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ESTIMATES FOR CONE MULTIPLIERS ASSOCIATED WITH HOMOGENEOUS FUNCTIONS

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ABSTRACT. Let $\rho \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be homogeneous of degree one. We show that the convolution operator

$$\widehat{T^\delta f}(\xi', \xi_{n+1}) = \left(1 - \frac{\rho(\xi')}{|\xi_{n+1}|}\right)_+^\delta \widehat{f}(\xi', \xi_{n+1}),$$
$$(\xi', \xi_{n+1}) \in \mathbf{R}^n \times \mathbf{R}^1$$

is bounded from Hardy spaces $H^p(\mathbf{R}^{n+1})$ to $L^p(\mathbf{R}^{n+1})$ for $\delta > n(1/p - 1/2) - 1/2$, $0 < p < 1$.

1. Introduction. Let $\widehat{f}$ be the Fourier transform of a Schwartz function f on $\mathbf{R}^n \times \mathbf{R}^1$. Let $\rho \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be homogeneous of degree one, $\rho(t\xi) = t\rho(\xi)$. We consider a family of convolution operators T^δ defined by

(1.1)
$$\widehat{T^\delta f}(\xi', \xi_{n+1}) = \left(1 - \frac{\rho(\xi')}{|\xi_{n+1}|}\right)_+^\delta \widehat{f}(\xi', \xi_{n+1}), \quad (\xi', \xi_{n+1}) \in \mathbf{R}^n \times \mathbf{R}^1$$

where $t_+^\delta = t^\delta$ for $t > 0$ and zero otherwise.

We are interested in obtaining the decay estimates for the kernel which is the inverse Fourier transform of the multiplier.

In the case of Riesz means, Randol [**7**] obtained a decay estimate for the kernel

$$\int_{\mathbf{R}^n} (1 - \rho(\xi))_+^\delta e^{i\langle x, \xi\rangle}\, d\xi, \quad \xi \in \mathbf{R}^n \tag{1.2}$$

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when ρ is convex and its boundary $\partial\rho$ is analytic by using asymptotic expansion. Svensson [**10**] investigated the case where $\partial\rho$ is flat of order $\geq [(n+1)/2]+1$ by applying a variant of the van der Corput lemma and the Hardy-Littlewood maximal inequality. Seeger [**8**] established the decay estimates for (1.2) when $\rho \in C^{n+2}(\mathbf{R}^n \setminus \{0\})$ is homogeneous of degree one without any finite type condition on ρ by using the decomposition by Córdoba in [**2**]. For the problems of a radial cone multiplier, Wolff in [**11**] provided a geometric decomposition of cone into rectangles on a piece of cone distant 1 from the boundary to obtain L^p boundedness on $\mathbf{R}^3$. Later, Łaba and Wolff in [**4**] extended it to higher dimensions $\mathbf{R}^n$, $n \geq 3$. One can also refer to Mockenhaupt, Seeger and Sogge [**6**] for related results on the wave equations.

We note that we only assume homogeneity and smoothness of ρ without any finite type condition. To do this we perform a careful analysis based on the geometry of our multiplier. We observe that our multiplier has singularities along the level surface $1-\rho(\xi')/|\xi_{n+1}|=0$. Since ρ is assumed to be homogeneous of degree one, the level surface is a cone. This leads us to a natural dyadic decomposition of the multiplier according to the height. To analyze the behavior of the operator caused by the singularities along the cone $\rho(\xi')=|\xi_{n+1}|$, we use dyadic decomposition with respect to the distance from the cone.

When ρ is radial, that is, $\rho(\xi)=|\xi|$, the kernel can be expressed by Bessel functions so the decay of the kernel can be obtained by the above decomposition and the properties of Bessel functions. However, expression of special functions for our kernel is not available because we don't have any symmetric assumption on our distance function ρ. For Riesz means, this type of decomposition has been used to obtain decay estimates for the kernel in general context in [**2**]. However, our multiplier has different feature caused by extra direction along a straight line on the cone. This forces us to perform additional analysis for the direction.

In this paper we obtain uniform boundedness of T^δ from $H^p(\mathbf{R}^{n+1})$ to $L^p(\mathbf{R}^{n+1})$ at the index $\delta > n(1/p-1/2)-1/2$, $0<p<1$, as a main application of the decay estimates for that kernel. We note that H^p are the standard real Hardy spaces as defined in [**9**].

Theorem 1. *Suppose $0 < p < 1$ and $\delta > n(1/p - 1/2) - 1/2$. Then T^δ is bounded from H^p to L^p in $\mathbf{R}^{n+1}$, i.e., for all $f \in H^p(\mathbf{R}^{n+1})$, there is a constant C so that*

$$\left\|T^\delta f\right\|_{L^p(\mathbf{R}^{n+1})} \leq C\,||f||_{H^p(\mathbf{R}^{n+1})}, \tag{1.3}$$

where the constant C is independent of f.

Remark 1. (i) As an example we introduce ρ as follows: let Ω be a convex set with a smooth boundary $\partial\Omega$ containing the origin. We may take

$$\rho(\xi) = \inf\{t : t^{-1}\xi \in \Omega\}.$$

Then it is obvious that ρ is homogeneous of degree one. If we take this ρ in (1.1), Theorem 1 still holds.

(ii) If $0 < p < 1$ and $\delta > n(1/p - 1/2) - 1/2$, one can easily show that T^δ is bounded from H^p to H^p in $\mathbf{R}^{n+1}$. To prove this, we apply Hölder's inequality, Plancherel's theorem, maximal characterization of H^p spaces, and kernel estimates in Lemma 1 of Section 2, see [**9**, Chapter 3].

(iii) When $\delta > (n-1)/2$, it is not hard to show that T^δ is of weak type $(1,1)$ by using Calderón-Zygmund theory, see [**9**, pages 16–22]. It is open whether T^δ is of weak type (p,p) at the critical index $\delta_p = n(1/p - 1/2) - 1/2$ on H^p spaces, $0 < p \leq 1$. In the case of radial cone multipliers, it was shown that T^δ is of weak type (p,p) at the critical index δ_p on H^p spaces, $0 < p < 1$ in [**3**].

(iv) For $p > 1$ we refer interested readers to [**1, 5, 6**].

In what follows, the letters C and c denote some positive constants that may not be the same at each occurrence.

2. Decay estimates. In this section we mainly obtain the decay estimates of the kernel. Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$ be supported in $(1/2, 2)$ such that $\sum_{j\in\mathbf{Z}} \varphi(2^j s) = 1$ and $\sum_{l\in\mathbf{Z}} \psi(2^{-l}t) = 1$ for $s,\, t > 0$. For $j > 1$, we set

$$\Psi_{j,0}(\xi', \xi_{n+1}) = \left(1 - \frac{\rho(\xi')}{|\xi_{n+1}|}\right)_+^\delta \varphi\left(2^j\left(1 - \frac{\rho(\xi')}{|\xi_{n+1}|}\right)\right)\psi(\xi_{n+1})$$

and

$$\Psi_{0,0}(\xi',\xi_{n+1}) = \left(1-\frac{\rho(\xi')}{|\xi_{n+1}|}\right)_+^\delta \psi(\xi_{n+1}) - \sum_{j>0} \Psi_{j,0}(\xi',\xi_{n+1}).$$

Since $\Psi_{0,0}$ behaves nicely, we only consider the case where $j>0$.

For each $j>0$, we choose $\chi \in C_0^\infty(B(0,1))$ such that $\sum_{\nu=1}^{2^{j(n-1)/2}} \times \chi(2^{j/2}(\xi'-\xi'_\nu)) = 1$ for all $\xi'_\nu, \xi' \in \{\xi' : \rho(\xi') = 1\}$, where $B(0,1)$ denotes the unit ball in $\mathbf{R}^n$ centered at the origin and $\{\xi'_\nu/|\xi'_\nu|\}_{\nu=1}^{2^{j(n-1)/2}}$ is a set of equally distributed unit vectors in $\mathbf{S}^{n-1}$.

For fixed j, ν and l we define

$$\Phi_{j,\nu,l}(\xi',\xi_{n+1}) = \varphi\left(2^j\left(1-\frac{\rho(\xi')}{|\xi_{n+1}|}\right)\right)\chi\left(2^{j/2}\left(\frac{\xi'}{\rho(\xi')}-\xi'_\nu\right)\right)\psi(2^{-l}\xi_{n+1}).$$

We shall need pointwise estimates for the kernels of

$$T^\delta_{j,\nu,l}f(x,t) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbf{R}^1}\int_{\mathbf{R}^n} K_{j,\nu,l}(x-y,t-w)f(y,w)\,dy\,dw$$

where

$$K_{j,\nu,0}(x,t) = \int_{\mathbf{R}^1}\int_{\mathbf{R}^n}\left(1-\frac{\rho(\xi')}{|\xi_{n+1}|}\right)_+^\delta \Phi_{j,\nu,0}(\xi',\xi_{n+1})e^{i\langle x,\xi'\rangle + it\xi_{n+1}}\,d\xi'\,d\xi_{n+1}.$$

We note that $K_{j,\nu,0}$ has the dilation property

$$K_{j,\nu,l}(\cdot,\cdot) = 2^{l(n+1)}\{K_{j,\nu,0}(2^l\cdot,2^l\cdot)\}. \tag{2.1}$$

We define

$$\Omega = \big\{(\xi',\xi_{n+1}) : 1/2 \le \rho(\xi')/\xi_{n+1} \le 2,\ 1/2 \le \xi_{n+1} \le 2\big\}.$$

In what follows we only consider a point $\xi = (\xi',\xi_{n+1})$ such that

$$\xi \in \Omega \cap \operatorname{supp}\Phi_{j,\nu,0}.$$

We shall use integration by parts along the three types of directions based on the geometry of the multiplier. To do this we precisely define those three directions. We define the first type of direction as

$$e_0(\xi) = \left(\frac{\nabla\rho(\xi')}{|\nabla\rho(\xi')|}, 0 \right)$$

which is the outer unit normal vector of $\Omega \cap \{\xi_{n+1} = 1\}$ at (ξ', ξ_{n+1}). The second type of directions is defined as unit vectors $e_d(\xi)$, $d = 1, \ldots, n-1$, satisfying

$$\langle e_d(\xi), e_0(\xi)\rangle = \langle e_d(\xi), \mathbf{e}_{n+1}\rangle = 0$$

where $\mathbf{e}_{n+1} = (0, \ldots, 0, 1)$. The third direction is defined as

$$\widetilde{e}(\xi) = -\xi/|\xi|$$

which is a tangential unit vector to the surface Ω from ξ to the origin.

We note that
$$\frac{1}{(1+|x|)^N} \leq \min\{1, |x|^{-N}\}.$$

Lemma 1. *There is an estimate as follows: for all $N \in \mathbf{N}$, $\gamma \in \mathbf{N}^{n+1}$ with $|\gamma| = 1, 2, 3, \ldots$ and for some $\xi_0 = (\xi_0', 1)$ in the support of $\Phi_{j,\nu,0}$*

$$\begin{aligned} |K_{j,\nu,0}(x,t)| &+ \sum_{|\gamma|=N} \left|(K_{j,\nu,0})^{(\gamma)}(x,t)\right| \\ &\leq C\, 2^{-j\delta} \frac{2^{-j}}{(1+2^{-j}|\langle x, e_0(\xi_0)\rangle|)^N} \\ &\times \frac{2^{-j(n-1)/2}}{\left(1+2^{-j/2}\sqrt{\sum_{d=1}^{n-1} |\langle x, e_d(\xi_0) > |^2}\right)^N} \frac{1}{(1+|ct+\langle x, \xi_0'\rangle|)^N} \end{aligned}$$

where the constant C is independent of the choice of ξ_0 and c is a constant between $1/2$ and 2.

Proof. We denote $m_{j,\nu,0}(\xi', \xi_{n+1}) = (1 - (\rho(\xi')/|\xi_{n+1}|))_+^\delta\, \Phi_{j,\nu,0} \times (\xi', \xi_{n+1})$. For a unit vector e, let $D_e f$ denote the directional derivative $\langle e, \nabla f\rangle$.

We first claim

$$|K_{j,\nu,0}(x,t)| \leq C\, 2^{-j\delta}\, \frac{2^{-j}}{(1+2^{-j}|\langle x, e_0(\xi_0)\rangle|)^N} \times \frac{2^{-j(n-1)/2}}{\left(1+2^{-j/2}\sqrt{\sum_{d=1}^{n-1}|\langle x, e_d(\xi_0)\rangle|^2}\right)^N}. \tag{2.2}$$

Since the righthand side of (2.2) is independent of the choice of $e_1(\xi_0), \dots, e_{n-1}(\xi_0)$, we may assume that $e_1(\xi_0)$ is on the plane spanned by $e_0(\xi_0)$ and x. We note that

$$|\langle x, e_1(\xi_0)\rangle| = \left(\sum_{d=1}^{n-1} |\rangle x, e_d(\xi_0)\langle|^2\right)^{1/2}$$

because $\langle x, e_2(\xi_0)\rangle = \cdots = \langle x, e_{n-1}(\xi_0)\rangle = 0$. We note that $\langle x, e_d(\xi_0)\rangle$ can be considered as an inner product in $\mathbf{R}^n$ because the last coordinate of $e_d(\xi_0)$ is zero. Furthermore, we may assume that $|e_d(\xi_0) - e_d(\xi)| \leq 2^{-j/2}/100$ for all $d = 0, \dots, n-1$ and $\xi \in \operatorname{supp} \Phi_{j,\nu,0}$.

Let γ be an angle between x and $e_0(\xi_0)$. We consider two cases: (i) $\pi - 2^{-j/2} < \gamma$ or $\gamma < 2^{-j/2}$ and (ii) $\pi - 2^{-j/2} \geq \gamma \geq 2^{-j/2}$.

Case (i). $\pi - 2^{-j/2} < \gamma$ or $\gamma < 2^{-j/2}$. In this case we note that

$$2^{-j}\,|\langle x, e_0(\xi_0)\rangle| \geq 2^{-j/2}\,|\langle x, e_1(\xi_0)\rangle| \tag{2.3}$$

and

$$2^{-j}\,|\langle x, e_0(\xi)\rangle - \langle x, e_0(\xi_0)\rangle| \leq \frac{1}{100}\, 2^{-j}\,|\langle x, e_0(\xi_0)\rangle|. \tag{2.4}$$

We integrate by parts with respect to the $e_0(\xi)$ direction to obtain

$$K_{j,\nu,0}(x,t) = \int_{\mathbf{R}^1}\int_{\mathbf{R}^n} e^{i\langle x,\xi'\rangle + it\xi_{n+1}}\, D^N_{e_0(\xi)}\left[\frac{m_{j,\nu,0}(\xi',\xi_{n+1})}{(i\langle x, e_0(\xi)\rangle)^N}\right] d\xi'\, d\xi_{n+1}.$$

By the product rule we write

$$D^N_{e_0(\xi)}\,[m_{j,\nu,0}(\xi',\xi_{n+1})/\langle x, e_0(\xi)\rangle^N] = \sum_{k_1+k_2=N} c_{k_1,k_2}\, D^{k_1}_{e_0(\xi)}\, m_{j,\nu,0}(\xi',\xi_{n+1})\, D^{k_2}_{e_0(\xi)}\left(\frac{1}{\langle x, e_0(\xi)\rangle^N}\right).$$

Due to the support of $m_{j,\nu,0}$ and the smoothness of $e_0(\xi)$, we obtain

$$|D^{k_1}_{e_0(\xi)}\, m_{j,\nu,0}(\xi', \xi_{n+1})| \leq C\, 2^{-j\delta}\, 2^{jN},$$

and use (2.4) to obtain

$$\left| D^{k_2}_{e_0(\xi)} \left(\frac{1}{\langle x, e_0(\xi)\rangle^N} \right) \right| \leq C\, \frac{1}{|\langle x, e_0(\xi_0)\rangle|^N}$$

where C is independent of ξ. After we integrate with respect to ξ' and ξ_{n+1}, we therefore have

$$\begin{aligned} \left| K_{j,\nu,0}(x,t) \right| &\leq C\, 2^{-j\delta}\, \frac{2^{-j}}{(1 + 2^{-j}|\langle x, e_0(\xi_0)\rangle|)^N}\; 2^{-j(n-1)/2} \\ &\leq C\, 2^{-j\delta}\, \frac{2^{-j}}{(1 + 2^{-j}|\langle x, e_0(\xi_0)\rangle|)^{N_1}} \\ &\quad \times \frac{2^{-j(n-1)/2}}{(1 + 2^{-j/2}|\langle x, e_1(\xi_0)\rangle|)^{N_2}}, \end{aligned}$$

where the last inequality follows from (2.3).

Case (ii). $\pi - 2^{-j/2} \geq \gamma \geq 2^{-j/2}$. In this case we note

$$2^{-j}\, |\langle x, e_0(\xi_0)\rangle| \leq 2^{-j/2}\, |\langle x, e_1(\xi_0)\rangle| \tag{2.5}$$

and

$$2^{-j/2}\, |\langle x, e_1(\xi)\rangle - \langle x, e_1(\xi_0)\rangle| \leq \frac{1}{100}\, 2^{-j/2}\, |\langle x, e_1(\xi_0)\rangle|.$$

We repeat the same argument as above to obtain

$$|D^{k_1}_{e_1(\xi)}\, m_{j,\nu,0}(\xi', \xi_{n+1})| \leq C\, 2^{-j\delta}\, 2^{jN/2},$$

and

$$\left| D^{k_2}_{e_1(\xi)} \left(\frac{1}{\langle x, e_1(\xi)\rangle^N} \right) \right| \leq C\, \frac{1}{|\langle x, e_1(\xi_0)\rangle|^N}.$$

Thus, after we integrate with respect to ξ' and ξ_{n+1}, we have

$$\begin{aligned}\big|K_{j,\nu,0}(x,t)\big| &\le C\,2^{-j\delta}\,\frac{2^{-j(n-1)/2}}{(1+2^{-j/2}|\langle x,e_1(\xi_0)\rangle|)^N}\;2^{-j}\\ &\le C\,2^{-j\delta}\,\frac{2^{-j}}{(1+2^{-j}|\langle x,e_0(\xi_0)\rangle|)^{N_1}}\\ &\quad\times\frac{2^{-j(n-1)/2}}{(1+2^{-j/2}|\langle x,e_1(\xi_0)\rangle|)^{N_2}},\end{aligned}$$

where the last inequality follows from (2.5). This completes the proof of (2.2).

Now we claim

$$|K_{j,\nu,0}(x,t)| \le C\,2^{-j\delta}\,2^{-j(n+1)/2}\,\frac{1}{(1+|ct+\langle x,\xi_0'\rangle|)^N}. \tag{2.6}$$

We choose an arbitrary $(\xi_0',1)\in\operatorname{supp}\Phi_{j,\nu,0}$ and consider two cases: (i) $|\langle x,\xi'-\xi_0'\rangle| \le (1/100)\,|\langle x,\xi_0'\rangle + t\xi_{n+1}|$, and (ii) $|\langle x,\xi'-\xi_0'\rangle| \ge (1/100)\,|\langle x,\xi_0'\rangle + t\xi_{n+1}|$.

Case (i). $|\langle x,\xi'-\xi_0'\rangle| \le (1/100)\,|\langle x,\xi_0'\rangle + t\xi_{n+1}|$. In this case

$$\frac{99}{100}\,|\langle x,\xi_0'\rangle + t\xi_{n+1}| \le |\langle x,\xi'\rangle + t\xi_{n+1}|. \tag{2.7}$$

We apply integration by parts with respect to the $\tilde e(\xi)$ direction to obtain

$$\begin{aligned}&K_{j,\nu,0}(x,t)\\ &= \int_{\mathbf{R}^1}\int_{\mathbf{R}^n} e^{i\langle x,\xi'\rangle+it\xi_{n+1}}\,D^N_{\tilde e(\xi)}\left[\frac{m_{j,\nu,0}(\xi',\xi_{n+1})|\xi|^N}{(-i\langle x,\xi'\rangle - it\xi_{n+1})^N}\right]d\xi'\,d\xi_{n+1}.\end{aligned}$$

By the product rule we write

$$\begin{aligned}&D^N_{\tilde e(\xi)}\left[\frac{m_{j,\nu,0}(\xi',\xi_{n+1})|\xi|^N}{(-\langle x,\xi'\rangle - t\xi_{n+1})^N}\right]\\ &= \sum_{k_1+k_2=N} c_{k_1,k_2}\,D^{k_1}_{\tilde e(\xi)}\,m_{j,\nu,0}(\xi',\xi_{n+1})\,D^{k_2}_{\tilde e(\xi)}\left(\frac{|\xi|^N}{(-\langle x,\xi'\rangle - t\xi_{n+1})^N}\right).\end{aligned}$$

To treat the first factor we recall Euler's differential equation for homogeneous functions:

$$\rho(\xi') = \sum_{\ell=1}^{n} \xi_\ell \frac{\partial \rho}{\partial \xi_\ell}(\xi').$$

By using this formula we can easily check that

$$D_{\tilde{e}(\xi)}\left[\varphi\left(2^j\left(1-\frac{\rho(\xi')}{|\xi_{n+1}|}\right)\right)\right] = \frac{2^j}{|\xi|}\,\varphi'\left(2^j\left(1-\frac{\rho(\xi')}{|\xi_{n+1}|}\right)\right)\frac{1}{|\xi_{n+1}|}\left[\rho(\xi')-\sum_{\ell=1}^{n}\xi_\ell\frac{\partial\rho}{\partial\xi_\ell}(\xi')\right]=0$$

and

$$D_{\tilde{e}(\xi)}\left[\chi\left(2^{j/2}\left(\frac{\xi'}{\rho(\xi')}-\xi'_\nu\right)\right)\right] = \frac{2^{j/2}}{|\xi|}\sum_{k=1}^{n}\left\{\chi_k\left(2^{j/2}\left(\frac{\xi'}{\rho(\xi')}-\xi'_\nu\right)\right)\xi_k\left[\frac{1}{\rho(\xi')}-\sum_{\ell=1}^{n}\frac{\xi_\ell}{\rho(\xi')^2}\frac{\partial\rho(\xi')}{\partial\xi_\ell}\right]\right\}=0,$$

where χ_k is the partial derivative of χ with respect to the kth variable. By using these two identities we obtain

$$\left|D^{k_1}_{\tilde{e}(\xi)}\, m_{j,\nu,0}(\xi',\xi_{n+1})\right| \le C\,2^{-j\delta}.$$

In view of the support of $m_{j,\nu,0}$ and (2.7), it is easy to see that

$$\left|D^{k_2}_{\tilde{e}(\xi)}\left(\frac{|\xi|^N}{(-\langle x,\xi'\rangle - t\xi_{n+1})^N}\right)\right| \le \frac{C}{|\langle x,\xi'_0\rangle + t\xi_{n+1}|^N}. \tag{2.8}$$

Case (ii). $|\langle x,\xi'-\xi'_0\rangle| \ge (1/100)\,|\langle x,\xi'_0\rangle + t\xi_{n+1}|$. In this case since, by the mean value theorem and the boundedness of the curvature

$$\langle x,\xi'-\xi'_0\rangle = \left\langle x, 2^{-j/2}\sum_{d=1}^{n-1} a_d e_d(\eta)\right\rangle,$$

where $\sum_{d=1}^{n-1} a_d^2 \leq 1$, and

$$(2.9)\qquad \begin{aligned} \frac{1}{100}\,|\langle x, \xi_0'\rangle + t\xi_{n+1}| &\leq |\langle x, \xi' - \xi_0'\rangle| \\ &\leq 2^{-j/2}\left(\sum_{d=1}^{n-1} |\langle x, e_d(\eta)\rangle|^2\right)^{1/2}. \end{aligned}$$

By combining the estimates (2.2) and (2.9), we obtain

$$(2.10)\qquad \begin{aligned} &\big|K_{j,\nu,0}(x,t)\big| \\ &\leq C\,2^{-j(\delta+1)} \int_{1/2}^{2} \frac{2^{-j(n-1)/2}}{\left(1 + 2^{-j/2}\sqrt{\sum_{d=1}^{n-1} |\langle x, e_d(\xi_0)\rangle|^2}\right)^N}\, d\xi_{n+1} \\ &\leq C\,2^{-j\{\delta+(n+1)/2\}} \int_{1/2}^{2} \frac{1}{|\langle x, \xi_0'\rangle + t\xi_{n+1}|^N}\, d\xi_{n+1}. \end{aligned}$$

By applying the integral version of mean value theorem to (2.8) and (2.10), we obtain (2.6).

Moreover, for $\gamma \in \mathbf{N}^{n+1}$ and $|\gamma| = 1, 2, 3, \ldots$, since $m_{j,\nu,0}$ is compactly supported,

$$(K_{j,\nu,0})^{(\gamma)} = \phi^{(\gamma)} * K_{j,\nu,0}$$

for some Schwartz function ϕ, and $(K_{j,\nu,0})^{(\gamma)}$ satisfies the same estimates as $K_{j,\nu,0}$. □

3. Estimates on $H^p(\mathbf{R}^{n+1})$, $0 < p < 1$. We define the characterization of the Hardy spaces via atomic decompositions.

Definition 1. Let $0 < p \leq 1$, and let k be an integer that satisfies $k \geq (n+1)(1/p - 1)$. Let Q be a cube in $\mathbf{R}^{n+1}$. We say that $\mathfrak{a}$ is a (p,k)-atom associated with Q if $\mathfrak{a}$ is supported on $Q \subset \mathbf{R}^{n+1}$ and satisfies

(i) $\|\mathfrak{a}\|_{L^\infty(\mathbf{R}^{n+1})} \leq |Q|^{-1/p}$;

(ii) $\int_{\mathbf{R}^{n+1}} \mathfrak{a}(x)\, x^\beta\, dx = 0$,

where $\beta = (\beta_1, \beta_2, \ldots, \beta_{n+1})$ is an $(n+1)$-tuple of nonnegative integers satisfying $|\beta| \leq \beta_1 + \beta_2 + \cdots + \beta_{n+1} \leq k$, and $x^\beta = x^{\beta_1} x^{\beta_2} \cdots x^{\beta_{n+1}}$.

If $\{\mathfrak{a}_j\}$ is a collection of (p,k)-atoms and $\{c_j\}$ is a sequence of complex numbers with $\sum_{j=1}^{\infty}|c_j|^p<\infty$, then the series $f=\sum_{j=1}^{\infty}c_j\mathfrak{a}_j$ converges in the sense of distributions, and its sum belongs to H^p with the quasi-norm

$$\|f\|_{H^p(\mathbf{R}^{n+1})}=\inf_{\sum_{j=1}^{\infty}c_j\mathfrak{a}_j=f}\left(\sum_{j=1}^{\infty}|c_j|^p\right)^{1/p},$$

see [**9**].

Now we shall prove the uniform estimates $T^\delta f$ when f is a (p,N)-atom with $N\geq(n+1)(1/p-1)$ defined on $\mathbf{R}^{n+1}$.

We denote $T^\delta_{j,\nu,l}f=K_{j,\nu,l}*f$ and $T^\delta_l=\sum_j\sum_\nu T^\delta_{j,\nu,l}$.

Proposition 1. *Suppose that* $\delta>n(1/p-1/2)-1/2$ *for* $0<p<1$ *and* f *is a* (p,N)*-atom,* $N\geq(n+1)(1/p-1)$*, on* $\mathbf{R}^{n+1}$*. Then the inequality*

$$\|T^\delta f\|_{L^p(\mathbf{R}^{n+1})}\leq C$$

holds where C *is independent of* f*.*

Proof. By translation invariance, we may assume that Q with diameter 1 is centered at the origin and write

$$\begin{aligned}\int_{\mathbf{R}^1}\int_{\mathbf{R}^n}&|T^\delta f(x,t)|^p\,dx\,dt\\&=\iint_{Q^*}|T^\delta f(x,t)|^p\,dx\,dt+\iint_{(Q^*)^c}|T^\delta f(x,t)|^p\,dx\,dt\\&=I+J.\end{aligned}$$

For I we take q such that $p/2+1/q=1$ and apply Plancherel's theorem and Hölder's inequality to obtain

$$I\leq C\,\|T^\delta f\|^p_{L^2(\mathbf{R}^{n+1})}\,|Q^*|^{1/q}\leq C\,\|f\|_2^p\,|Q^*|^{1/q}\leq C$$

where Q^* is a cube centered at the origin with diameter C_0 which will be chosen to be large enough that the following arguments hold. For J

we write

$$J \le \sum_{l=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2^{j(n-1)/2}} \|T^\delta_{j,\nu,l} f\|^p_{L^p}.$$

Since the domain of the integration J is $(Q^*)^c$, we may assume that $|x| + |t| \ge C_0$ when we treat J. We first consider the case where $l > 0$. We have

$$\begin{aligned} |T^\delta_{j,\nu,l} f(x,t)| &= 2^{l(n+1)} \iint_Q |K_{j,\nu,0}(2^l(x-y), 2^l(t-s))||f(y,s)|\, dy\, ds \\ &\le C\, 2^{-j\delta}\, 2^{l(n+1)} \iint_Q \frac{2^{-j}}{\Big(1 + 2^{-j} 2^l |\langle x-y, e_0(\xi_0)\rangle|\Big)^N} \\ &\quad \times \frac{2^{-j(n-1)/2}}{\Big(1 + 2^{-j/2} 2^l \sqrt{\sum_{d=1}^{n-1} |\langle x-y, e_d(\xi_0)\rangle|^2}\Big)^N} \\ &\quad \times \frac{1}{(1 + 2^l |c(t-s) + \langle x-y, \xi_0'\rangle|)^N}\, dy\, ds. \end{aligned}$$

Since $|x| + |t| \ge C_0$ and $\{e_0(\xi_0), \dots, e_{n-1}(\xi_0), \xi_0\}$ is linearly independent,

$$|\langle x, e_0(\xi_0)\rangle| > 2, \qquad \left(\sum_{d=1}^{n-1} |\langle x, e_d(\xi_0)\rangle|^2\right)^{1/2} > 2,$$

or

$$|ct + \langle x, \xi_0'\rangle| > 2.$$

We shall only treat the case where $|\langle x, e_0(\xi_0)\rangle| > 2$. Arguments for other cases are similar so we leave the remaining cases to interested readers. In this case, since $|\langle y, e_0(\xi_0)\rangle| \le 1$,

$$|\langle x-y, e_0(\xi_0)\rangle| \ge \frac{1}{2} |\langle x, e_0(\xi_0)\rangle|.$$

We therefore obtain

$$\begin{aligned} |T^\delta_{j,\nu,l} f(x,t)| &\le C\, 2^{-j\delta}\, 2^{l(n+1)} \frac{2^{-j}}{(1 + 2^{-j} 2^l |\langle x, e_0(\xi_0)\rangle|)^N} \\ &\quad \times \iint_Q \frac{2^{-j(n-1)/2}}{\Big(1 + 2^{-j/2} 2^l \sqrt{\sum_{d=1}^{n-1} |\langle x-y, e_d(\xi_0)\rangle|^2}\Big)^N} \\ &\quad \times \frac{1}{(1 + 2^l |c(t-s) + \langle x-y, \xi_0'\rangle|)^N}\, dy\, ds. \end{aligned}$$

By using a change of variables we obtain

$$\begin{aligned}|T^{\delta}_{j,\nu,l}f(x,t)| &\\ &\le \frac{C\,2^{-j\delta}\,2^{l(n+1)}\,2^{-j}}{(1+2^{-j}2^{l}|\langle x,e_0(\xi_0)\rangle|)^N}\,2^{-j(n-1)/2}\,(2^{-j/2}2^{l})^{-n+1}\,(2^{l})^{-1}\\ &= C\,2^{-j\delta}\,2^{l}\,\frac{2^{-j}}{(1+2^{-j}2^{l}|\langle x,e_0(\xi_0)\rangle|)^N}.\end{aligned}$$

Hence,

$$\iint_{\mathbf{R}^{n+1}}|T^{\delta}_{j,\nu,l}f(x,t)|^p\,dx\,dt \le C\,2^{l(p-1)}\,2^{-j\{\delta p+((n+1)p/2)-(n+1/2)\}}.$$

Since $\delta > n(1/p-1/2)-1/2$, for $l>0$ we obtain

$$\begin{aligned}\|T^{\delta}_{l}f\|^p_{L^p} &\le C\,2^{l(p-1)}\sum_{j=1}^{\infty}\sum_{\nu=1}^{2^{j(n-1)/2}}2^{-j\{\delta p+((n+1)p/2)-(n+1/2)\}}\\ &\le C\,2^{l(p-1)}.\end{aligned}\tag{3.1}$$

Now we fix $l \le 0$. Let $P_{j,\nu,l}(y,s)$ denote the Nth order Taylor polynomial of the function $(y,s)\to K_{j,\nu,l}(x-y,t-s)$ expanded about the origin, the center of the cube. Now $P_{j,\nu,l}=2^{l(n+1)}P_{j,\nu,0}(2^{l}\cdot,2^{l}\cdot)$ for fixed j,ν and l. Then, using the moment conditions on f,

$$\begin{aligned}T^{\delta}_{j,\nu,l}f(x,t) &= \iint_Q f(y,s)\,2^{l(n+1)}\,K_{j,\nu,0}(2^{l}(x-y),2^{l}(t-s))\,dy\,ds\\ &= \iint_Q f(y,s)\,2^{l(n+1)}\\ &\quad\times[K_{j,\nu,0}(2^{l}(x-y),2^{l}(t-s))-P_{j,\nu,0}(2^{l}y,2^{l}s)]\,dy\,ds.\end{aligned}$$

A straightforward calculation shows that the absolute value of the last term is dominated by

$$C\iint_Q|f(y,s)|\,2^{l(n+1)}\times\sum_{|\gamma|=N+1}|K^{(\gamma)}_{j,\nu,0}(2^{l}(x-y),2^{l}(t-s))|\,|2^{l}(y,s)|^{N+1}\,dy\,ds.$$

We apply Lemma 1, (2.1), and the same arguments as above, and thus obtain

$$\begin{aligned}|T^{\delta}_{j,\nu,l}f(x,t)| &= 2^{l(n+2+N)}\iint_Q \sum_{|\gamma|=N+1} \\ &\quad\times |K^{(\gamma)}_{j,\nu,0}(2^l(x-y),2^l(t-s))||f(y,s)|\,dy\,ds \\ &\le C\,2^{-j\delta}\,2^{l(n+2+N)}\iint_Q \frac{2^{-j}}{(1+2^{-j}2^l|\langle (x-y),e_0(\xi_0)\rangle|)^N} \\ &\quad\times \frac{2^{-j(n-1)/2}}{\Big(1+2^{-j/2}2^l\sqrt{\sum_{d=1}^{n-1}|\langle (x-y),e_d(\xi_0)\rangle|^2}\Big)^N} \\ &\quad\times \frac{1}{(1+2^l|c(t-s)+\langle (x-y),\xi_0'\rangle|)^N}\,dy\,ds.\end{aligned}$$

We use the same arguments as above and use the assumption $\delta > n(1/p-1/2)-1/2$ to obtain

$$\begin{aligned}\|T^{\delta}_l f\|^p_{L^p} &\le C\,2^{\{l(n+2+N)p-(n+1)\}} \\ &\quad\times \sum_{j=1}^{\infty}\sum_{\nu=1}^{2^{j(n-1)/2}} 2^{-j\{\delta p+((n+1)p/2)-(n+1/2)\}} \\ &\le C\,2^{\{l(n+2+N)p-(n+1)\}}.\end{aligned}\tag{3.2}$$

We combine (3.1) and (3.2) and use the assumption $N \ge (n+1)((1/p)-1)$ to obtain

$$\begin{aligned}\|T^{\delta}f\|^p_{L^p} &\le \sum_{l=-\infty}^{\infty}\|T^{\delta}_l f\|^p_{L^p} \\ &\le C\sum_{l=-\infty}^{0}2^{\{l(n+2+N)p-(n+1)\}} + C\sum_{l=1}^{\infty}2^{l(p-1)} \le C.\end{aligned}$$

Now we suppose that f is a (p,N)-atom $(N \ge (n+1)((1/p)-1))$, supported in a cube Q of diameter 2^R centered at (x_Q,t_Q). By translation invariance, we can assume $(x_Q,t_Q)=(0,0)$. Let $h(x,t) = 2^{R(n+1)/p}f(2^Rx,2^Rt)$. Then h is an atom supported in the cube of diameter 1 centered at $(0,0)$ and we write

$$\begin{aligned}T^\delta f(x,t) &= \int_{r^1}\int_{\mathbf{R}^n} 2^{-R(n+1)/p}\, h(2^{-R}(x-y), 2^{-R}(t-s))K_l(y,s)\,dy\,ds \\ &= 2^{-R(n+1)/p}\; T^\delta_{l+R}\; h(2^{-R}x, 2^{-R}t),\end{aligned}$$

which implies

$$T^\delta f(x,t) = 2^{-R(n+1)/p} \sum_{l+R} T^\delta_{l+R}\, h(2^{-R}x, 2^{-R}t).$$

We therefore have

$$\begin{aligned}\|T^\delta f\|^p_{L^p} &= \Big\| 2^{-R(n+1)/p} \sum_{l+R} T^\delta_{l+R}\, h(2^{-R}\cdot, 2^{-R}\cdot)\Big\|^p_{L^p} \\ &\le C\,(2^{R(n+1)/p})^{-p}\, 2^{R(n+1)}.\end{aligned}$$

This completes the proof. □

Therefore, from Definition 1 for H^p spaces and Proposition 1, one can easily see that Theorem 1 holds.

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