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A COMBINATORIAL CHARACTERIZATION OF JACOBI FORMS

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ABSTRACT. We present a combinatorial characterization of Jacobi forms à la Henri Cohen.

1. Introduction. Cohen [**3**] establishes a lift from elliptic modular forms to Hilbert modular forms over real quadratic fields. A key tool in Cohen's proof is a combinatorial characterization of modular forms in two variables. In this paper, we extend Cohen's combinatorial characterization of modular forms to Jacobi forms on $\mathbf{H}^2\times\mathbf{C}^2$. It would be interesting to see if our result could be used to construct a lift from Jacobi forms in the sense of Eichler and Zagier [**4**] to Jacobi forms over real quadratic fields.

2. Notation and statement of results. Let $\mathbf{H}\subset\mathbf{C}$ be the usual complex upper half plane. We write variables $\vec{\tau}=(\tau_1,\tau_2)\in\mathbf{H}^2$ and $\vec{z}=(z_1,z_2)\in\mathbf{C}^2$. We define the following slash operator on functions $f:\mathbf{H}^2\to\mathbf{C}$. For fixed $\mathfrak{k}=(k_1,k_2)\in\mathbf{N}_0^2$, where $\mathbf{N}_0$ denotes the set of nonnegative integers, set

$$(1)\qquad \left(f\ |_{\mathfrak{k}}\ M\right)(\vec{\tau})=\prod_{j=1}^{2}(c_j\tau_j+d_j)^{-k_j}f\left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1},\frac{a_2\tau_2+b_2}{c_2\tau_2+d_2}\right)$$

for $M=\left(\left(\begin{smallmatrix}a_1&b_1\\c_1&d_1\end{smallmatrix}\right),\left(\begin{smallmatrix}a_2&b_2\\c_2&d_2\end{smallmatrix}\right)\right)\in\mathrm{SL}_2(\mathbf{R})^2$. Furthermore, we define the following two slash operators on functions $f:\mathbf{H}^2\times\mathbf{C}^2\to\mathbf{C}$. For a

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fixed $\mathfrak{k} = (k_1, k_2) \in \mathbf{N}_0^2$ and $\mathfrak{m} = (m_1, m_2) \in \mathbf{N}_0^2$, set
(2)

$$(f \mid_{\mathfrak{k},\mathfrak{m}} M)(\vec{\tau}, \vec{z}) = \prod_{j=1}^{2} (c_j\tau_j + d_j)^{-k_j} \exp\left\{ -2\pi i \sum_{j=1}^{2} m_j \frac{c_j z_j^2}{c_j\tau_j + d_j} \right\} \cdot f\left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \frac{a_2\tau_2 + b_2}{c_2\tau_2 + d_2}, \frac{z_1}{c_1\tau_1 + d_1}, \frac{z_2}{c_2\tau_2 + d_2} \right)$$

for $M = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix} \right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix} \right) \right) \in \mathrm{SL}_2(\mathbf{R})^2$, and also

$$(3) \quad (f \mid_{\mathfrak{m}} [\lambda, \mu])(\vec{\tau}, \vec{z}) = \exp\left\{ 2\pi i \sum_{j=1}^{2} m_j(\lambda_j^2 \tau_j + 2\lambda_j z_j) \right\} \cdot f(\tau_1, \tau_2, z_1 + \lambda_1\tau_1 + \mu_1, z_2 + \lambda_2\tau_2 + \mu_2)$$

for $[\lambda, \mu] = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbf{R}^4$.

Of particular interest are the elements invariant under the actions in (1), (2) and (3). The most important case occurs when one restricts to $M \in \Gamma$ and $[\lambda, \mu] \in \mathcal{O}_K^2$, where Γ is a subgroup of finite index of $\mathrm{SL}_2(\mathcal{O}_K)$ and where $\mathcal{O}_K$ is the ring of integers of a real quadratic field K. In that case, a Hilbert modular form $f : \mathbf{H}^2 \to \mathbf{C}$ is invariant under the action (1) and a Hilbert Jacobi form $f : \mathbf{H}^2 \times \mathbf{C}^2 \to \mathbf{C}$ is invariant under (2) and (3). For more details, see Freitag [**5**], Garrett [**6**] and Skogman [**7**].

We now introduce the function on $\mathbf{H} \times \mathbf{C}$ which will play the main role in our characterization of Jacobi forms. Let $f : \mathbf{H}^2 \times \mathbf{C}^2 \to \mathbf{C}$ be holomorphic and $k, m \in \mathbf{N}_0$. Set $\vec{k} = (k, k)$, $\vec{m} = (m, m)$,

$$\alpha = k - \frac{1}{2} \quad \text{and} \quad L_j = 8\pi i m \partial_{\tau_j} - \partial_{z_j}^2 \quad \text{(heat operator)},$$

where

$$\partial_{\tau_j} = \frac{\partial}{\partial \tau_j}, \quad \partial_{z_j} = \frac{\partial}{\partial z_j},$$

and where $j = 1, 2$. If $x \in \mathbf{C}$, then we write $x! = \Gamma(x + 1)$, where $\Gamma(\cdot)$ denotes the Gamma function. For every $v \in \mathbf{N}_0$, define $\Phi_v[f] : \mathbf{H} \times \mathbf{C} \to \mathbf{C}$ by

$$(4) \quad \Phi_v[f](\tau, z) = \sum_{v_1 + v_2 = v} (-1)^{v_1} \frac{(L_1^{v_1} L_2^{v_2} f)(\tau, \tau, z, z)}{(v_1)!(v_2)!(\alpha + v_1 - 1)!(\alpha + v_2 - 1)!}.$$

The following theorem gives a combinatorial characterization of Jacobi forms on $\mathbf{H}^2 \times \mathbf{C}^2$ in terms of functions on $\mathbf{H}$ and on $\mathbf{H} \times \mathbf{C}$. In particular, we extend Theorem 2.2 in Cohen [**3**] to Jacobi forms.

Theorem 1. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}),$$

and set

$$M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Then statements (G1) *and* (G2) *are equivalent*:

(G1):

$$\begin{cases} \text{(i)} \quad \textit{For all } v \in \mathbf{N}_0, \\ \Phi_v[f]\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^{2k+2v} \exp\left\{4\pi i m \, \frac{cz^2}{c\tau+d}\right\} \Phi_v[f](\tau, z); \\ \text{(ii)} \quad \textit{For all } u_1, u_2, v_1, v_2 \in \mathbf{N}_0 \textit{ and } r, s \in \{0,1\}, \\ \quad \left(\partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^{r} \partial_{z_2}^{s} L_1^{v_1} L_2^{v_2}[f \mid_{\vec{k}, \vec{m}} M - f]\right)(\tau, \tau, 0, 0) \\ \quad = \left(\partial_{\tau_1}^{u_2} \partial_{\tau_2}^{u_1} \partial_{z_1}^{s} \partial_{z_2}^{r} L_1^{v_1} L_2^{v_2}[f \mid_{\vec{k}, \vec{m}} M - f]\right)(\tau, \tau, 0, 0); \\ \text{(iii)} \quad \textit{If } v_1, v_2 \in \mathbf{N}_0 \textit{ such that} \\ \quad \left(L_1^{v_1} L_2^{v_2}[f \mid_{\vec{k}, \vec{m}} M - f]\right)(\tau, \tau, z, z) = 0, \\ \quad \textit{then } \left(\partial_{z_1} \partial_{z_2} L_1^{v_1} L_2^{v_2}[f \mid_{\vec{k}, \vec{m}} M - f]\right)(\tau, \tau, z, z) = 0. \end{cases}$$

(G2):

$$\left(f \mid_{\vec{k}, \vec{m}} M\right)(\tau_1, \tau_2, z_1, z_2) = f(\tau_1, \tau_2, z_1, z_2).$$

Let $(\lambda, \mu) \in \mathbf{Z}^2$ *and* $[\lambda, \mu] = (\lambda, \lambda, \mu, \mu)$. *Then statements* (H1) *and* (H2) *are equivalent*:

(H1):

$$\begin{cases} \text{(i)} \quad \textit{For all } v \in \mathbf{N}_0, \\ \quad \Phi_v[f]\,(\tau, z + \lambda\tau + \mu) = \exp\left\{-4\pi i m(\lambda^2\tau + 2\lambda z)\right\} \Phi_v[f](\tau, z); \\ \text{(ii)} \quad \textit{For all } u_1, u_2, v_1, v_2 \in \mathbf{N}_0 \textit{ and } r, s \in \{0,1\}, \\ \quad \left(\partial_{\tau_1}^{u_1}\partial_{\tau_2}^{u_2}\partial_{z_1}^{r}\partial_{z_2}^{s} L_1^{v_1} L_2^{v_2}[f\,|_{\vec{m}}\,[\lambda,\mu] - f]\right)(\tau,\tau,0,0) \\ \quad = \left(\partial_{\tau_1}^{u_2}\partial_{\tau_2}^{u_1}\partial_{z_1}^{s}\partial_{z_2}^{r} L_1^{v_1} L_2^{v_2}[f\,|_{\vec{m}}\,[\lambda,\mu] - f]\right)(\tau,\tau,0,0); \\ \text{(iii)} \quad \textit{If } v_1, v_2 \in \mathbf{N}_0 \textit{ such that} \\ \quad \left(L_1^{v_1} L_2^{v_2}[f\,|_{\vec{m}}\,[\lambda,\mu] - f]\right)(\tau,\tau,z,z) = 0, \\ \quad \textit{then } \left(\partial_{z_1}\partial_{z_2} L_1^{v_1} L_2^{v_2}[f\,|_{\vec{m}}\,[\lambda,\mu] - f]\right)(\tau,\tau,z,z) = 0. \end{cases}$$

(H2):

$$\left(f\,|_{\vec{m}}\,[\lambda,\mu]\right)(\tau_1,\tau_2,z_1,z_2) = f(\tau_1,\tau_2,z_1,z_2).$$

In particular, f is a Jacobi form of weight $\vec{k}$ and index $\vec{m}$ with respect to the subgroup $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ (*acting diagonally*) *of the Hilbert Jacobi group of a real quadratic field if and only if f satisfies* (G1) *and* (H1) *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ *and* $(\lambda,\mu) \in \mathbf{Z}^2$.

Remark 1. Note that all of the conditions in (G1) and (H1) are necessary. For example,

$$f_1(\tau_1,\tau_2,z_1,z_2) = (z_1 - z_2)^2,$$
$$f_2(\tau_1,\tau_2,z_1,z_2) = z_1^2 + z_2^2,$$

and

$$f_3(\tau_1,\tau_2,z_1,z_2) = \frac{1}{4\pi i m}(\tau_1 - \tau_2) + z_1^2 - z_2^2$$

do not satisfy (G2) and (H2). On the other hand:

(1) f_1 satisfies (G1) (i), (G1) (ii), (H1) (i) and (H1) (ii), but f_1 does not satisfy (G1) (iii) and (H1) (iii) for $v_1 = v_2 = 0$.

(2) f_2 satisfies (G1) (ii), (G1) (iii), (H1) (ii) and (H1) (iii), but f_2 does not satisfy (G1) (i) and (H1) (i) for $v = 0$.

(3) f_3 satisfies (G1) (i), (G1) (iii), (H1) (i) and (H1) (iii), but f_3 does not satisfy (G1) (ii) and (H1) (ii) for $u_2 = r = s = v_1 = v_2 = 0$ and $u_1 = 1$.

As a conclusion, none of the conditions in (G1) and (H1) can be omitted. However, if one additionally assumes that $f(\tau_1, \tau_2, z_1, z_2) = f(\tau_2, \tau_1, z_2, z_1)$, then conditions (G1) (ii) and (H1) (ii) are automatically satisfied. Hence conditions (G1) (i), (G1) (iii), (H1) (i) and (H1) (iii) alone already characterize such symmetric Jacobi forms on $\mathbf{H}^2 \times \mathbf{C}^2$.

3. Proof of Theorem 1. First we need the following proposition:

Proposition 1. *Let* $M = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \mathrm{SL}_2(\mathbf{R})^2$, *and suppose that* $f : \mathbf{H}^2 \times \mathbf{C}^2 \to \mathbf{C}$ *satisfies*

$$(f \mid_{\mathfrak{k},\mathfrak{m}} M)(\vec{\tau}, \vec{z}) = f(\vec{\tau}, \vec{z}) \tag{5}$$

for some $\mathfrak{k} = (k_1, k_2)$, $\mathfrak{m} = (m_1, m_2) \in \mathbf{N}_0^2$. *Define*

$$\tilde{f}(\vec{\tau}, \vec{z}, \vec{x}) = \sum_{v_1+v_2 \geq 0} \frac{L_{m_1}^{v_1} L_{m_2}^{v_2}(f)\big(\vec{\tau}, \vec{z}\big)}{(v_1)!(v_2)!(\alpha_1 + v_1 - 1)!(\alpha_2 + v_2 - 1)!} \, x_1^{v_1} x_2^{v_2}, \tag{6}$$

where $\vec{x} = (x_1, x_2) \in \mathbf{C}^2$,

$$a_j = k_j - \frac{1}{2}$$

and $L_{m_j} = 8\pi i m_j \partial_{\tau_j} - \partial_{z_j}^2$ *for* $j = 1, 2$. *Then*

$$\begin{aligned} &\tilde{f}\left(\tfrac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \tfrac{a_2\tau_2+b_2}{c_2\tau_2+d_2}, \tfrac{z_1}{c_1\tau_1+d_1}, \tfrac{z_2}{c_2\tau_2+d_2}, \tfrac{x_1}{(c_1\tau_1+d_1)^2}, \tfrac{x_2}{(c_2\tau_2+d_2)^2} \right) \\ &\quad = \prod_{j=1}^{2} (c_j\tau_j + d_j)^{k_j} \exp\left\{ 2\pi i \sum_{j=1}^{2} m_j \left(\frac{c_j(z_j^2 + 4x_j)}{c_j\tau_j + d_j} \right) \right\} \tilde{f}(\vec{\tau}, \vec{z}, \vec{x}). \end{aligned} \tag{7}$$

Proof. Let $\mathfrak{v} = (v_1, v_2) \in \mathbf{N}_0^2$. An induction on $v_1 + v_2$ shows that

$$(8) \quad \left(L_{m_1}^{v_1} L_{m_2}^{v_2}(f) \mid_{\mathfrak{k}+2\mathfrak{v},\mathfrak{m}} M\right)(\vec{\tau}, \vec{z}) = \sum_{\substack{(l_1, l_2)\in \mathbf{N}_0^2 \\ 0\le l_j \le v_j}} \left(\prod_{j=1}^{2} \frac{(v_j)!}{(l_j)!} \binom{\alpha_j + v_j - 1}{v_j - l_j} \left(\frac{8\pi i m_j c_j}{c_j \tau_j + d_j} \right)^{v_j - l_j} \right) \cdot L_{m_1}^{l_1} L_{m_2}^{l_2}(f)(\vec{\tau}, \vec{z}),$$

where the sum on the right side is over all vectors $(l_1, l_2) \in \mathbf{N}_0^2$ such that $0 \le l_j \le v_j$ for $j = 1, 2$. It is easy to check that (8) implies (7). □

Remark 2. (1) Proposition 1 can obviously be extended to functions $f : \mathbf{H}^n \times \mathbf{C}^n \to \mathbf{C}$, for any $n \ge 2$.

(2) Let f_1 and f_2 be Hilbert Jacobi forms over a real quadratic field K. Then f_1 and f_2 satisfy (5) for all $M \in \mathrm{SL}_2(\mathcal{O}_K)$ and computing the coefficient of $x_1^{v_1} x_2^{v_2}$ in $\tilde{f}_1(\vec{\tau}, \vec{z}, \mathfrak{m}_2\, \vec{x}) \tilde{f}_2(\vec{\tau}, \vec{z}, -\mathfrak{m}_1\, \vec{x})$ yields an explicit formula for the Rankin-Cohen bracket for Hilbert Jacobi forms. For details on Rankin-Cohen brackets for Jacobi forms on $\mathbf{H} \times \mathbf{C}$, see [**1, 2**].

Now we turn to the proof of Theorem 1. Assume that (G2) is true. Then (G1) (i) follows from Proposition 1 after setting $\tau = \tau_1 = \tau_2$, $z = z_1 = z_2$, $k = k_1 = k_2$, $m = m_1 = m_2$, $x_1 = -mx$ and $x_2 = mx$. Clearly, (G1) (ii) and (G1) (iii) hold.

Conversely, suppose that (G1) is true. Define $g : \mathbf{H}^2 \times \mathbf{C}^2 \to \mathbf{C}$ by

$$(9) \quad g(\tau_1, \tau_2, z_1, z_2) = \left(f \mid_{\vec{k},\vec{m}} M\right)(\tau_1, \tau_2, z_1, z_2) - f(\tau_1, \tau_2, z_1, z_2).$$

Consider the Taylor expansion of g around $(\tau_1, \tau_2, 0, 0)$ to see that

$$g \equiv 0 \Longleftrightarrow \text{For all } \tau \in \mathbf{H}, u_1, u_2, v_1, v_2 \in \mathbf{N}_0, \text{ and } r, s \in \{0, 1\}, \\ (\partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^{r} \partial_{z_2}^{s} L_1^{v_1} L_2^{v_2} g)(\tau, \tau, 0, 0) = 0.$$

Now we come to the key step. We will verify that $(L_1^{v_1} L_2^{v-v_1} g)(\tau, \tau, z, z) = 0$ for all $\tau \in \mathbf{H}$, $z \in \mathbf{C}$ and $v_1 \le v$ by inducting on v. Set

$v = 0$ in (G1) (i) to see that $g(\tau,\tau,z,z) = 0$. Suppose that $(L_1^{v_1}L_2^{v-1-v_1}g)(\tau,\tau,z,z) = 0$. With the help of (G1) (i) and (G1) (iii), one may check that

$$(L_1^{v_1}L_2^{v-v_1}g)(\tau,\tau,z,z) = (-1)^{v_1}(L_2^{v}g)(\tau,\tau,z,z). \tag{10}$$

An induction on $v_1 + v_2$ shows that

$$\begin{aligned} &L_1^{v_1}L_2^{v_2}g(\tau_1,\tau_2,z_1,z_2) + L_1^{v_1}L_2^{v_2}(f)(\vec{\tau},\vec{z}) \\ &= \sum_{\substack{(l_1,l_2)\in\mathbf{N}_0^2 \\ 0\le l_j\le v_j}} \left(\prod_{j=1}^{2} \binom{v_j}{l_j} \tfrac{(\alpha+v_j-1)!}{(\alpha+l_j-1)!} \tfrac{(-8\pi imc)^{v_j-l_j}}{(c\tau_j+d)^{v_j+l_j+k}} \exp\left\{ -2\pi im\, \tfrac{cz_j^2}{c\tau_j+d} \right\} \right) \\ &\quad \cdot L_1^{l_1}L_2^{l_2}(f)\Big(\tfrac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \tfrac{a_2\tau_2+b_2}{c_2\tau_2+d_2}, \tfrac{z_1}{c_1\tau_1+d_1}, \tfrac{z_2}{c_2\tau_2+d_2} \Big), \end{aligned}$$

which, in combination with (G1) (i), yields (with a computation like in [**3**]) that

$$\begin{aligned} 0 &= \sum_{v_1+v_2=v} (-1)^{v_1} \frac{(L_1^{v_1}L_2^{v_2}g)(\tau,\tau,z,z)}{(v_1)!(v_2)!(\alpha+v_1-1)!(\alpha+v_2-1)!} \\ &\underset{(10)}{=} \left(\sum_{v_1+v_2=v} \frac{1}{(v_1)!(v_2)!(\alpha+v_1-1)!(\alpha+v_2-1)!} \right) (L_2^{v}g)(\tau,\tau,z,z). \end{aligned}$$

Hence, $0 = (L_2^{v}g)(\tau,\tau,z,z) \underset{(10)}{=} (L_1^{v_1}L_2^{v-v_1}g)(\tau,\tau,z,z)$. Then (G1) (ii) and (G1) (iii) imply that $(\partial_{\tau_1}^{u_1}\partial_{\tau_2}^{u_2}\partial_{z_1}^{r}\partial_{z_2}^{s}L_1^{v_1}L_2^{v_2}g)(\tau,\tau,0,0) = 0$ for all $\tau \in \mathbf{H}$, $u_1,u_2,v_1,v_2 \in \mathbf{N}_0$ and $r,s \in \{0,1\}$. We conclude that (G2) holds.

The proof of the equivalence of (H1) and (H2) is similar and we omit the detailed proof. □

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