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A PRIME GEODESIC THEOREM FOR HIGHER RANK II: SINGULAR GEODESICS

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ABSTRACT. A prime geodesic theorem for singular geodesics in a locally symmetric space is proved. As an application, an asymptotic formula for units in number fields is given.

1. Introduction. The prime geodesic theorem gives an asymptotic growth for the number of closed geodesics counted by their lengths [**11, 18, 23–27, 31**]. Before the paper [**7**], it has only been proven for manifolds of strictly negative curvature. For manifolds containing higher dimensional flats it is not a priori clear what a prime geodesic theorem might look like. In the paper [**7**] the author has given such a theorem for regular geodesics in a locally symmetric space. Regular geodesics give points in a higher dimensional Weyl cone, and the prime geodesic theorem describes the distribution of these points. In the current paper we turn to the remaining, i.e., singular geodesics. As already mentioned in [**7**], there are serious obstacles to giving an asymptotical formula in general, but if one imposes extra regularity conditions on the space, then these obstacles disappear and one can derive an asymptotical formula for singular geodesics.

We describe the main result of the paper. One of the various equivalent formulations of the prime geodesic theorem for locally symmetric spaces of rank one is the following. Let $\overline{X}$ be a compact locally symmetric space with universal covering of rank one. For $T > 0$, let

$$\psi(T) = \sum_{c\,:\,e^{l(c)}\leq T} l(c_0).$$

Here the sum runs over all closed geodesics c such that $e^{l(c)} \leq T$, where $l(c)$ is the length of the geodesic c, and c_0 is the prime geodesic underlying c. Then, under a suitable scaling of the metric, as $T \to \infty$,

$$\psi(T) \sim T.$$

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We now replace the space $\overline{X}$ by an arbitrary compact locally symmetric space which is a quotient of a globally symmetric space $X = G/K$ where G is a semi-simple Lie group of split-rank r and K is a maximal compact subgroup. So the space under consideration is $\Gamma\backslash X = \Gamma\backslash G/K$, where $\Gamma \subset G$ is a torsion-free discrete subgroup. The extra regularity condition one has to put on this space is that Γ be a *regular* group, see Section 1. In geometric terms regularity means that each closed geodesic in the space $\Gamma \setminus G/K$ has highly nontrivial monodromy. A closed geodesic c gives rise to a point a_c in the closure of the negative Weyl chamber A_0^- of a maximal split torus A_0. We pick a wall A^- of A_0^- which might be equal to A_0^- or of smaller dimension. We consider all geodesics c that give points a_c in A^-. Let r be the dimension of A^-. For $T_1, \dots, T_r > 0$, let

$$\psi(T_1, \dots, T_r) = \sum_{c\,:\,a_{c,j} \leq T_j} \lambda_c,$$

where λ_c is the volume of the unique maximal flat c lies in and $a_{c,j}$ are the coordinates of a_c with respect to a canonical coordinate system on A^- given by the roots. The sum runs over all closed geodesics c with $a_c \in A^-$ modulo homotopy. The main result of this paper is that, as T_j tends to infinity for every j,

$$\psi(T_1, \dots, T_r) \sim T_1 \cdots T_r.$$

The proof is based on a Lefschetz formula similar to the one [**7**], but at various places one has to argue in a fashion different to the previous case.

The restriction that the space be regular is a strong one, but fortunately the most important application which is an asymptotic formula for units in orders of number fields, can be derived in this context if the degree of the number field is a prime.

1. The Lefschetz formula. In this section we give a Lefschetz formula for regular locally symmetric spaces. Let G be a connected semi-simple Lie group with finite center, and choose a maximal compact subgroup K with Cartan involution θ, i.e., K is the group of fixed points of θ. Let P be a cuspidal parabolic subgroup with Langlands decomposition $P = MAN$. Cuspidality here means that the group

M admits a compact Cartan subgroup. Modulo conjugation we can assume that A and M are stable under θ. The centralizer of A is AM. Let $W(A,G)$ be the *Weyl group* of A, i.e. $W(A,G)$ is the quotient of the normalizer of A by the centralizer. This is a finite group acting on A.

We have to fix Haar measures. We use the normalization of Harish-Chandra [**15**]. Note that this normalization depends on the choice of an invariant bilinear form B on $\mathfrak{g}_{\mathbf{R}}$ which we keep at our disposal until later. Changing B amounts to scaling the metric of the symmetric space. Note further that in this normalization of Haar measures the compact groups K and M have total volume 1.

We write $\mathfrak{g}_{\mathbf{R}}, \mathfrak{k}_{\mathbf{R}}, \mathfrak{a}_{\mathbf{R}}, \mathfrak{m}_{\mathbf{R}}, \mathfrak{n}_{\mathbf{R}}$ for the real Lie algebras of G, K, A, M, N and $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{m}, \mathfrak{n}$ for their complexifications. $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. This algebra is isomorphic to the algebra of all left invariant differential operators on G with complex coefficients. Pick a compact Cartan subgroup T of M, and let $\mathfrak{t}$ be its complexified Lie algebra. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $W(\mathfrak{h}, \mathfrak{g})$ be the corresponding absolute Weyl group.

Let $\mathfrak{a}^*$ denote the dual space of the complex vector space $\mathfrak{a}$. Let $\mathfrak{a}^*_{\mathbf{R}}$ be the real dual of $\mathfrak{a}_{\mathbf{R}}$. We identify $\mathfrak{a}^*_{\mathbf{R}}$ with the real vector space of all $\lambda \in \mathfrak{a}^*$ that map $\mathfrak{a}_{\mathbf{R}}$ to $\mathbf{R}$. Let $\Phi \subset \mathfrak{a}^*$ be the set of all roots of the pair $(\mathfrak{a}, \mathfrak{g})$, and let Φ^+ be the subset of positive roots with respect to P. Let $\Delta \subset \Phi^+$ be the set of simple roots. Then Δ is a basis of $\mathfrak{a}^*$. The open *negative Weyl chamber* $\mathfrak{a}^-_{\mathbf{R}} \subset \mathfrak{a}_{\mathbf{R}}$ is the cone of all $X \in \mathfrak{a}_{\mathbf{R}}$ with $\alpha(X) < 0$ for every $\alpha \in \Delta$. Let $\overline{\mathfrak{a}^-_{\mathbf{R}}}$ be the closure of $\mathfrak{a}^-_{\mathbf{R}}$.

The bilinear form B is indefinite on $\mathfrak{g}_{\mathbf{R}}$, but the form

$$\langle X, Y \rangle \stackrel{\text{def}}{=} -B(X, \theta(Y))$$

is positive definite, i.e., an inner product on $\mathfrak{g}_{\mathbf{R}}$. We extend it to an inner product on the complexification $\mathfrak{g}$. Let $\|X\| = \sqrt{\langle X, X \rangle}$ be the corresponding norm. The form B, being nondegenerate, identifies $\mathfrak{g}$ to its dual space $\mathfrak{g}^*$. In this way we also define an inner product $\langle ., . \rangle$ and the corresponding norm on $\mathfrak{g}^*$. Furthermore, if $V \subset \mathfrak{g}$ is any subspace on which B is nondegenerate, then B gives an identification of V^* with V and so one gets an inner product and a norm on V^*. This in particular applies to $V = \mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$, which is defined over $\mathbf{R}$.

Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. We are interested in the closed geodesics on the locally symmetric space $X_\Gamma = \Gamma \setminus X = \Gamma \setminus G/K$. Note that every locally symmetric space of nonpositive curvature without Euclidean factors is of this form. Every closed geodesic c lifts to a Γ-orbit of geodesics on X and gives a Γ-conjugacy class $[\gamma_c]$ of elements closing the particular geodesics. This induces a bijection between the set of all homotopy classes of closed geodesics in X_Γ and the set of all nontrivial conjugacy classes in Γ, see [**10**].

Let C be an arbitrary Cartan subgroup of G. The *regular elements* of C are

$$C^{\mathrm{reg}} \stackrel{\mathrm{def}}{=} \{x \in C : G_x = C\},$$

where G_x denotes the centralizer of x in G. Then G^{reg} is by definition the union of all C^{reg} over all Cartan subgroups. This is an open dense set in G. The group Γ is called *regular* if

$$\Gamma \setminus \{1\} \subset G^{\mathrm{reg}}.$$

We will from now on assume that Γ is regular.

To give an example of a regular group, let d be a prime ≥ 3, and let D be a division algebra of degree d over $\mathbf{Q}$. Assume that D splits at infinity, i.e., that $D \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{Mat}_d(\mathbf{R})$. Choose a maximal order $D(\mathbf{Z})$ in D and define for any ring R,

$$D(R) \stackrel{\mathrm{def}}{=} D(\mathbf{Z}) \otimes_{\mathbf{Z}} R.$$

The reduced norm defines a multiplicative homomorphism $\det\colon D(R) \to R$, and we let

$$\mathcal{G}(R) \stackrel{\mathrm{def}}{=} \{x \in D(R) : \det(x) = 1\}.$$

Then $\mathcal{G}$ is a linear algebraic group defined over $\mathbf{Z}$ with $\mathcal{G}(\mathbf{R}) \cong \mathrm{SL}_d(\mathbf{R})$. Let $\Gamma \stackrel{\mathrm{def}}{=} \mathcal{G}(\mathbf{Z})$; then Γ is a discrete cocompact subgroup of $G = \mathcal{G}(\mathbf{R})$. We show that it is regular. For this let $\gamma \in \Gamma \setminus \{1\}$. It follows that $\gamma \notin \mathbf{Q} \cdot 1_D$, hence the division algebra $D_\gamma =$ centralizer of γ in D is neither $\mathbf{Q}$ nor D. Since the degree of D is a prime, D_γ must be a subfield, i.e., commutative. So the centralizer of γ in $\mathcal{G}$ is a torus, i.e., γ is regular, so the group Γ is a regular group.

1.1. Lefschetz numbers. Let Γ be a torsion-free regular subgroup of G. Let $\mathcal{E}_P(\Gamma)$ denote the set of all Γ-conjugacy classes $[\gamma]$ such that γ is G-conjugate to an element $a_\gamma t_\gamma$ of A^-T. Let $\mathcal{E}_P^0(\Gamma)$ be the set of all Γ-conjugacy classes $[\gamma]$ such that γ is G-conjugate to an element $a_\gamma t_\gamma$ of $A^-\widetilde{T}$, where $\widetilde{T}$ is the intersection of T with the connected component M^0 of the unit in M. Then $\mathcal{E}_P^0(\Gamma)$ is a subset of $\mathcal{E}_P(\Gamma)$. Let $n = \#(T/\widetilde{T}) \in \mathbf{N}$; then for every $[\gamma] \in \mathcal{E}_P(\Gamma)$ we have $[\gamma^n] \in \mathcal{E}_P^0(\Gamma)$.

Let $[\gamma] \in \mathcal{E}_P(\Gamma)$. There is a closed geodesic c in the Riemannian manifold $\Gamma \setminus G/K$ which gets closed by γ. This means that there is a lift $\tilde{c}$ to the universal covering G/K which is preserved by γ and γ acts on $\tilde{c}$ by a translation. The closed geodesic c is not unique in general. Since Γ is regular, there is a unique maximal flat F_c containing c. By *maximal flat* we here mean a flat, totally geodesic submanifold which is maximal with these properties with respect to inclusion. Note that other authors sometimes insist that a maximal flat should be of maximal dimension which we do not. Let λ_γ be the volume of that flat,

$$\lambda_\gamma \stackrel{\text{def}}{=} \mathrm{vol}\,(F_c).$$

As the notation indicates, this number only depends on γ and not on c.

Let $\mathfrak{n}$ denote the complexified Lie algebra of N. For any $\mathfrak{n}$-module V let $H_q(\mathfrak{n}, V)$ and $H^q(\mathfrak{n}, V)$ for $q = 0, \ldots, \dim \mathfrak{n}$ be the Lie algebra homology and cohomology [**3**]. Let $\widehat{G}$ denote the unitary dual of G, i.e., the set of isomorphism classes of irreducible unitary representations of G. For $\pi \in \widehat{G}$, let π_K be the $(\mathfrak{g}, K)$-module of K-finite vectors. If $\pi \in \widehat{G}$, then $H_q(\mathfrak{n}, \pi_K)$ and $H^q(\mathfrak{n}, \pi_K)$ are admissible $(\mathfrak{a} \oplus \mathfrak{m}, M)$-modules of finite length [**17**].

Note that AM acts on the Lie algebra $\mathfrak{n}$ of N by the adjoint representation. Let $[\gamma] \in \mathcal{E}_P(\Gamma)$. Since $a_\gamma \in A^-$ it follows that every eigenvalue of $a_\gamma t_\gamma$ on $\mathfrak{n}$ is of absolute value < 1. Therefore, $\det(1 - a_\gamma t_\gamma \mid \mathfrak{n}) \neq 0$.

For $[\gamma] \in \mathcal{E}_P(\Gamma)$, let

$$\mathrm{ind}\,(\gamma) = \frac{\lambda_\gamma}{\det(1 - a_\gamma t_\gamma \mid \mathfrak{n})} > 0,$$

where $r = \dim A$. Since Γ is cocompact, the unitary G-representation on $L^2(\Gamma \setminus G)$ splits discretely with finite multiplicities

$$L^2(\Gamma \setminus G) = \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi)\pi,$$

where $N_\Gamma(\pi)$ is a nonnegative integer and $\widehat{G}$ is the unitary dual of G. A *quasi-character* of A is a continuous group homomorphism to $\mathbf{C}^\times$. Via differentiation the set of quasi-characters can be identified with the dual space $\mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$ we write $a \mapsto a^\lambda$ for the corresponding quasi-character on A. We denote by $\rho \in \mathfrak{a}^*$ the modular shift with respect to P, i.e., for $a \in A$ we have $\det(a \mid \mathfrak{n}) = a^{2\rho}$.

For a complex vector space V on which A acts linearly and $\lambda \in \mathfrak{a}^*$ let $(V)_\lambda$ denote the generalized $(\lambda + \rho)$-eigenspace, i.e.,

$$(V)_\lambda = \{v \in V \mid (a - a^{\lambda+\rho}\mathrm{Id})^n v = 0 \text{ for some } n \in \mathbf{N}\}.$$

Since $H^p(\mathfrak{n}, \pi_K)$ is of finite length as an $(\mathfrak{a} \oplus \mathfrak{m}, K_M)$-module, one has

$$H^p(\mathfrak{n}, \pi_K) = \bigoplus_{\nu \in \mathfrak{a}^*} H^p(\mathfrak{n}, \pi_k)_\nu.$$

Let T be a compact Cartan subgroup of M, and let $\mathfrak{t}$ be its complex Lie algebra. Then AT is a Cartan subgroup of G. Let $K_M = M \cap K$. This is a maximal compact subgroup of M. Let $\Lambda_\pi \in (\mathfrak{a} \oplus \mathfrak{t})^*$ be a representative of the infinitesimal character of π. By [**17**, Corollary 3.32], it follows

$$H_p(\mathfrak{n}, \pi_K) = \bigoplus_{\nu = w\Lambda_\pi|_{\mathfrak{a}}} H_p(\mathfrak{n}, \pi_K)_\nu,$$

where w ranges over $W(\mathfrak{g}, \mathfrak{h})$.

Lemma 1.1. *For* $0 \le p \le d = \dim(\mathfrak{n})$ *we have*

$$H_p(\mathfrak{n}, \pi_K) \;\cong\; H^{d-p}(\mathfrak{n}, \pi_K) \otimes \det(\mathfrak{n}),$$

where the determinant of a finite dimensional space is the top exterior power. So $\det(\mathfrak{n})$ *is a one dimensional* AM*-module on which* AM *acts via the quasi-character* $am \mapsto \det(am \mid \mathfrak{n}) = a^{2\rho}$*. This in particular implies*

$$H^p(\mathfrak{n}, \pi_K) = \bigoplus_{\nu = w\Lambda_\pi|_{\mathfrak{a}}} H^p(\mathfrak{n}, \pi_K)_{\nu - 2\rho}.$$

Proof. The first part follows straight from the definition of Lie algebra cohomology. The second part follows by [**17**, Corollary 3.32]. □

Let $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{p}_M$ be the Cartan decomposition of the complex Lie algebra $\mathfrak{m}$ of M with respect to K_M. For $\lambda \in \mathfrak{a}^*$ and $\pi \in \widehat{G}$, let

$$m_\lambda(\pi) = \sum_{q=0}^{\dim \mathfrak{n}} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^{q+\dim \mathfrak{n}} \dim \left(H^q(\mathfrak{n}, \pi_K)_\lambda \otimes \bigwedge^p \mathfrak{p}_M \right)^{K_M},$$

where the superscript K_M indicates the subspace of K_M-invariants. Then $m_\lambda(\pi)$ is an integer and, by the above, the set of λ for which $m_\lambda(\pi) \neq 0$ for a given π has at most $|W(\mathfrak{g}, \mathfrak{h})|$ many elements.

Likewise, define

$$m_\lambda^0(\pi) = \sum_{q=0}^{\dim \mathfrak{n}} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^{q+\dim \mathfrak{n}} \dim \left(H^q(\mathfrak{n}, \pi_K)_\lambda \otimes \bigwedge^p \mathfrak{p}_M \right)^{K_M^0},$$

where K_M^0 is the connected component of the unit in K_M, or $K_M^0 = K_M \cap M^0$.

1.2. The formula. For $\mu \in \mathfrak{a}^*$ and $j \in \mathbf{N}$, let $\mathcal{C}^{j,\mu,-}(A)$ denote the space of functions φ on A which

- are j-times continuously differentiable on A,
- are zero outside A^-,
- are such that $a^{-\mu} D\varphi(a)$ is bounded on A for every invariant differential operator D on A of degree $\leq j$.

For every invariant differential operator D of degree $\leq j$ let $N_D(\varphi) = \sup_{a\in A} |a^{-\mu} D\varphi(a)|$. Then N_D is a semi-norm. Let $D_1, \dots, D_n$ be a basis of the space of invariant differential operators of degree $\leq j$, then $N(\varphi) = \sum_{j=1}^n N_{D_j}(\varphi)$ is a norm that makes $\mathcal{C}^{j,\mu,-}(A)$ into a Banach space. A different choice of basis will give an equivalent norm.

Let $f_{EP} \in C_c^\infty(M)$ denote an Euler-Poincaré function on M. This means that for every irreducible unitary representation η of M one has

$$\operatorname{tr} \eta(f_{EP}) = \sum_{q=0}^{\dim \mathfrak{p}_M} (-1)^q \dim \left(\eta \otimes \bigwedge^p \mathfrak{p}_M \right)^{K_M}.$$

Euler-Poincaré functions have the property that their orbital integrals filter out elliptic elements, i.e., for $x \in M$ a regular element one has

$$\mathcal{O}_x^M(f_{EP}) = \int_{M/M_x} f_{EP}(yxy^{-1})\, dy$$

equals 1 if x is elliptic and zero otherwise.

Theorem 1.2 (Lefschetz formula). *Assume Γ is regular and torsion-free. There exists $j \in \mathbf{N}$ and $\mu \in \mathfrak{a}^*$ such that for any $\varphi \in \mathcal{C}^{j,\mu,-}(A)$, we have*

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) \int_{A^-} \varphi(a) a^{\lambda+\rho}\, da = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \mathrm{ind}\,(\gamma)\, \varphi(a_\gamma),$$

where all sums and integrals converge absolutely. The inner sum on the left is always finite; *more precisely, it has length $\leq |W(\mathfrak{h}, \mathfrak{g})|$. The lefthand side is called the* global side *and the other the* local side *of the Lefschetz formula. Both sides of the formula give a continuous linear functional on the Banach space $\mathcal{C}^{j,\mu,-}(A)$.*

We also obtain a weak Lefschetz formula *as follows.*

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_\lambda^0(\pi) \int_{A^-} \varphi(a) a^{\lambda+\rho}\, da$$
$$= [M : M^0] \sum_{[\gamma] \in \mathcal{E}_P^0(\Gamma)} \mathrm{ind}\,(\gamma)\, \varphi(a_\gamma),$$

Proof. The proof is in [**8**, Section 4] or, in a special case, in [**7**]. The proof of the weak version is a variant of that proof where one replaces the Euler-Poincaré function of M with the Euler-Poincaré function of the connected component M^0. □

2. The Dirichlet series. We keep assuming that the torsion-free group Γ also is regular. Let $r = \dim A$ and, for $k = 1, \ldots, r$, let α_k be a positive real multiple of a simple root of (A, P) such that the modular shift ρ satisfies

$$2\rho = \alpha_1 + \cdots + \alpha_r.$$

This defines $\alpha_1, \dots, \alpha_r$ uniquely up to order.

We fix a form B, i.e., a Haar measure, such that the subset of A,

$$\{a \in A \mid 0 \le \alpha_k(\log a) \le 1,\ k = 1, \dots, r\}$$

has volume 1.

For $a \in A$ and $k = 1, \dots r$, let $l_k(a) = |\alpha_k(\log a)|$ and $l(a) = l_1(a) \cdots l_r(a)$. For $s = (s_1, \dots, s_r) \in \mathbf{C}^r$ and $j \in \mathbf{N}$, define

$$L^j(s) = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \operatorname{ind}(\gamma)\, l(a_\gamma)^{j+1}\, a_\gamma^{s \cdot \alpha},$$

where $s \cdot \alpha = s_1\alpha_1 + \cdots + s_r\alpha_r$. We will show that this series converges if $\operatorname{Re}(s_k) > 1$ for $k = 1, \dots, r$. Likewise, we define

$$L^{0,j}(s) = [M : M^0] \sum_{[\gamma] \in \mathcal{E}_P^0(\Gamma)} \operatorname{ind}(\gamma)\, l(a_\gamma)^{j+1}\, a_\gamma^{s \cdot \alpha}.$$

Let D denote the differential operator

$$D = (-1)^r \left(\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_r} \right).$$

Let $\widehat{G}(\Gamma)$ denote the set of all $\pi \in \widehat{G}$, $\pi \neq \text{triv}$ with $N_\Gamma(\pi) \neq 0$. For a given $\pi \in \widehat{G}$, let $\Lambda(\pi)$ denote the set of all $\lambda \in \mathfrak{a}^*$ with $m_{\lambda-\rho}(\pi) \neq 0$. Then $\Lambda(\pi)$ has at most $|W(\mathfrak{h}, \mathfrak{g})|$ elements.

Let $\lambda \in \mathfrak{a}^*$. Since $\alpha_1, \dots, \alpha_r$ is a basis of $\mathfrak{a}^*$ we can write $\lambda = \lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r$ for uniquely determined $\lambda_k \in \mathbf{C}$.

Let $R_k(s)$, $k \in \mathbf{N}$ be a sequence of rational functions on $\mathbf{C}^r$. For an open set $U \subset \mathbf{C}^r$, let $\mathbf{N}(U)$ be the set of natural numbers k such that the pole-divisor of R_k does not intersect U. We say that the series

$$\sum_k R_k(s)$$

converges weakly locally uniformly on $\mathbf{C}^r$ if, for every open $U \subset \mathbf{C}^r$, the series

$$\sum_{k\in \mathbf{N}(U)} R_k(s)$$

converges locally uniformly on U.

Let

$$q_M \stackrel{\text{def}}{=} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^p \dim \left(\bigwedge^p \mathfrak{p}_M\right)^{K_M},$$

and

$$q_M^0 \stackrel{\text{def}}{=} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^p \dim \left(\bigwedge^p \mathfrak{p}_M\right)^{K_M^0}.$$

Note that if $A = A_0$, then M is compact and $q_M = q_M^0 = 1$.

Theorem 2.1. *For $j \in \mathbf{N}$ large enough the series $L^j(s)$ converges locally uniformly in the set*

$$\{s \in \mathbf{C} : \mathrm{Re}\,(s_k) > 1,\ k = 1, \dots, r\}.$$

The function $L^j(s)$ can be written as the Mittag-Leffler series,

$$L^j(s) = D^{j+1} \frac{q_M}{(s_1-1)\cdots(s_r-1)} + \sum_{\pi\in\widehat{G}(\Gamma)} N_\Gamma(\pi) \sum_{\lambda\in\Lambda(\pi)} m_{\lambda-\rho}(\pi) D^{j+1} \frac{1}{(s_1+\lambda_1)\cdots(s_r+\lambda_r)}.$$

The double series converges weakly locally uniformly on $\mathbf{C}^r$. For $\pi \neq \mathrm{triv}$ and $\lambda \in \Lambda(\pi)$ we have $\mathrm{Re}\,(\lambda_k) > -1$ for $k = 1, \dots, r$. So in particular, the double series converges locally uniformly on $\{\mathrm{Re}\,(s_k) > 1\}$.

The same holds for $L^{0,j}(s)$ which satisfies

$$L^{0,j}(s) = D^{j+1} \frac{q_M^0}{(s_1-1)\cdots(s_r-1)} + \sum_{\pi\in\widehat{G}(\Gamma)} N_\Gamma(\pi) \sum_{\lambda\in\Lambda(\pi)} m^0_{\lambda-\rho}(\pi) D^{j+1} \frac{1}{(s_1+\lambda_1)\cdots(s_r+\lambda_r)}.$$

The integers q_M, q_M^0 *satisfy*

$$q_M[M : M^0] \geq q_M^0 > 0.$$

The proof will occupy the rest of this section. We will show that the series $L^j(s)$ converges if the real parts $\mathrm{Re}\,(s_k)$ are sufficiently large for $k = 1, \dots, r$. Since $L^j(s)$ is a Dirichlet series with positive coefficients, the convergence in the set $\{\mathrm{Re}\,(s_k) > 1\}$ will follow, once we have established holomorphy there. This holomorphy will in turn follow from the convergence of the Mittag-Leffler series.

Since the sum defining $L^{0,j}$ runs over a smaller set, we have for $(s_j) > 1$, that $L^j(s)[M : M^0] \geq L^{0,j}(s)$. Approaching the pole at $s_1 = s_2 = \cdots = s_r = 1$ from above, say along $s_1 = s_2 = \cdots = s_r = t$ and $t \downarrow 1$, we infer $q_M[M : M^0] \geq q_M^0$. To see $q_M^0 > 0$, let $M_{\mathbf{C}}$ be the complexification of M, and let $M_d \subset M_{\mathbf{C}}$ be a compact form containing K_M. Then $X_{M,d} = M_d/K_M$ is the dual symmetric space to M/K_M. The Betti numbers of $X_{M,d}$ can be computed using the complex $\Omega^\bullet(X_{M,d})^{M_d^0}$ of M_d^0-invariant differential forms, where M_d^0 is the connected component of the unit. This complex is isomorphic to

$$\begin{aligned}\Omega^\bullet(X_{M,d})^{M_d^0} &\cong \left(C^\infty(M_d^0) \otimes \bigwedge^\bullet \mathfrak{p}_M\right)^{M_d^0 \times K_M^0}\\ &\cong \left(\bigwedge^\bullet \mathfrak{p}_M\right)^{K_M^0}.\end{aligned}$$

Thus we see that q_M^0 equals the Euler characteristic $\chi(X_{M,d})$. It is known that Euler characteristics of compact symmetric spaces are positive, so $q_M^0 > 0$. Note that this deduction of $q_M > 0$ is the sole reason for introducing m_λ^0, q_M^0, and $L^{0,j}(s)$.

Let

$$\mathfrak{a}_{\mathbf{R}}^{*,+} = \{\lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r \mid \lambda_1, \dots, \lambda_r > 0\}$$

be the dual positive cone. Let $\overline{\mathfrak{a}_{\mathbf{R}}^{*,+}}$ be the closure of $\mathfrak{a}_{\mathbf{R}}^{*,+}$ in $\mathfrak{a}_{\mathbf{R}}^*$.

Proposition 2.2. *Let* $\pi \in \widehat{G}$, $\lambda \in \mathfrak{a}^*$ *with* $m_\lambda(\pi) \neq 0$. *Then* $\mathrm{Re}\,(\lambda)$ *lies in the set*

$$C = -3\rho + \overline{\mathfrak{a}_{\mathbf{R}}^{*,+}}.$$

For $\pi \in \widehat{G}$ and λ in the boundary of C, we have $m_\lambda(\pi) = 0$ unless π is the trivial representation and $\lambda = -3\rho$ in which case $m_\lambda(\pi) = q_M$. The same assertion holds for $m_\lambda(\pi)$ replaced with $m_\lambda^0(\pi)$, only then the integer q_M changes to q_M^0.

Proof. We introduce a partial order on $\mathfrak{a}^*$ by

$$\mu > \nu \iff \mu - \nu \quad \text{is a linear combination,}$$
$$\text{with positive integral coefficients, of roots in } \Phi^+. \quad \square$$

Lemma 2.3. *Let $p \in \mathbf{N}$, let $\pi \in \widehat{G}$ and $\mu \in \mathfrak{a}^*$ be such that $H_p(\mathfrak{n}, \pi_K)_\mu \neq 0$. Then there exists a $\nu \in \mathfrak{a}^*$ with $\nu < \mu$ and $H_0(\mathfrak{n}, \pi_K)_\nu \neq 0$.*

Equivalently, if $0 \leq p < d = \dim(\mathfrak{n})$ and $H^p(\mathfrak{n}, \pi_K)_\mu \neq 0$, then there exists $\eta \in \mathfrak{a}^*$ with $\eta < \mu$ and $H^d(\mathfrak{n}, \pi_K)_\eta \neq 0$.

Proof. The first assertion is a weak version of Proposition 2.32 in [**17**] and the second follows from the first and Lemma 1.1. $\square$

To prove Proposition 2.2, we consider the trivial representation $\pi =$ triv first. Using the definition of Lie algebra homology, it is easy to show that $m_{-3\rho}(\text{triv}) = q_M$ and the other λ with $m_\lambda(\text{triv}) \neq 0$ lie in $-3\rho + \mathfrak{a}_R^{*,+}$. Likewise for $m_\lambda^0(\pi)$.

For $\pi \neq$ triv we show the stronger statement that if $H^p(\mathfrak{n}, \pi_K)_\lambda \neq 0$, then $\operatorname{Re}(\lambda) \in -3\rho + \mathfrak{a}_{\mathbf{R}}^{*,+}$. We start with the case of P being a minimal parabolic. Then M is compact, i.e., $M = K_M$ and $q_M = 1$. Using Lemma 1.1 we see that it suffices to show that if $H_0(\mathfrak{n}, \pi_K)_\lambda \neq 0$, then $\operatorname{Re}(\lambda) \in -\rho + \mathfrak{a}_{\mathbf{R}}^{*,+}$. So assume $H_0(\mathfrak{n}, \pi_K)_\lambda \neq 0$ and π is nontrivial. Theorems 4.16 and 4.25 of [**17**] imply that λ is a leading coefficient of the asymptotic of matrix coefficients of π. By the Howe-Moore theorem [**19**], these matrix coefficients vanish at infinity on G, and this implies that $\operatorname{Re}(\lambda + \rho) \in \mathfrak{a}_{\mathbf{R}}^{*,+}$. The case of a minimal parabolic is settled.

In general, there is a minimal parabolic $P_0 = M_0 A_0 N_0 \subset P = MAN$ such that $M_0 \subset M$, $A_0 \supset A$, and $N_0 \supset N$. Let $\mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0$ be the Lie algebras of M_0, A_0, and N_0. Then

$$\mathfrak{n}_0 = \mathfrak{n} \oplus \mathfrak{n}_M, \qquad \mathfrak{a}_0 = \mathfrak{a} \oplus \mathfrak{a}_M,$$

where $\mathfrak{n}_M = \mathfrak{n}_0 \cap \mathfrak{m}$ and $\mathfrak{a}_M = \mathfrak{a}_0 \cap \mathfrak{m}$. Note that $\mathfrak{n}$ is an ideal in $\mathfrak{n}_0$.

In light of Lemma 2.3 it suffices to show that if $H^d(\mathfrak{n},\pi_K)\neq 0$, then $\mathrm{Re}\,(\lambda)\in -3\rho+\mathfrak{a}_{\mathbf{R}}^{*,+}$. Let $d_M=\dim\mathfrak{n}_M$ and $d_0=\dim\mathfrak{n}_0$. Then $d_0=d+d_M$. Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q}=H^p(\mathfrak{n}_M,H^q(\mathfrak{n},\pi_K))$$

which abuts to $H^{p+q}(\mathfrak{n}_0,\pi_K)$. We assume that $H^d(\mathfrak{n},\pi_K)_\lambda\neq 0$. Then

$$H^{d_M}(\mathfrak{n}_M,H^d(\mathfrak{n},\pi_K)_\lambda)\neq 0$$

as well and thus there exists $\lambda_M\in\mathfrak{a}_M^*$ with

$$H^{d_M}(\mathfrak{n}_M,H^d(\mathfrak{n},\pi_K)_\lambda)_{\lambda_M}\neq 0.$$

Since A acts trivially on $\mathfrak{n}_M$, the latter equals

$$H^{d_M}(\mathfrak{n}_M,H^d(\mathfrak{n},\pi_K))_{\lambda+\lambda_M}=(E_2^{d_M,d})_{\lambda+\lambda_M},$$

where we view $\lambda+\lambda_M$ as an element of $\mathfrak{a}_0^*=\mathfrak{a}^*\oplus\mathfrak{a}_M^*$. The spectral sequence E is supported in the set of indices $0\leq p\leq d_M$, $0\leq q\leq d$ and its differentials are A_0-homomorphisms. So $E_2^{d_M,d}$ is the right top corner of this spectral sequence, hence equals $E_\infty^{d_M,d}$ which in this case is $H^{d_0}(\mathfrak{n}_0,\pi_K)$. It follows that $H^{d_0}(\mathfrak{n}_0,\pi_K)_{\lambda+\lambda_M}\neq 0$ and hence, by the above,

$$\mathrm{Re}\,(\lambda+\lambda_M)\in -3\rho_0+\mathfrak{a}_{0,\mathbf{R}}^{*,+},$$

which by projection implies $\mathrm{Re}\,(\lambda)\in -3\rho+\mathfrak{a}_{\mathbf{R}}^{*,+}$. Proposition 2.2 is proved. □

We continue the proof of Theorem 2.1. For $a\in A$ set

$$\varphi(a)=l(a)^{j+1}\ a^{s\cdot\alpha}.$$

For $\mathrm{Re}\,(s_k)\gg 0$, $k=1,\dots,r$, the Lefschetz formula is valid for this test function. The local side of the Lefschetz formula equals

$$\sum_{[\gamma]\in\mathcal{E}_P(\Gamma)}\mathrm{ind}\,(\gamma)\,l(a_\gamma)^{j+1}\ a_\gamma^{s\cdot\alpha}=L^j(s).$$

The convergence assertion in the Lefschetz formula implies that the series converges absolutely if $\mathrm{Re}\,(s_k)$ is sufficiently large for every

$k = 1, \dots, r$. We will show that it extends to a holomorphic function in the set $\mathrm{Re}(s_k) > 1$, $k = 1, \dots, r$. Since $L^j(s)$ is a Dirichlet series with positive coefficients, it must therefore converge in that region.

With our given test function and the Haar measure chosen, we compute

$$\begin{aligned}\int_{A^-} \varphi(a) a^\lambda \, da &= (-1)^{r(j+1)} \int_{A^-} (\alpha_1(\log a) \cdots \alpha_r(\log a))^{j+1} a^{s\cdot\alpha+\lambda} \, da \\ &= (-1)^{r(j+1)} \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_r)^{j+1} \\ &\qquad\qquad \times e^{-((s_1+\lambda_1)t_1+\cdots+(s_r+\lambda_r)t_r)} dt_1 \dots dt_r \\ &= D^{j+1} \int_0^\infty \cdots \int_0^\infty e^{-((s_1+\lambda_1)t_1+\cdots+(s_r+\lambda_r)t_r)} \, dt_1 \dots dt_r \\ &= D^{j+1} \frac{1}{(s_1+\lambda_1)\dots(s_r+\lambda_r)}.\end{aligned}$$

Performing a ρ-shift, we see that the Lefschetz formula gives

$$\begin{aligned}L^j(s) &= \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_{\lambda-\rho}(\pi) \, D^{j+1} \frac{q_M}{(s_1+\lambda_1)\cdots(s_r+\lambda_r)} \\ &= \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_{\lambda-\rho}(\pi) \, \frac{q_M((j+1)!)^r}{(s_1+\lambda_1)^{j+2}\cdots(s_r+\lambda_r)^{j+2}}\end{aligned}$$

for $\mathrm{Re}(s_k) \gg 0$. For every $\pi \in \widehat{G}$ we fix a representative $\Lambda_\pi \in (\mathfrak{a}+\mathfrak{t})^*$ of the infinitesimal character of π. According to Lemma 1.1, if $m_{\lambda-\rho}(\pi) \neq 0$, then $\lambda = w\Lambda_\pi|_\mathfrak{a} - \rho$ for some $w \in W(\mathfrak{h}, \mathfrak{g})$. By abuse of notation, we will write $w\Lambda_\pi$ instead of $w\Lambda_\pi\,|_\mathfrak{a}$. Hence, we get

$$L^j(s) = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \sum_{w \in W(\mathfrak{h},\mathfrak{g})} m_{w\Lambda_\pi - 2\rho}(\pi) \, D^{j+1} \frac{q_M}{(s_1+\lambda_1)\cdots(s_r+\lambda_r)}.$$

For $\lambda \in \mathfrak{a}^*$, let $\|\lambda\|$ be the norm given by the form B as explained in the beginning of Section 1.

Proposition 2.4. *There are $m \in \mathbf{N}$, $C > 0$, such that for every $\pi \in \widehat{G}$ and every $\lambda \in \mathfrak{a}^*$, one has*

$$|m_{\lambda-\rho}(\pi)| \leq \ C(1 + \|\lambda\|)^m.$$

Proof. Harish-Chandra has shown that there is a locally integrable function Θ_π^G on G, called the global character of π, such that

$$\operatorname{tr} \pi(h) = \int_G h(x) \Theta_\pi^G(x)\, dx$$

for every $h \in C_c^\infty$. It follows that Θ_π^G is invariant under conjugation. Hecht and Schmid have shown in [**17**] that, for $at \in A^- T$,

$$\Theta_\pi^G(at) = \frac{\sum_{q=0}^{\dim \mathfrak{n}} (-1)^q \Theta^{AM}_{H_q(\mathfrak{n}, \pi_K)}(at)}{\det(1 - at \mid \mathfrak{n})},$$

where Θ^{AM} is the corresponding global character on the group AM.

Let $T = C_1, \dots, C_r$ be a set of representatives of the Cartan subgroups of M modulo M-conjugation. Choose a set of positive roots $\phi_j^+ \subset \phi(\mathbf{C}_j, \mathfrak{m})$ for each j. Let $\rho_j = \sum_{\alpha \in \phi_j^+} \alpha/2$. For $x \in C_j$, set

$$D_{C_j}(x) = x^{\rho_j} \prod_{\alpha \in \phi_j^+} (1 - x^{-\alpha}).$$

This is the Weyl denominator. By the Weyl integration formula, the integral

$$\int_M f_{EP}(m) \sum_{q=0}^{\dim \mathfrak{n}} (-1)^{q + \dim \mathfrak{n}}\, \Theta^{AM}_{H^q(\mathfrak{n}, \pi_K)}(am)\, dm$$

equals

$$\sum_{j=1}^r \frac{(-1)^{\dim \mathfrak{n}}}{|W(C_j, M)|} \int_{C_j^{reg}} D_{C_j}(x) \mathcal{O}_x(f_{EP}) \sum_{q=0}^{\dim \mathfrak{n}} (-1)^q \Theta^{AM}_{H^q(\mathfrak{n}\pi_K)}(ax)\, dx,$$

where f_{EP} is the Euler-Poincaré function on M and $\mathcal{O}$ denotes the orbital integral. Since the orbital integral of the Euler-Poincaré function vanishes unless x is elliptic, in which it equals 1 for regular x, we see that this equals

$$\frac{(-1)^{\dim \mathfrak{n}}}{|W(T, M)|} \int_{T^{\mathrm{reg}}} D_T(t) \sum_{q=0}^{\dim \mathfrak{n}} (-1)^q \Theta^{AM}_{H^q(\mathfrak{n}\pi_K)}(at)\, dt.$$

On the other hand, by the defining property of the Euler-Poincaré function we get that

$$\int_M f_{EP}(m) \sum_{q=0}^{\dim \mathfrak{n}} (-1)^{q+\dim \mathfrak{n}} \Theta^{AM}_{H^q(\mathfrak{n},\pi_K)}(am)\, dm$$

equals

$$\sum_{p,q\geq 0} (-1)^{q+\dim \mathfrak{n}} \dim \left(H^q(\mathfrak{n},\pi_K) \otimes \bigwedge^p \mathfrak{p}_M \right)^{K_M} = \sum_{\alpha\in\mathfrak{a}^*} m_{\lambda-\rho}(\pi) a^\lambda.$$

We put this together and use the result of Hecht and Schmid to infer

$$\begin{aligned}
&\sum_{\alpha\in\mathfrak{a}^*} m_{\lambda-\rho}(\pi) a^\lambda \\
&\quad = \int_M f_{EP}(m) \sum_{q=0}^{\dim \mathfrak{n}} (-1)^{q+\dim \mathfrak{n}} \Theta^{AM}_{H^q(\mathfrak{n},\pi_K)}(am)\, dm \\
&\quad = \frac{(-1)^{\dim \mathfrak{n}}}{|W(T,M)|} \int_{T^{\mathrm{reg}}} |D_T(t)|^2 \sum_{q\geq 0} (-1)^q \Theta^{AM}_{H^q(\mathfrak{n},\pi_K)}(at)\, dt \\
&\quad = \frac{(-1)^{\dim \mathfrak{n}}}{|W(T,M)|} a^{-2\rho} \int_{T^{\mathrm{reg}}} |D_T(t)|^2 \sum_{q\geq 0} (-1)^q \Theta^{AM}_{H_q(\mathfrak{n},\pi_K)}(at)\, dt \\
&\quad = \frac{(-1)^{\dim \mathfrak{n}}}{|W(T,M)|} a^{-2\rho} \int_{T^{\mathrm{reg}}} |D_T(t)|^2 \det(1-at|\mathfrak{n}) \Theta^G_\pi(at)\, dt.
\end{aligned}$$

The function $(-1)^{\dim \mathfrak{n}} a^{2\rho} \det(1 - am \mid \mathfrak{n}) D_T(t)$ equals the Weyl denominator for $H = AT$. By [**22**, Theorems 10.35 and 10.48] there are constants c_w, $w \in W(\mathfrak{h},\mathfrak{g})$ such that

$$(-1)^{\dim \mathfrak{n}} a^{-2\rho} \det(1-am \mid \mathfrak{n}) D_T(t) \Theta^G_\pi(at) = \sum_{w\in W(\mathfrak{h}\mathfrak{g})} c_w\, (at)^{w\Lambda_\pi}.$$

We thus have proved the following lemma.

Lemma 2.5. *For $a \in A^-$,*

$$\sum_{\lambda\in\mathfrak{a}^*} m_{\lambda-\rho}(\pi)\, a^\lambda = \sum_{w\in W(\mathfrak{h},\mathfrak{g})} \frac{c_w}{|W(T,M)|}\, a^{w\Lambda_\pi} \times \int_{T^{\mathrm{reg}}} t^{w\Lambda_\pi - \rho_M} \prod_{\alpha\in\phi^+(\mathfrak{t},\mathfrak{m})} (1-t^\alpha)\, dt.$$

Proposition 2.4 will follow from explicit formulae for the global character Θ_π^G (see below) which give bounds on the c_w. Another remarkable consequence of Lemma 2.5 is the fact that there is a finite set $E \subset \mathfrak{t}^*$ such that whenever $m_{\lambda-\rho}(\lambda) \neq 0$ for some $\lambda \in \mathfrak{a}^*$ it follows $\Lambda_\pi \mid_{\mathfrak{t}} \in E$. Hence Proposition 2.4 will follow from the estimate

$$|m_{\lambda-\rho}(\pi)| \leq C(1 + \|\Lambda_\pi\|)^m.$$

In [**13**] Harish-Chandra gives an explicit formula for characters of discrete series representations which imply the sharper estimate $|m_{\lambda-\rho}(\pi)| \leq C$ for the discrete series representations. From Harish-Chandra's paper, a similar formula can be deduced for limit of discrete series representations. Alternatively, one can use Zuckerman tensoring ([**22**, Proposition 10.44]) to deduce the estimate for limits of discrete series representations. Next, if $\pi = \pi_{\sigma,\nu}$ is induced from some parabolic $P_1 = M_1 A_1 N_1$, then the character of π can be computed from the character of σ and ν, see [**22**, formula (10.27)]. From this it follows that the claim holds for standard representations, i.e., admissible representations which are induced from discrete series or limit of discrete series representations.

Lemma 2.6. *There are natural numbers n, m and a constant $d > 0$ such that for every $\pi \in \widehat{G}$ there are standard representations $\pi_1, \ldots, \pi_n$ and integers $c_1, \ldots, c_n$ with*

$$\Theta_\pi = \sum_{k=1}^{n} c_k \, \Theta_{\pi_k}$$

and $|c_k| \leq d(1 + \|\Lambda_\pi\|^m)$ for $k = 1, \ldots, n$.

Proof. This is Lemma 2.6 of [**7**]. □

It remains to deduce Theorem 2.1. Since the coefficients $m_{\lambda-\rho}(\pi)$ grow at most like a power of $\|\Lambda_\pi\|$, the convergence assertion in Theorem 2.1 will be implied by the following lemma.

Lemma 2.7. *Let S denote the set of all pairs $(\pi, \lambda) \in \widehat{G} \times \mathfrak{a}^*$ such that $m_{\lambda-\rho}(\pi) \neq 0$. There is an $m_1 \in \mathbf{N}$ such that*

$$\sum_{(\pi,\lambda)\in S} \frac{N_\Gamma(\pi)}{(1+\|\lambda\|)^{m_1}} < \infty.$$

Proof. By the remark following Lemma 2.5 it suffices to show that there is an $m \in \mathbf{N}$ such that

$$\sum_{(\pi,\lambda)\in S} \frac{N_\Gamma(\pi)}{(1+\|\Lambda_\pi\|)^{m_1}} < \infty.$$

Let $\pi \in \widehat{G}$. The restriction of π to the maximal compact subgroup K decomposes into finite dimensional isotypes

$$\pi\mid_K = \bigoplus_{\tau\in\widehat{K}} \pi(\tau).$$

Let C_K be the Casimir operator of K, and let

$$\Delta_G \stackrel{\text{def}}{=} -C + 2C_K.$$

Then Δ_G is the Laplacian on G given by the left invariant metric which at the point $e \in G$ is given by $\langle .,. \rangle = -B(.,\theta(.))$. Since Δ_G is left invariant it induces an operator on $\Gamma \setminus G$ denoted by the same letter. This operator is ≥ 0 and elliptic, so there is a natural number k such that $(1+\Delta_G)^{-k}$ is of trace class on $L^2(\Gamma \setminus G)$. Hence,

$$\begin{aligned}
\infty &> \operatorname{tr}(1+\Delta_G)^{-k} \\
&= \sum_{\pi\in\widehat{G}} N_\Gamma(\pi) \sum_{\tau\in\widehat{K}} (1-\pi(C)+2\tau(C_K))^{-k} \dim \pi(\tau) \\
&\geq \sum_{\pi\in\widehat{G}} \frac{N_\Gamma(\pi)}{(1-\pi(C)+2\tau_\pi(C_K))^k},
\end{aligned}$$

where for each $\pi \in \widehat{G}$ we fix a minimal K-type τ_π. Since the infinitesimal character of the minimal K-type grows like the infinitesimal character of π, the lemma follows. □

Proof of Theorem 2.1. Finally, let $U \in \mathbf{C}^r$ be open. Let $S(U)$ be the set of all pairs $(\pi, \lambda) \in \widehat{G} \times \mathfrak{a}^*$ such that $m_{\lambda-\rho}(\pi) \neq 0$ and the pole divisor of

$$\frac{1}{(s_1+\lambda_1)\cdots(s_r+\lambda_r)}$$

does not intersect U. Let $V \subset U$ be a compact subset. We have to show that for some $j \in \mathbf{N}$ which does not depend on U or V,

$$\sup_{s\in V} \sum_{(\pi,\lambda)\in S(U)} \left| \frac{N_\Gamma(\pi)\, m_{\lambda-\rho}(\pi)}{(s_1+\lambda_1)^{j+2}\cdots(s_r+\lambda_r)^{j+2}} \right| < \infty.$$

Let m be as in Lemma 2.4, and let m_1 be as in Lemma 2.7. Then let $j \geq m + m_1 - 2$. Since $V \subset U$ and V is compact there is an $\varepsilon > 0$ such that $s \in V$ and $(\pi, \lambda) \in S(U)$ implies $|s_k + \lambda_k| \geq \varepsilon$ for every $k = 1, \ldots, r$. Hence, there is a $c > 0$ such that for every $s \in V$ and every $(\pi, \lambda) \in S(U)$,

$$|(s_1+\lambda_1)\cdots(s_r+\lambda_r)| \;\geq\; c(1+\|\lambda\|).$$

This implies

$$\begin{aligned}\left| \frac{m_{\lambda-\rho}(\pi)}{(s_1+\lambda_1)^{j+2}\cdots(s_r+\lambda_r)^{j+2}} \right| &\leq \frac{1}{c^{j+2}}\; \frac{|m_{\lambda-\rho}(\pi)|}{(1+\|\lambda\|)^{j+2}} \\ &\leq \frac{C}{c^{j+2}}\; \frac{1}{(1+\|\lambda\|)^{j+2-m}} \\ &\leq \frac{C}{c^{j+2}}\; \frac{1}{(1+\|\lambda\|)^{m_1}}.\end{aligned}$$

The claim now follows from Lemma 2.7. The proof of Theorem 2.1 is finished. The version for $L^{0,j}$ is analogous. □

3. The prime geodesic theorem. We now give the two main results of the paper.

Theorem 3.1 (Prime geodesic theorem). *For* $T_1, \ldots, T_r > 0$*, let*

$$\Psi(T_1, \ldots, T_r) = \sum_{\substack{[\gamma]\in\mathcal{E}_P(\Gamma)\\ a_\gamma^{-\alpha_k} \leq T_k,\\ k=1,\ldots,r}} \lambda_\gamma.$$

Then, as $T_k \to \infty$, *for* $k = 1, \dots, r$, *we have*

$$\Psi(T_1, \dots, T_r) \sim q_M T_1 \cdots T_r.$$

Proof. Using Theorem 2.1 the proof is the same as the proof of Theorem 3.1 in [**7**]. □

Finally, we give a new asymptotic formula for class numbers in number fields. It is quite different from known results like Siegel's theorem [**1**, Theorem 6.2]. The asymptotic is in several variables and thus contains more information than a single variable one. In a sense it states that the units of the orders are equally distributed in different directions if only one averages over sufficiently many orders.

Let d be a prime number ≥ 3. Let $r, s \geq 0$ be integers with $d = r + 2s$. A number field F is said to be *of type* (r, s) if F has r real and $2s$ complex embeddings. Let S be a finite set of primes with $|S| \geq 2$. Let $C_{r,s}(S)$ be the set of all number fields F of type (r, s) with the property $p \in S \Rightarrow p$ is nondecomposed in F.

Let $O_{r,s}(S)$ denote the set of all orders $\mathcal{O}$ in number fields $F \in C_{r,s}(S)$ which are maximal at each $p \in S$. For such an order $\mathcal{O}$, let $h(\mathcal{O})$ be its class number, $R(\mathcal{O})$ its regulator and $\lambda_S(\mathcal{O}) = \prod_{p \in S} f_p$, where f_p is the inertia degree of p in $F = \mathcal{O} \otimes \mathcal{Q}$. Then $f_p \in \{1, d\}$ for every $p \in S$.

For $\lambda \in \mathcal{O}^\times$, let $\rho_1, \dots, \rho_r$ denote the real embeddings of F ordered in a way such that $|\rho_k(\lambda)| \geq |\rho_{k+1}(\lambda)|$ holds for $k = 1, \dots, r-1$. For the same λ, let $\sigma_1 \dots \sigma_s$ be pairwise nonconjugate complex embeddings ordered in such a way that $|\sigma_k(\lambda)| \geq |\sigma_{k+1}(\lambda)|$ holds for $k = 1, \dots, s-1$.

For $k = 1, \dots s-1$, let

$$\alpha_k(\lambda) \stackrel{\text{def}}{=} 2k(d-2k) \log\left(\frac{|\sigma_k(\lambda)|}{|\sigma_{k+1}(\lambda)|}\right).$$

If $s > 0$, let

$$\alpha_s(\lambda) \stackrel{\text{def}}{=} 2rs \log\left(\frac{|\sigma_s(\lambda)|}{|\rho_1(\lambda)|}\right).$$

For $k = s+1, \dots, r+s-1$, let

$$\alpha_k(\lambda) \stackrel{\text{def}}{=} (k+s)(r+s-k) \log\left(\frac{|\rho_{k-s}(\lambda)|}{|\rho_{k-s+1}(\lambda)|}\right).$$

For $T_1, \dots, T_{r+s-1} > 0$, set

$$v_{\mathcal{O}}(T_1, \dots, T_{r+s-1}) \stackrel{\text{def}}{=} \#\{\lambda \in \mathcal{O}^\times / \pm 1 \mid 0 < \alpha_k(\lambda) \le T_k,\ k = 1, \dots, r+s-1\}.$$

Let

$$c = (\sqrt{2})^{1-r-s} \left(\prod_{k=1}^{s-1} (4k(d-2k)) \right) 2rs \left(\prod_{k=s+1}^{r+s-1} 2(k+s)(r+s-k) \right),$$

where the factor $2rs$ only occurs if $rs \neq 0$. So $c > 0$ and it comes about as correctional factor between the Haar measure normalization used in the prime geodesic theorem and the normalization used in the definition of the regulator.

Theorem 3.2. *With*

$$\vartheta_S(T) \stackrel{\text{def}}{=} \sum_{\mathcal{O} \in O(S)} v_{\mathcal{O}}(T)\, R(\mathcal{O})\, h(\mathcal{O})\, \lambda_S(\mathcal{O}),$$

we have, as $T_1, \dots, T_{r+s-1} \to \infty$,

$$\vartheta(T_1, \dots, T_{r+s-1}) \sim \frac{c}{\sqrt{r+s}}\, T_1 \cdots T_{r+s-1}.$$

Proof. For a given S, there is a division algebra D over $\mathbf{Q}$ of degree p which splits exactly outside S. Fix a maximal order $D(\mathbf{Z})$ in D and for any ring R define $D(R) \stackrel{\text{def}}{=} D(\mathbf{Z}) \otimes R$. Let $\det : D(R) \to R$ denote the reduced norm. Then

$$\mathcal{G}(R) \stackrel{\text{def}}{=} \{x \in D(R) \mid \det(x) = 1\}$$

defines a group scheme over $\mathbf{Z}$ with $\mathcal{G}(\mathbf{R}) \cong \mathrm{SL}_d(\mathbf{R}) = G$. Then $\Gamma = \mathcal{G}(\mathbf{Z})$ is a cocompact discrete regular torsion-free subgroup of G, see Section 1. In this case we choose $A = \mathrm{diag}\,(a_1, a_1, \dots, a_s, a_s, a_{s+1}, \dots, a_{s+r})$. Then $M \cong S((\mathrm{SL}_2^{\pm})^s \times \{\pm 1\}^r)$ and $q_M = 1$. The factor c is the correction one has to put in when comparing the Haar measure on G to the measure defining the regulator. As can be seen in [**6**], the theorem can be deduced from the prime geodesic theorem. □

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