is onto, let $f \in Q^\star(X) \cap UC(X)$. By Theorem 2.6, $\overline{\mathrm{Gr}\, f}$ is a minimal USCO map. The upper semi-continuity of f guarantees the equality $f(x) = \sup\{y : (x,y) \in \overline{\mathrm{Gr}\, f}\}$ for every $x \in X$, i.e., $\Omega(\overline{\mathrm{Gr}\, f}) = f$. Thus, Ω is onto.

If $F \in D^\star(X)$, then by Proposition 3.2, $f^F \in A(X)$. Of course, f^F is upper semi-continuous and locally bounded, i.e., $\Omega(D^\star(X)) \subset A^\star(X) \cap UC(X)$. Now we prove the equality. Let $f \in A^\star(X) \cap UC(X)$. By Proposition 3.2, $\overline{\mathrm{Gr}\, f} \in D(X)$ and since f is locally bounded $\overline{\mathrm{Gr}\, f} \in D^\star(X)$. The upper semi-continuity of f guarantees the equality $f(x) = \sup\{y : (x,y) \in \overline{\mathrm{Gr}\, f}\}$ for every $x \in X$. Thus, $\Omega(\overline{\mathrm{Gr}\, f}) = f$. $\square$

Of course, for a map $F : X \to \mathbf{R}$ with bounded values we can define also

$$f_F(x) = \inf\{y : y \in F(x)\}.$$

If F is a USCO map, then f_F is lower semi-continuous. We can give a similar result for lower semi-continuous functions as we gave above for upper semi-continuous functions. The result for lower semi-continuous functions is dual.

Denote by $LC(X)$ the set of all lower semi-continuous functions, and define the mapping $\mathcal{S} : M(X) \to Q^\star(X) \cap LC(X)$ by $\mathcal{S}(F) = f_F$.

Proposition 3.7. *The mapping* $\mathcal{S} : M(X) \to Q^\star(X) \cap LC(X)$ *is a bijection, and* $\mathcal{S}(D^\star(X)) = A^\star(X) \cap LC(X)$.

4. Let (X,d) be a metric space. The open d-ball with center $z_0 \in \mathbf{R}$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$, and the ε-parallel body $\cup_{a\in A} S_\varepsilon(a)$ for a subset A of $\mathbf{R}$ will be denoted by $S_\varepsilon(A)$.

We denote by $2^{\mathbf{R}}$ the space of all closed subsets of $\mathbf{R}$, by $CL(\mathbf{R})$ the space of all nonempty closed subsets of $\mathbf{R}$. By $\mathfrak{K}(X)$ and $\mathfrak{F}(X)$ we mean the family of all nonempty compact and finite subsets of X, respectively.

If $A \in CL(\mathbf{R})$, the distance functional $d(.,A) : \mathbf{R} \mapsto [0,\infty)$ is described by the familiar formula

$$d(z,A) = \inf\{d(z,a) :\ a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on $2^{\mathbf{R}}$ [**2**] is defined by

$$H_d(A,B) = \max\{\sup\{d(a,B) :\ a \in A\},\ \sup\{d(b,A) :\ b \in B\}\},$$

if A and B are nonempty. If $A \neq \varnothing$, take $H_d(A,\varnothing) = H_d(\varnothing,A) = \infty$. We will often use the following equality on $CL(\mathbf{R})$:

$$H_d(A,B) = \inf\{\varepsilon > 0 :\ A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

Denote by $F(X)$ the set of all maps from a topological space X to $\mathbf{R}$ with closed values.

Following [**9**] we will define the topology τ_p of pointwise convergence on $F(X)$. The topology τ_p of pointwise convergence on $F(X)$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A,\varepsilon) = \{(\Phi,\Psi) :\ \text{for all } x \in A,\ H_d(\Phi(x),\Psi(x)) < \varepsilon\}$$

where $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. The general τ_p-basic neighborhood of $\Phi \in F(X)$ will be denoted by $W(\Phi,A,\varepsilon)$, i.e., $W(\Phi,A,\varepsilon) = W(A,\varepsilon)[\Phi] = \{\Psi : H_d(\Phi(x),\Psi(x)) < \varepsilon$ for every $x \in A\}$. If $A = \{a\}$, we will write $W(\Phi,a,\varepsilon)$ instead of $W(\Phi,\{a\},\varepsilon)$. The space $D^\star(X)$ $(M(X))$ with the induced topology τ_p will be denoted by $D^\star_p(X)$ $(M_p(X))$ for short.

We will define the topology τ_K of uniform convergence on compact sets on $F(X)$ [**9**]. This topology is induced by the uniformity $\mathfrak{U}_K$ which has a base consisting of sets of the form

$$W(K,\varepsilon) = \{(\Phi,\Psi) :\ \text{for all } x \in K,\ H_d(\Phi(x),\Psi(x)) < \varepsilon\},$$

where $A \in \mathfrak{K}(X)$ and $\varepsilon > 0$. The general τ_K-basic neighborhood of $\Phi \in F(X)$ will be denoted by $W(\Phi, K, \varepsilon)$, i.e., $W(\Phi, K, \varepsilon) = W(K, \varepsilon)[\Phi]$. The space $D^\star(X)$ $(M(X))$ with the induced topology τ_K will be denoted by $D^\star_K(X)$ $(M_K(X))$.

Finally we will define the topology τ_{UC} of uniform convergence on $F(X)$ [**9**]. Let e be the (extended-valued) metric on $F(X)$ defined by

$$e(\Phi, \Psi) = \sup\{H_d(\Phi(x), \Psi(x)) : \ x \in X\}$$

for each $\Phi, \Psi \in F(X)$. Then the topology of uniform convergence for the space $F(X)$ is the topology generated by the metric e. The space $D^\star(X)$ $(M(X))$ with the induced topology τ_{UC} will be denoted by $D^\star_{UC}(X)$ $(M_{UC}(X))$.

We use the symbols τ_p $(\mathfrak{U}_p)$, τ_K $(\mathfrak{U}_K)$ and τ_{UC} $(\mathfrak{U}_{UC})$ also for the topology (uniformity) of pointwise convergence, the topology (uniformity) of uniform convergence on compacta and the topology (uniformity) of uniform convergence on the space of all functions from X to $\mathbf{R}$, respectively.

Remark 4.1. It is easy to see that if A and B are nonempty compact subsets of $\mathbf{R}$, then $d(\sup A, \sup B) \leqq H_d(A, B)$.

Proposition 4.2. *Let X be a topological space. Then the mapping Ω from $(M(X), \mathfrak{U}_p)$ onto $(Q^\star(X) \cap UC(X), \mathfrak{U}_p)$ is uniformly continuous.*

Proof. The proof follows from Remark 4.1. □

The following example shows that even τ_K-convergence in $Q^\star(X) \cap UC(X)$ does not imply the convergence in $M_p(X)$.

Example 4.3. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1 with the usual topology. Let L be the set of all limit ordinal numbers different from ω_1. Put $X = W \setminus L$ and equip X with the induced topology from W.

If λ is a nonlimit number, there are a unique integer $I(\lambda) \in Z^+$ and a limit number β such that $\lambda = \beta + I(\lambda)$.

For every $n \in Z^+$, put $C_n = \{\lambda \in X \setminus \omega_1 : I(\lambda) = n\}$. Then $\omega_1 \in \overline{C_n}$ for every $n \in Z^+$. Further, for every $n \in Z^+$, let $f_n \in Q^\star(X) \cap UC(X)$ be defined as follows: $f_n(x) = 0$ if $x \in C_n$ and $f_n(x) = 1$ otherwise. It is easy to verify that $\{f_n\}$ τ_K-converges to the function f identically equal to 1. However, the sequence $\{\Omega^{-1}(f_n)\}$ fails to converge to $\Omega^{-1}(f)$ in $M_p(X)$ since $\Omega^{-1}(f_n) = \overline{\operatorname{Gr} f_n}$ takes the value $\{0,1\}$ at ω_1 for every $n \in Z^+$ and $\Omega^{-1}(f)(\omega_1) = \{1\}$.

Theorem 4.4. *Let X be a topological space. Then the spaces $(M(X), e)$ and $(Q^\star(X) \cap UC(X), e)$ are uniformly isomorphic. Also the spaces $(D^\star(X), e)$ and $(A^\star(X) \cap UC(X), e)$ are uniformly isomorphic.*

Proof. As we proved above, the mapping Ω from $M(X)$ to $Q^\star(X) \cap UC(X)$ is a bijection. By Remark 4.1 we have that $\Omega : (M(X), e) \to (Q^\star(X) \cap UC(X), e)$ is uniformly continuous. To prove that also Ω^{-1} is uniformly continuous, it is sufficient to show that if, for $f, g \in Q^\star(X) \cap UC(X)$, $d(f(x), g(x)) < \varepsilon$ for every $x \in X$, then $H_d(\overline{\operatorname{Gr} f}(x), \overline{\operatorname{Gr} g}(x)) \leq \varepsilon$ for every $x \in X$.

Suppose that this is not true. Then there exists an $x_0 \in X$ such that $H_d(\overline{\operatorname{Gr} f}(x_0), \overline{\operatorname{Gr} g}(x_0)) > \varepsilon$. There is an $r \in \overline{\operatorname{Gr} f}(x_0)$ such that $d(r, \overline{\operatorname{Gr}(g)}(x_0)) > \varepsilon$, or there is an $s \in \overline{\operatorname{Gr} g}(x_0)$ such that $d(s, \overline{\operatorname{Gr} f}(x_0)) > \varepsilon$. Suppose the first case occurs; the proof of the second one is analogous. Put $\beta = d(r, \overline{\operatorname{Gr} g}(x_0)) - \varepsilon$. Let $\{x_\sigma : \sigma \in \Sigma\}$ be a net in X converging to x_0, such that the net $\{f(x_\sigma) : \sigma \in \Sigma\}$ converges to r. Then for $\beta/4$ there is a $\sigma_0 \in \Sigma$ such that $f(x_\sigma) \in S_{\beta/4}(r)$ for all $\sigma > \sigma_0$. The upper semi-continuity of $\overline{\operatorname{Gr} g}$ at x_0 implies that there is a $U \in \mathcal{U}(x_0)$ such that $\overline{\operatorname{Gr} g}(x) \subset S_{\beta/4}(\overline{\operatorname{Gr} g}(x_0))$ for all $x \in U$. Let $\sigma \in \Sigma$ be such that $\sigma > \sigma_0$ and $x_\sigma \in U$. Then of course $d(f(x_\sigma), g(x_\sigma)) > \varepsilon$, a contradiction.

Concerning the proof of the second statement of the theorem, by Proposition 3.6 we have that $\Omega(D^\star(X)) = A^\star(X) \cap UC(X)$, and by the above we know that $\Omega : (M(X), e) \to (Q^\star(X) \cap UC(X), e)$ is uniformly isomorphic. Thus, also the restriction of Ω on $D^\star(X)$ to $A^\star(X) \cap UC(X)$ is uniformly isomorphic. □

Theorem 4.5. *Let X be a locally compact topological space. The spaces $(M(X), \mathfrak{U}_K)$ and $(Q^\star(X) \cap UC(X), \mathfrak{U}_K)$ are uniformly isomor-*

phic. Also the spaces $(D^\star(X), \mathfrak{U}_K)$ *and* $(A^\star(X) \cap UC(X), \mathfrak{U}_K)$ *are uniformly isomorphic.*

Proof. As we proved above, the mapping Ω from $M(X)$ to $Q^\star(X) \cap UC(X)$ is a bijection. By Remark 4.1, we have that $\Omega : (M(X), \mathfrak{U}_K) \to (O^\star(X) \cap UC(X), \mathfrak{U}_K)$ is uniformly continuous. To prove that also Ω^{-1} is uniformly continuous, let $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. The local compactness of X implies that there is an open set G in X such that $K \subset G$ and $\overline{G}$ is compact. Let $f, g \in Q^\star(X) \cap UC(X)$ be such that $d(f(x), g(x)) < \varepsilon$ for every $x \in \overline{G}$. To prove that $H_d(\overline{\mathrm{Gr}\, f}(x), \overline{\mathrm{Gr}\, g}(x)) \leq \varepsilon$ for every $x \in K$, we can use a similar idea as in the proof of Theorem 4.4. □

The cardinal function properties of character, pseudo character, density, weight, net weight and cellularity on $D_p^\star(X)$ and $D_k^\star(X)$ were studied in the papers [**11, 12**].

We will end our paper with some results concerning cardinal invariants on $D_{UC}^\star(X)$.

Let X be a topological space. Since $D_{UC}^\star(X)$ is always metrizable, we have

$$c(D_{UC}^\star(X)) = d(D_{UC}^\star(X)) = nw(D_{UC}^\star(X)) = w(D_{UC}^\star(X)).$$

We give some estimates for cellularity of $D_{UC}^\star(X)$, i.e., also for its density, net weight and weight.

Let us recall that for cardinals μ, η, it is customary to put

$$\mu^{<\eta} = \sup\{\mu^\alpha : \alpha < \eta, \alpha \text{ cardinal}\}.$$

Proposition 4.6. *For every space* X, $2^{<c(X)} \leq c(D_{UC}^\star(X)) \leq 2^{w(X)}$.

Proof. First we prove that $2^{<c(X)} \leq c(D_{UC}^\star(X))$. We will use some ideas from [**11**]. Let $\mathcal{G}$ be a family of pairwise disjoint nonempty open subsets of X. Let $2^{\mathcal{G}}$ denote the set of all functions from $\mathcal{G}$ to $\{0, 1\}$. For each $\Psi \in 2^{\mathcal{G}}$, define $f_\Psi : X \to \mathbf{R}$ as follows: $f_\Psi(x) = \Psi(G)$ if $x \in G$ for some $G \in \mathcal{G}$ and $f_\Psi(x) = 0$ otherwise.

Of course, for each $\Psi \in 2^{\mathcal{G}}$, the set $C(f_\Psi)$ is dense in X. So each $F_\Psi = \overline{\mathrm{Gr}\,(f_\Psi\lceil C(f_\Psi))}$, is a member of $D^\star(X)$. Now, for each $\Psi \in 2^{\mathcal{G}}$, define $B_\Psi = \{\Phi \in D^\star(X) : e(F_\Psi, \Phi) < 1/4\}$. Then $\{B_\Psi : \Psi \in 2^{\mathcal{G}}\}$ is a pairwise disjoint family of nonempty open subsets of $D^\star_{UC}(X)$. Therefore,

$$2^{|\mathcal{G}|} \leq c(D^\star_{UC}(X)).$$

Since $\mathcal{G}$ is any family of pairwise disjoint nonempty open subsets of X we have

$$2^{<c(X)} \leq c(D^\star_{UC}(X)).$$

Now $c(D^\star_{UC}(X)) \leq d(D^\star_{UC}(X)) \leq |D^\star(X)|$. By [**16**, Proposition 5.3], $|D^\star(X)| \leq 2^{w(X)}$, so we are done. □

Remark 4.7. It is easy to verify from the proof of Proposition 4.6 that if X is a topological space in which there is a family of pairwise disjoint nonempty open sets with the cardinality of $c(X)$, then

$$2^{c(X)} \leq c(D^\star_{UC}(X)) \leq 2^{w(X)}.$$

Corollary 4.8. *If X is a metrizable space, then*

$$2^{c(X)} = c(D^\star_{UC}(X)) = w(D^\star_{UC}(X)) = 2^{w(X)}.$$

Proof. If X is a metrizable space, then we have $c(X) = d(X) = nw(X) = w(X)$. Moreover, if X is a metrizable space of weight m, there exists a family of pairwise disjoint nonempty open sets which has cardinality m [**6**]. □

Of course the assumption of the metrizability of X in the above corollary is essential.

Example 4.9. Let $D(m)$ be the discrete space of the cardinality m, and let $\beta D(m)$ be the Čech-Stone compactification of $D(m)$. Then we have

$$c(D^\star_{UC}(\beta D(m)) = 2^m = 2^{c(\beta D(m))} < 2^{w(\beta D(m))} = 2^{2^m}.$$

Proof. It is sufficient to realize that we have $d(D^\star_{UC}(\beta D(m))) \leq |D^\star_{UC}(\beta D(m))| = |C(\beta D(m))| \leq (2^{\aleph_0})^m = 2^m$. $\square$

There are also nonmetrizable spaces for which the equality in Corollary 4.8 holds.

Example 4.10. Let $X = \beta N \setminus N$. Then $2^{\mathfrak{c}} = 2^{c(X)} = c(D^\star_{UC}(X)) = w(D^\star_{UC}(X)) = 2^{w(X)}$. It is known [**6**] that $\mathfrak{c} = c(X) = w(X)$ and that X contains a family of cardinality $\mathfrak{c}$ consisting of pairwise disjoint nonempty open sets.

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