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OSCILLATION OF FIRST-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY AND EULER FORM

KAIZHONG GUAN AND JIANHUA SHEN

ABSTRACT. In this paper, we investigate a first-order neutral differential equation with unbounded delay and Euler form. Some necessary and sufficient as well as explicit sufficient conditions for the oscillation of all solutions are established.

1. Introduction. The oscillation theory of delay differential equations and neutral differential equations has drawn much attention in recent years. This is evidenced by extensive references in the books of Györi and Ladas [**3**], Erbe et al. [**2**] and Ladde et al. [**6**].

For the following delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0, \tag{1.1}$$

where $p_i > 0$, $\tau_i > 0$ are constants, $i = 1, 2, \dots, n$. In 1983, Ladas and Stavroulakis [**5**] established the following well-known oscillation result:

Theorem A. *All solutions of* (1.1) *oscillate if and only if*

$$-\lambda + \sum_{i=1}^{n} p_i e^{\lambda \tau_i} > 0, \quad \textit{for all } \lambda > 0. \tag{1.2}$$

Recently, Ran [**7**] discussed a class of delay differential equations with unbounded delays of the form

$$x'(t) + \frac{1}{t} \sum_{i=1}^{n} p_i x(\alpha_i t) = 0, \quad t \geq t_0 > 0, \tag{1.3}$$

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where $p_i > 0$ and $0 < \alpha_i < 1$ are constants, $i = 1, 2, \dots, n$. The author established the following necessary and sufficient condition which is similar to Theorem A for the oscillation of every solution of equation (1.3).

Theorem B. *Every solution of equation* (1.3) *oscillates if and only if*

$$-\lambda + \sum_{i=1}^{n} p_i \alpha_i^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{1.4}$$

Some explicit sufficient conditions for the oscillation of all solutions of (1.3) are also obtained therein.

On the other hand, for the neutral differential equation with constant delay

$$(x(t) - cx(t-\tau))' + px(t-\sigma) = 0, \quad t \geq t_0, \tag{1.5}$$

where $\tau, p \in (0, \infty)$, $\tau \in [0, \infty)$ and $0 \leq c < 1$. It is well known that [**3**] all solutions of equation (1.5) oscillate if and only if

$$-\lambda + c\lambda e^{\lambda\tau} + pe^{\lambda\sigma} > 0, \quad \text{for all } \lambda > 0. \tag{1.6}$$

And for the differential equation with several constant delays,

$$(x(t) - cx(t-\tau))' + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \quad t \geq t_0, \tag{1.7}$$

where $0 \leq c < 1$, $\tau, p_i \in (0, \infty)$ and $\tau_i \in [0, \infty)$, $i = 1, 2, \dots, n$, it is also known that [**4**] every solution of equation (1.7) oscillates if and only if

$$-\lambda + c\lambda e^{\lambda\tau} + \sum_{i=1}^{n} p_i e^{\lambda\tau_i} > 0, \quad \text{for all } \lambda > 0. \tag{1.8}$$

There are also a few results on the oscillation behavior of solutions of equation (1.7). See, for example, [**1, 8, 9**] and the references cited therein.

In this paper we consider the following first order neutral type differential equation with unbounded delay

$$\frac{d}{dt}\left(x(t) - cx(\alpha t)\right) + \frac{p}{t}x(\beta t) = 0, \quad t \geq t_0 > 0, \tag{1.9}$$

where $0 \leq c < 1$, $0 < \alpha$, $\beta < 1$, $p > 0$, and the more general one

$$\frac{d}{dt}\left(x(t) - cx(\alpha t)\right) + \frac{1}{t}\sum_{i=1}^{n} p_i x(\beta_i t) = 0, \quad t \geq t_0 > 0, \tag{1.10}$$

where $0 \leq c < 1$, $0 < \alpha < 1$, $0 < \beta_i < 1$, $p_i > 0$, $i = 1, 2, \ldots, n$.

By a solution of equation (1.9) (or equation (1.10)) we mean a function $x(t) \in C([\rho\bar{t}, \infty), R)$ for some $\bar{t} \geq t_0$, such that $x(t) - cx(\alpha t)$ is continuously differentiable, and $x(t)$ satisfies equation (1.9) (or equation (1.10)) for all $t \geq \bar{t}$, where $\rho = \min\{\alpha, \beta\}$ (or $\min\{\alpha, \min_{1 \leq i \leq n} \beta_i\}$).

As usual, a solution of equation (1.9) (or equation (1.10)) is called oscillatory if it has arbitrarily large zeroes and nonoscillatory if it is eventually positive or eventually negative.

It should be mentioned that equation (1.3) is a differential equation with variable coefficients and unbounded delays which is similar to the first order ordinary Euler equation. Thus, one may call equation (1.3) a delay Euler equation with unbounded delays. It should be noted that there are few results concerning the oscillation theory of solutions to such equations though there are many good results on the qualitative properties of ordinary Euler equations. In particular, to the best of our knowledge, there is little in the way of results for the oscillation of neutral differential equations (1.9) and (1.10) with unbounded delays (and Euler form). Thus, there is strong interest in investigating such equations.

Our main purpose is to establish some oscillation criteria for all solutions of (1.9) and (1.10). For equation (1.9), we establish the sufficient and necessary conditions for the oscillation of all solutions which is similar to (1.6); furthermore, some explicit sufficient conditions that every solution of (1.9) oscillates are obtained. And for equation (1.10), the explicit sufficient conditions for every solution to oscillate are established.

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large t.

2. Lemmas. We need the following lemmas for the proofs of our main results.

Lemma 2.1 [**7**]. *Suppose that $p > 0$, $0 < \alpha < 1$, and that $x(t)$ is an eventually positive solution of the delay differential inequality*

$$x'(t) + \frac{p}{t}x(\alpha t) \leq 0. \tag{2.1}$$

Then

$$x(\alpha t) \leq \frac{1}{(p \ln \alpha/2)^2} x(t). \tag{2.2}$$

Lemma 2.2. *Let $v(t)$ be a positive and continuously differentiable function on some interval $[t_0, \infty)$. Assume that there exist constants A and $\alpha \in (0,1)$ such that for sufficiently large t*

$$v'(t) \leq 0 \quad \text{and} \quad v(\alpha t) < Av(t). \tag{2.3}$$

Set

$$\Lambda = \left\{ \lambda \geq 0 : v'(t) + \frac{\lambda}{t} v(t) \leq 0 \ \text{eventually} \right\}. \tag{2.4}$$

Then $A > 1$ and $\lambda_0 = \ln A/|\ln \alpha| \overline{\in} \Lambda$.

Proof. Obviously, $A > 1$ if (2.3) holds. Below, for the sake of contradiction, assume that $\lambda_0 \in \Lambda$. Then eventually,

$$\frac{d}{dt}[t^{\lambda_0} v(t)] = t^{\lambda_0} \left[v'(t) + \frac{\lambda_0}{t} v(t) \right] \leq 0,$$

which implies that the function $u(t) = t^{\lambda_0} v(t)$ is eventually decreasing. Hence, for sufficiently large t, we have

$$(\alpha t)^{\lambda_0} v(\alpha t) \geq t^{\lambda_0} v(t),$$

or

$$v(\alpha t) \geq \alpha^{-\lambda_0} v(t) = Av(t),$$

which contradicts (2.3), and the proof of the lemma is complete. □

Lemma 2.3. *Let $y(t)$ be an eventually positive solution of equation* (1.9)*, and set*

$$z(t) = y(t) - cy(\alpha t), \tag{2.5}$$

$$w(t) = z(t) - cz(\alpha t) \tag{2.6}$$

Then

(1) *$z(t)$ and $w(t)$ are also solutions of equation* (1.9); *furthermore, $z(t)$ is a differential solution, while $w(t)$ is twice differentiable*;

(2) *$z'(t) < 0$ and $z(t) > 0$*;

(3) *$w(t) > 0$, $w'(t) < 0$ and $w''(t) > 0$.*

Proof. (1) This can be done by substituting $z(t)$ and $w(t)$ into equation (1.9) respectively and simply calculating.

(2) Let $t_1 \geq t_0$ be such that $y(t) > 0$, $y(\beta t) > 0$, and $y(\alpha t) > 0$ for $t \geq t_1$. Then, by (1.9), we find

$$z'(t) = -\frac{p}{t} y(\beta t) < 0, \quad \text{for } t \geq t_1,$$

which implies that $z(t)$ is eventually decreasing. Hence, if $z(t) > 0$ does not hold eventually, then eventually $z(t) < 0$, and so there exist $t_2 \geq t_1$ and $\mu > 0$ such that $z(t) \leq -\mu$ for $t \geq t_2$. In view of (2.5), we obtain

$$y(t) = z(t) + cy(\alpha t) < -\mu + cy(\alpha t) < -\mu + y(\alpha t), \quad t \geq t_2. \tag{2.7}$$

If we choose a $\bar{t}$ such that $\alpha \bar{t} > t_2$, we thus have

$$y\left(\frac{\bar{t}}{\alpha^k}\right) < -\mu + y\left(\frac{\bar{t}}{\alpha^{k-1}}\right),$$

for $k = 1, 2, \ldots$. Then by induction,

$$y\left(\frac{\bar{t}}{\alpha^k}\right) < -(k+1)\mu + y(\alpha \bar{t}),$$

for $k = 1, 2, \dots$, and in view of this we have $y(\bar{t}/\alpha^k) \to -\infty$ as $k \to \infty$. This is a contradiction and so $z(t)$ is eventually positive.

(3) In view of (2), $w'(t) = -(p/t)z(\beta t)$ implies that $w'(t) < 0$ eventually. Similar to the proof of (2), we can prove that $w(t) > 0$. Again,

$$\begin{aligned} w''(t) &= \frac{d}{dt}\bigg(-\frac{p}{t}z(\beta t)\bigg) = -\frac{p}{t^2}[\beta t z'(\beta t) - z(\beta t)] \\ &= \frac{p}{t^2}[py(\beta^2 t) + z(\beta t)] > 0. \end{aligned}$$

Thus, the proof of the lemma is complete. □

3. Main results. In this section, we establish necessary and sufficient as well as some explicit sufficient conditions for the oscillation of all solutions of equation (1.9). An explicit sufficient condition for every solution of equation (1.10) to oscillate is also obtained.

Theorem 3.1. *Assume that $p > 0$, $0 \le c < 1$, and $0 < \beta < \alpha < 1$. Then every solution of equation* (1.9) *oscillates if and only if*

$$F(\lambda) = -\lambda + c\lambda\alpha^{-\lambda} + p\beta^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{3.1}$$

Proof. Let $c = 0$. Equation (1.9) is reduced to equation (1.3), and so the theorem is Theorem 1 of paper [**7**]. Below we consider the case $0 < c < 1$.

Assume firstly that (3.1) does not hold. We may then choose a $\lambda_0 > 0$ such that

$$F(\lambda_0) = -\lambda_0 + c\lambda_0\alpha^{-\lambda_0} + p\beta^{-\lambda_0} = 0.$$

But then $x(t) = t^{-\lambda_0}$ is a nonoscillatory solution of equation (1.9), a contradiction.

Assume conversely that not all solutions of equation (1.9) oscillate. This would imply that there exists at least one nonoscillatory solution of equation (1.9). Without loss of generality we assume that $y(t)$ is an eventually positive solution. Let

$$z(t) = y(t) - cy(\alpha t), \qquad w(t) = z(t) - cz(\alpha t).$$

By Lemma 2.3 then, eventually,

$$w(t) > 0, \qquad w'(t) < 0, \qquad w''(t) > 0. \tag{3.2}$$

Put

$$w_n(t) = \begin{cases} w(t) & n = 0, \\ w_{n-1}(t) - cw_{n-1}(\alpha t) & n = 1, 2, \dots . \end{cases} \tag{3.3}$$

From (3.3) and Lemma 2.3 it follows that for $n = 1, 2, \dots,$

$$\frac{d}{dt}\left(w_n(t) - cw_n(\alpha t)\right) + \frac{p}{t} w_n(\beta t) = 0, \quad t \geq t_0 > 0, \tag{3.4}$$

$$w_n'(t) = -\frac{p}{t} w_{n-1}(\beta t), \tag{3.5}$$

and

$$w_n(t) > 0, \qquad w_n'(t) < 0, \qquad w_n''(t) > 0. \tag{3.6}$$

Define the sets

$$\Lambda_n = \left\{\lambda \geq 0 : w_n'(t) + \frac{\lambda}{t} w_n(t) < 0\right\}, \quad n = 0, 1, 2, \dots .$$

Note that $0 \in \Lambda_n$ for all n. Also, if $0 \leq a \leq b$ and $b \in \Lambda_n$, then $a \in \Lambda_n$. That is, Λ_n is a nonempty subinterval of nonnegative real numbers. The proof is completed by showing that the following contradictory properties hold:

(P_1) There exist nonnegative numbers λ_1, λ_2 such that $\lambda_1 \in \cap_{n=1}^{\infty} \Lambda_n$ and $\lambda_2 \overline{\in} \cup_{n=1}^{\infty} \Lambda_n$.

(P_2) There exists a positive numbers μ such that if $\lambda \in \Lambda_n$, then $(\lambda + \mu) \in \Lambda_{n+1}$.

In view of (3.1) and $F(0) = p > 0$ there exists an $m > 0$ such that

$$F(\lambda) = -\lambda + c\lambda\alpha^{-\lambda} + p\beta^{-\lambda} > m, \quad \text{for all } \lambda > 0. \tag{3.7}$$

From (3.4) and (3.6) it follows that

$$(1 - c\alpha) w_n'(t) + \frac{\alpha p}{t} w_n\left(\frac{\beta}{\alpha} t\right) \leq 0,$$

or

$$w_n'(t) + \frac{\alpha p}{(1-c\alpha)t} w_n\left(\frac{\beta}{\alpha}t\right) \leq 0.$$

This implies that $\lambda_1 = \alpha p/(1-c\alpha) \in \cap_{n=1}^{\infty}\Lambda_n$. By Lemma 2.1 and Lemma 2.2, we have

$$\lambda_2 = \ln\left(\frac{2(1-c\alpha)}{\alpha p \ln(\beta/\alpha)}\right)^2 \Big/ \left|\ln\frac{\beta}{\alpha}\right| \overline{\in} \bigcup_{n=1}^{\infty} \Lambda_n.$$

Let $\lambda \in \Lambda_n$ and set $\phi(t) = t^{\lambda} w_n(t)$. By (3.5), (3.7) and noting the fact that $\phi(t)$ is decreasing, we have

$$\begin{aligned} w_{n+1}'(t) &+ \frac{\lambda+m}{t} w_{n+1}(t) \\ &= -\frac{pt^{\lambda-1}}{\beta^{\lambda}} \cdot \phi(\beta t) + (\lambda+m)t^{\lambda-1} \cdot [\phi(t) - c\alpha^{-\lambda}\phi(\alpha t)] \\ &\leq t^{\lambda-1}\phi(t)[-p\beta^{-\lambda} + (\lambda+m)(1-c\alpha^{-\lambda})] \\ &= t^{\lambda-1}\phi(t)[-p\beta^{-\lambda} + \lambda - c\lambda\alpha^{-\lambda} + m(1-c\alpha^{-\lambda})] \\ &\leq t^{\lambda-1}\phi(t)[-m + m(1-c\alpha^{-\lambda})] < 0, \end{aligned}$$

which implies $(\lambda+m) \in \Lambda_{n+1}$. Thus, the proof is complete. □

Theorem 3.1 is of theoretical interest. But the assumptions are not easy to verify. Hence, it is necessary to establish some explicit sufficient conditions for the oscillation of every solution of equation (1.9). Now we give the following theorem.

Theorem 3.2. *Assume that $p > 0$, $0 \leq c < 1$, and $0 < \beta < \alpha < 1$. Then every solution of equation* (1.9) *oscillates if*

$$pe \ln\left(\frac{1}{\beta}\right) \geq 1 - c\alpha^{-p/(1-c)}. \tag{3.8}$$

Proof. One can easily find that (3.1) is equivalent to

$$g(\lambda) = -1 + c\alpha^{-\lambda} + \frac{p}{\lambda}\beta^{-\lambda} > 0, \quad \text{for all } \lambda > 0. \tag{3.9}$$

By Theorem 3.1, we only need to prove that (3.9) is true when (3.8) holds. To this end, let $f_1(\lambda) = (p/\lambda)\beta^{-\lambda}$ and $f_2(\lambda) = 1 - c\alpha^{-\lambda}$. It suffices to show that $f_1(\lambda) > f_2(\lambda)$ for $\lambda > 0$. One can easily find that $f_1(\lambda)$ has only a global minimum at $\lambda_0 = \ln(1/\beta) = -\ln\beta$ and the minimum value is $pe\ln(1/\beta)$. In addition,

$$g(\lambda) = f_1(\lambda) - f_2(\lambda) = \frac{p}{\lambda}\beta^{-\lambda} + c\alpha^{-\lambda} - 1.$$

Calculating the value of $f_1(\lambda) - f_2(\lambda)$ at $\lambda = -1/(t\ln\beta)$ for $0 < t < \infty$, we have

$$\begin{aligned}[f_1(\lambda) - f_2(\lambda)]|_{\lambda=-1/(t\ln\beta)} &= pt\ln\left(\frac{1}{\beta}\right)\cdot\beta^{1/(t\ln\beta)} + c\alpha^{1/(t\ln\beta)} - 1\\ &> pt\ln\left(\frac{1}{\beta}\right) + c - 1.\end{aligned}$$

So, if $t > -(1-c)/(p\ln\beta)$, then $f_1(\lambda) - f_2(\lambda) > 0$. Hence, for $\lambda \in (0, p/(1-c))$, $f_1(\lambda) - f_2(\lambda) > 0$. We are now in a position to consider $\lambda \geq p/(1-c)$ and note,

$$\begin{aligned}[f_1(\lambda) - f_2(\lambda)]|_{\lambda\geq p/(1-c)} &\geq pe\ln\left(\frac{1}{\beta}\right) - (1 - c\alpha^{-\lambda})\\ &\geq pe\ln\left(\frac{1}{\beta}\right) + c\alpha^{-p/(1-c)} - 1.\end{aligned}$$

By assumption (3.8), it follows that

$$f_1(\lambda) - f_2(\lambda) > 0, \quad \text{for } \lambda \geq \frac{p}{1-c}.$$

Up to now, we have shown that

$$f_1(\lambda) - f_2(\lambda) > 0, \quad \text{for } \lambda > 0.$$

Therefore, (3.9) holds and so every solution of equation (1.9) oscillates. The proof is complete. □

Remark 3.3. The condition (3.8) is the "best possible" in the sense that when $c = 0$ condition (3.8) reduces to $p\ln(1/\beta) > e^{-1}$ which is a necessary and sufficient condition for the oscillation of all solutions of the differential equation:

$$x'(t) + \frac{p}{t} x(\beta t) = 0, \quad t \geq t_0 > 0.$$

For equation (1.10), whether one can establish the conclusion which is similar to Theorem 3.1 is a problem to be investigated further. However, we can establish the following explicit sufficient condition for the oscillation of all solutions of equation (1.10).

Theorem 3.4. *Assume that*

(i) $0 < c < 1$, $0 < \beta_i \leq \alpha < 1$, $p_i > 0$, $i = 1, 2, \ldots, n$;

(ii) $(\sum_{i=1}^n p_i) \ln(1/\alpha) > F(\bar{l}) = (1 - c\bar{l})^2 / \bar{l}$,

where $\bar{l}$ is the unique real root of the equation

$$1 - cl = \ln l, \quad 1 \leq l \leq \frac{1}{c}.$$

Then every solution of equation (1.10) *is oscillatory.*

Proof. Assume that the conclusion of Theorem 3.4 is false and, without loss of generality, assume that there exists an eventually positive solution $y(t)$ for $t \geq T$. Set

$$z(t) = y(t) - cy(\alpha t) \quad \text{and} \quad h(t) = \frac{z(\alpha t)}{z(t)} \quad \text{for} \quad t \geq \frac{T}{\alpha^2}.$$

Similar to the proof of Lemma 2.3, one can easily find that $z(t)$ is a positive and decreasing solution of

$$\frac{d}{dt}\left(z(t) - cz(\alpha t)\right) + \frac{1}{t} \sum_{i=1}^n p_i z(\beta_i t) = 0.$$

As $\beta_i t \leq \alpha t$, $i = 1, 2, \ldots, n$, we see that

$$\frac{d}{dt}\left(z(t) - cz(\alpha t)\right) + \frac{1}{t}\left(\sum_{i=1}^n p_i\right) z(\alpha t) \leq 0, \tag{3.10}$$

and in particular (as $z'(\alpha t) < 0$)

$$z'(t) + \frac{\sum_{i=1}^{n} p_i}{t} z(\alpha t) \leq 0. \tag{3.11}$$

Dividing both sides of (3.10) by $z(t)$ and integrating from αt to t, we find that

$$\int_{\alpha t}^{t} \frac{z'(s)}{z(s)}\,ds - \int_{\alpha t}^{t} c\alpha \frac{z'(\alpha s)}{z(s)}\,ds + \left(\sum_{i=1}^{n} p_i\right) \int_{\alpha t}^{t} \frac{z(\alpha s)}{sz(s)}\,ds \leq 0,$$

or

$$\ln h(t) \geq \left(\sum_{i=1}^{n} p_i\right) \int_{\alpha t}^{t} \frac{h(s)}{s}\,ds - c \int_{\alpha t}^{t} h(s) \frac{d}{ds}(\ln z(\alpha s))\,ds. \tag{3.12}$$

From (3.11) and Lemma 2.1, it follows that

$$z(\alpha t) < Bz(t),$$

where

$$B = \frac{4}{\left(\left(\sum_{i=1}^{n} p_i\right) \ln \alpha\right)^2}.$$

Hence,

$$1 \leq h(t) \leq B.$$

Let

$$l = \liminf_{t \to \infty} h(t).$$

Then, it follows from (3.12) that for $\varepsilon > 0$ sufficiently small,

$$\ln(l + \varepsilon) \geq \left(\sum_{i=1}^{n} p_i\right)(l - \varepsilon) \ln\left(\frac{1}{\alpha}\right) + c(l - \varepsilon)\ln(l - \varepsilon).$$

As ε is arbitrary, we have

$$\left(\sum_{i=1}^{n} p_i\right) \ln\left(\frac{1}{\alpha}\right) \leq \frac{(1 - cl)\ln l}{l}.$$

Set

$$F(l) = \frac{(1-cl)\ln l}{l}, 1 \le l \le B.$$

Then

$$F'(l) = \frac{1-cl-\ln l}{l^2}.$$

Let $\bar{l}$ be the unique real root of the equation

$$1 - cl = \ln l, \quad l \in [1, 1/c].$$

Then

$$\max_{l \ge 1} F(l) = F(\bar{l}) = \frac{(1-c\bar{l})^2}{\bar{l}}.$$

Hence,

$$\left(\sum_{i=1}^{n} p_i\right) \ln\left(\frac{1}{\alpha}\right) \le \frac{(1-c\bar{l})^2}{\bar{l}},$$

which contradicts (ii), and so the proof is complete.

Remark 3.5. $\bar{l}$ is a unique simple real root of the equation $1-cl = \ln l$ which is easily solved using the iterative argument or solution through diagrams.

4. Examples. In this section, two examples are given to illustrate the applications of our results.

Example 4.1. Consider the neutral differential equation

$$\frac{d}{dt}\left(x(t) - cx(e^{-\pi}t)\right) + \frac{1+c}{t}x(e^{-(5/2)\pi}t) = 0, \quad t \ge 2, \tag{4.1}$$

where $0 \le c < 1$ is constant.

It is easy to see that equation (4.1) satisfies the assumptions of Theorem 3.2, and so every solution of equation (4.1) is oscillatory. Indeed, $x(t) = \sin(\ln t)$ is such a solution.

Example 4.2. Consider the neutral differential equation

$$\frac{d}{dt}\left(x(t) - \frac{1}{2}x(e^{-\pi}t)\right) + \frac{2}{t}x(e^{-(5/2)\pi}t) + \frac{1}{2t}x(e^{-(3/2)\pi}t) = 0, \quad t \ge 2. \tag{4.2}$$

Clearly, $(p_1+p_2)\ln(1/\alpha) = (5/2)\pi$. Let $\bar{l}$ be the unique real root of the equation

$$1 - \frac{l}{2} = \ln l, \quad 1 \leq l \leq 2.$$

One can easily find that $4/3 < \bar{l} < 3/2$ and $F(\bar{l}) \leq 1/12$. Thus, equation (4.2) satisfies the conditions of Theorem 3.4, and so all solutions of equation (4.2) oscillate. Indeed, $x(t) = \sin(\ln t)$ is such a solution.

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Department of Mathematics, University of South China, Hengyang, Hunan 421001, China and Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China
Email address: kaizhongguan@yahoo.com.cn

Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China and Department of Mathematics, College of Huaihua, Huaihua, Hunan 418008, China
Email address: jhshen2ca@yahoo.com