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CONVEX POLYTOPES AND FACTORIZATION PROPERTIES IN GENERALIZED POWER SERIES DOMAINS

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ABSTRACT. It is shown how to associate to any polytope that is not a simplex and any field K, a commutative integral domain D which has no irreducible elements and which is not pre-Schreier. The integral domain D is a generalized power series ring over K.

Let R be an integral domain with quotient field K. Recall that $a \in R \setminus \{0\}$ is said to be *irreducible*, or an *atom*, if a is not the product of two nonunits of R, and that a is said to be *prime* if, for all $b, c \in R$, $a \mid bc$ implies $a \mid b$ or $a \mid c$. It is easy to show that any prime element is irreducible, and much research has been done into the question of when the converse is true.

For example, in any pre-Schreier domain, all irreducible elements are prime, and so we recall the definition: An element a of an integral domain R is *primal* if, whenever a divides bc with b and c in R, then $a = b'c'$ for some $b', c' \in R$ where b' divides b and c' divides c. An integral domain in which each element is primal is said to be *pre-Schreier*. (A *Schreier* domain is a pre-Schreier domain which is also integrally closed.) Such rings have been studied by many authors, for example, [**6, 7, 9, 13, 18, 22**].

It is immediate that, in a pre-Schreier domain, each irreducible element is prime. On the other hand, there exist examples of integral domains which are not pre-Schreier, but in which each irreducible element is prime, see [**18**, Example 3.7]. (See also [**1**] for a comparison of these properties and several related ones.)

In [**21**], Waterhouse shows that, if each quadratic polynomial $f \in R[X]$ factors into linear polynomials in $R[X]$ whenever it factors into linear polynomials in $K[X]$, then every irreducible element in R is prime. The relation between this result and the pre-Schreier condition was explored in [**18**] where it is shown that R is pre-Schreier if and

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only if each quadratic polynomial $f \in R[X]$ of certain type factors into linear polynomials in $R[X]$ whenever it factors into linear polynomials in $K[X]$, see [**18**, Theorem 1.2]. Since this last result holds even for domains without irreducible elements, for which Waterhouse's result says nothing, this leads to the question of whether domains without irreducible elements are necessarily pre-Schreier. Integral domains having no irreducible elements are considered in [**8**], where they are called *antimatter domains*. Several constructions of antimatter domains are given in [**8**], but it is not evident that these rings can fail to be pre-Schreier. Indeed those produced by the Krull-Jaffard-Kaplansky-Ohm theorem are Bezout, and thus pre-Schreier. In [**4**] the authors showed how to produce examples of antimatter domains which are not pre-Schreier by reducing the construction of such domains, via generalized power series rings, to the construction of torsion-free monoids with the analogous properties. This preprint [**4**] was posted on the first author's web site for a time and is referenced in [**1**], but it was never submitted for publication.

The purpose of this note, which supersedes [**4**], is to expand on our previous preprint by using ideas from [**11**] to associate a monoid $M = \mathcal{M}_s(C)$ to a convex polytope $C \subseteq \mathbf{R}^n$ such that $R = K[[M, \leq]]$ is an antimatter domain and is pre-Schreier only if C is a simplex. (The example given in [**4**] is the case that C is a square in $\mathbf{R}^2$, which is mentioned here as Example 3.2.) Questions on factorization properties in generalized power series have been studied for some time in different contexts. For example, see [**2, 12, 14, 17, 23**] and the references listed there. Also, although polytopes often occur in Noetherian commutative ring theory, (see, for example, [**19, 20**]), their occurrence is much less frequent in non-Noetherian commutative ring theory. Therefore, although it is now known that antimatter domains which are not pre-Schreier can also be obtained by a pull-back construction [**1**], we think our construction still holds some interest.

We review in Section 1 the definition and basic properties of generalized power series rings and show that, if M is a conical cancellative torsion-free monoid, $\leq$ is the natural preorder on M, K is a field, and $R = K[[M, \leq]]$ is the generalized power series ring, then

(1) R is an antimatter domain, if $M^* = M \setminus \{0\}$ is strictly downward directed (that is, for each $x, y \in M^*$ there exists $z \in M^*$ with $z < x$ and $z < y$), and

(2) R is not pre-Schreier, if M does not have decomposition. (M has decomposition means that for all x, $y_1, y_2 \in M$ such that $x \leq y_1 + y_2$, there are $z_1, z_2 \in M$ such that $x = z_1 + z_2$ and $z_1 \leq y_1$ and $z_2 \leq y_2$.)

Thus, our ring question is reduced to finding a strictly downward directed conical monoid which does not have decomposition. In Section 2, using ideas from [11], we associate a strictly downward directed conical monoid $\mathcal{M}_s(C)$, to a convex polytope $C \subseteq \mathbf{R}^n$ and show that $\mathcal{M}_s(C)$ has decomposition if and only if C is a simplex. In Section 3, we put the above results together to get generalized power series rings $K[[M, \leq]]$ which are antimatter domains but are not pre-Schreier.

1. Generalized power series rings. In this section we define generalized power series rings and discuss when such rings are antimatter domains or pre-Schreier domains.

Let $(M, \leq)$ be a *strictly ordered monoid*; that is, M is a commutative monoid and $\leq$ is a partial order on M such that $x < y$ implies $x + z < y + z$ for all $x, y, z \in M$. A subset N of M is said to be *narrow* if each subset of N consisting of pairwise order-incomparable elements in the $\leq$ order is finite.

Let R be a commutative ring. For a function $f : M \to R$ the *support of* f is defined as $\operatorname{supp}(f) = \{x \in M \mid f(x) \neq 0\}$. Then the *generalized power series ring* $R[[M, \leq]]$ is the set of all such functions whose support is Artinian and narrow in the $\leq$ partial ordering. Addition is defined by $(f + g)(x) = f(x) + g(x)$ and multiplication by $(fg)(x) = \sum_{x_1+x_2=x} f(x_1)g(x_2)$ for $x \in M$. Since M is strictly ordered, the sum is in fact finite. See [**16**] for the details of this construction.

For $f \in R[[M, \leq]]$, we will write $\operatorname{minsupp}(f)$ for the (finite) set of minimal elements in the support of f. For $x \in M$, we write X^x for the function such that $\operatorname{supp}(X^x) = \{x\}$ and $X^x(x) = 1$. Thus $X^x X^y = X^{x+y}$ for all $x, y \in M$.

Any monoid M has a preorder, defined by $x \leq y$ if $x + z = y$ for some $z \in M$, which we call the *natural preorder*. In general, $x \leq y \leq x$ does not imply $x = y$, so the natural preorder is not always a partial order on M.

A monoid M is *conical* if $x + y = 0$ in M implies $x = y = 0$. It is easy to show that if $\leq$ is the natural preorder on a conical cancellative monoid, then $(M, \leq)$ is a strictly ordered monoid.

Proposition 1.1. *Let K be a field. Let M be a conical cancellative torsion-free monoid and $\leq$ the natural preorder on M. Then the following hold*:

(1) *$K[[M,\leq]]$ is a domain.*

(2) *$f \in K[[M,\leq]]$ is a unit if and only if $f(0) \neq 0$.*

(3) *There is a total order $\preceq$ on M such that $a \leq b$ implies $a \preceq b$ for all a, b in M. For $0 \neq f \in K[[M,\leq]]$, define* $\deg(f)$ *to be the least element in* $\operatorname{supp}(f)$ *in the $\preceq$ order. Then for all nonzero $f, g \in K[[M,\leq]]$ we have* $\deg(fg) = \deg(f) + \deg(g)$. *In particular, if $f \mid g$, then* $\deg(f) \leq \deg(g)$.

Proof. (1) Since M is cancellative and torsion-free, this follows from [**16**, 1.20].

(2) Since $0 \leq x$ for all $x \in M$, this follows from [**16**, 2.3].

(3) Since M is cancellative and torsion-free, the existence of $\preceq$ comes from [**16**, 1.10]. If $A \subseteq M$ is Artinian and narrow in the order $\leq$, then A is Artinian and narrow in the order $\preceq$ [**16**, 1.7]. Thus, $\deg(f)$ is defined for $f \in K[[M,\leq]]$. The properties of $\deg(f)$ are in [**15**, 4.2]. □

Proposition 1.2. *Let K and $(M,\leq)$ be as in Proposition* 1.1. *Suppose that $M^* = M \setminus \{0\}$ is strictly downward directed*; *that is, for each x, $y \in M^*$, there exists $z \in M^*$ with $z < x$ and $z < y$. Then $K[[M,\leq]]$ is an antimatter domain.*

Proof. We must show that all nonzero nonunits of $K[[M,\leq]]$ are reducible. If $0 \neq f \in K[[M,\leq]]$ is not a unit, then $f(0) = 0$ by Proposition 1.1; that is, $\operatorname{minsupp}(f) \subseteq M^*$. Since M^* is strictly downward directed, there exists $y \in M^*$ such that $y < x$ for each $x \in \operatorname{minsupp}(f)$. In fact, the same is true of all elements of $\operatorname{supp}(f)$ since each is greater than an element of $\operatorname{minsupp}(f)$; that is, for each $x \in \operatorname{supp}(f)$, there is $x' \neq 0$ such that $x = y + x'$.

Define $g : M \to K$ by $g(x) = f(x+y)$ for $x \in M$. Since M is cancellative, the map $x' \mapsto y + x'$ is an order isomorphism from $\operatorname{supp}(g)$ to $\operatorname{supp}(f)$. Since $\operatorname{supp}(f)$ is narrow and Artinian, so is $\operatorname{supp}(g)$ and hence $g \in K[[M,\leq]]$. Moreover, $g(0) = f(y) = 0$ since $y \notin \operatorname{supp}(f)$, and so g is not a unit. Since $y \neq 0$, X^y is not a unit either. It is easy to check that $f = X^y g$, and so we have shown that f is reducible. □

To discuss the pre-Schreier property in generalized power series rings, we need to define the corresponding monoid property: A partially ordered monoid $(M, \leq)$ has *decomposition* (or is a *decomposition* monoid) if, for all $x, y_1, y_2 \in M$ such that $x \leq y_1 + y_2$, there are $z_1, z_2 \in M$ such that $x = z_1 + z_2$ and $z_1 \leq y_1$ and $z_2 \leq y_2$ (see, for example, [**3, 10**], [**11**, Proposition 2.1]). Notice that a domain R is pre-Schreier if and only if the multiplicative monoid $R^* = R \setminus \{0\}$ has decomposition, see [**5**].

Proposition 1.3. *Let K and $(M, \leq)$ be as in Proposition* 1.1. *If $K[[M, \leq]]$ is pre-Schreier, then M has decomposition.*

Proof. Suppose $x \leq y_1 + y_2$ in M. Then $x + z = y_1 + y_2$ for some $z \in M$ and so $X^x X^z = X^{y_1} X^{y_2}$. In particular, $X^x | X^{y_1} X^{y_2}$. Since $K[[M, \leq]]$ is pre-Schreier, there are $f_1, f_2 \in K[[M, \leq]]$ such that $X^x = f_1 f_2$ with $f_1 | X^{y_1}$ and $f_2 \mid X^{y_2}$ in $K[[M, \leq]]$. The functions f_1 and f_2 must be nonzero since $X^x \neq 0$. So we can set $z_1 = \deg(f_1)$ and $z_2 = \deg(f_2)$. Then, using Proposition 1.1 (3), we get $x = \deg(X^x) = \deg(f_1) + \deg(f_2) = z_1 + z_2$ and $z_1 = \deg(f_1) \leq \deg(X^{y_1}) = y_1$ and similarly $z_2 \leq y_2$. □

2. Monoids and simplices. Let $\mathbf{R}$ be the set of real numbers and $\mathbf{R}^+ = \{a \in \mathbf{R} \mid 0 \leq a\}$. Let V be an $\mathbf{R}$-vector space. If $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in \mathbf{R}^+$ are such that $\sum_i a_i = 1$, then $\sum_i a_i x_i$ is called a *convex combination* of the elements $x_1, x_2, \ldots, x_n$. A subset C of V is *convex* if it is closed under convex combinations of finite subsets of C. Since V is convex, and any intersection of convex subsets is convex, any subset X of V is contained in a smallest convex subset, its *convex hull*, written $\langle X \rangle$. A *polytope* is the convex hull of a finite subset X of V.

We use the following notation and terminology from [**11**]. A subset $X \subseteq V$ is said to be *affinely dependent* if $x_0 = \sum_{i=1}^n a_i x_i$ for some $x_0, x_1, \ldots, x_n \in X$, $a_1, \ldots, a_n \in \mathbf{R}$ with $\sum_{i=1}^n a_i = 1$. Otherwise, X is said to be *affinely independent.* A (*classical*) *simplex* in a real vector space V is a convex subset of V that is the convex hull of a finite set of affinely independent points of V. If C_1 and C_2 are convex subsets of the real vector spaces V_1 and V_2 respectively, a map $f : C_1 \to C_2$ is said to be *affine* if f preserves convex combinations.

Let $(X, \leq)$ be a partially ordered set, let Y be an arbitrary set and denote by X^Y the set of functions from Y to X. The *pointwise ordering* $\leq$ on the set X^Y is defined by $f \leq g$ if $f(y) \leq g(y)$ for each $y \in Y$. We also define $f \ll g$ if and only if $f(y) < g(y)$ for each $y \in Y$. The corresponding partial ordering $\lll$ on X^Y is called the *strict ordering*. That is, $f \lll g$ if and only if either $f \ll g$ or $f = g$.

If C is a compact convex set in a real vector space V, we denote by $\mathrm{Aff}\,(C)$ the group of all affine continuous real-valued functions with pointwise addition. Then $\mathrm{Aff}\,(C)$ is an ordered group under the pointwise ordering, and also under the strict ordering. In this section we give an exposition of some needed results on the ordered groups $(\mathrm{Aff}\,(C), \leq)$ and $(\mathrm{Aff}\,(C), \lll)$.

For a convex subset C of the vector space V, let $\mathcal{M}(C)$ and $\mathcal{M}_s(C)$ be the positive cones of $(\mathrm{Aff}\,(C), \leq)$ and $(\mathrm{Aff}\,(C), \lll)$ respectively. That is, $\mathcal{M}(C) = \{f \in \mathrm{Aff}\,(C) \mid f \geq 0\}$ and $\mathcal{M}_s(C) = \{f \in \mathrm{Aff}\,(C) \mid f \gg 0\} \cup \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the zero function in $\mathrm{Aff}\,(C)$. The natural preorders on these monoids are $\leq$ and $\lll$, respectively. Since $\mathcal{M}(C)$ and $\mathcal{M}_s(C)$ are obviously conical, they are strictly ordered monoids under these orderings.

Lemma 2.1. *Let $C = \langle X \rangle$ for some $X \subseteq V$. Then the map $\Phi : \mathcal{M}(C) \to (\mathbf{R}^+)^X$ defined by $\Phi(f) = (f(x))_{x \in X}$, is an injective monoid homomorphism. Moreover, $f \leq g$ if and only if $\Phi(f) \leq \Phi(g)$ and, if f, $g \in \mathcal{M}_s(C)$, then $f \ll g$ if and only if $\Phi(f) \ll \Phi(g)$.*

Proof. This is a special case of [**11**, Theorem 5.20]. □

Suppose again that $C = \langle X \rangle$ for some $X \subseteq V$. For $f \in \mathcal{M}(C)$, define the *support of* f by $\mathrm{supp}_X(f) = \{x \in X \mid f(x) > 0\}$, and let $\Gamma(C) = \{\mathrm{supp}_X(f) \mid f \in \mathcal{M}(C)\}$. Of course, $\varnothing = \mathrm{supp}_X(0) \in \Gamma(C)$, and, if $A = \mathrm{supp}_X(f)$ and $B = \mathrm{supp}_X(g)$ for $f, g \in \mathcal{M}(C)$, then

$$A \cup B = \mathrm{supp}_X(f + g) \in \Gamma(C).$$

Thus $(\Gamma(C), \cup, \varnothing)$ is a monoid.

An element $x \in C$ is an *extreme point* of C if $\langle C \setminus \{x\} \rangle \neq C$, or equivalently x is not a convex combination of other elements of C. Obviously, if $C = \langle X \rangle$, then all extreme points of C must be in X, and any finite subset of nonextreme points can be removed from X without changing its convex hull.

Lemma 2.2. *If $C = \langle X \rangle$ for some $X \subseteq V$, then $x \in X$ is an extreme point if and only if $X \setminus \{x\} \in \Gamma(C)$.*

Proof. If $x \in X$ is an extreme point, then, by [**11**, Theorem 5.14], there is some $f \in \mathcal{M}(C)$ such that $f^{-1}(0) = \{x\}$; that is, $\operatorname{supp}_X(f) = X \setminus \{x\}$. Conversely, if $\operatorname{supp}_X(f) = X \setminus \{x\} \in \Gamma(C)$, $f \in \mathcal{M}(C)$, and x is not an extreme point of X, then $x = \sum_{y \in X \setminus \{x\}} a_y y$ with $\sum_{y \in X \setminus \{x\}} a_y = 1$. But since $\operatorname{supp}_X(f) = X \setminus \{x\}$, $0 = f(x) = \sum_{y \in X \setminus \{x\}} a_y f(y)$. But this is impossible since $f(y) > 0$ for all $y \in X \setminus \{x\}$, $a_y \geq 0$ for all $y \in X$ and $\sum_{y \in X \setminus \{x\}} a_y = 1$. □

Proposition 2.3. *Let X be a finite subset of V, let $C = \langle X \rangle$, and assume that all elements of X are extreme points of C. The following are equivalent*:

(1) *C is a simplex.*

(2) *$\mathcal{M}(C) \cong ((\mathbf{R}^+)^n, \leq)$ for some $n \in \mathbf{N}$.*

(3) *$\mathcal{M}(C)$ has decomposition.*

(3′) *$\mathcal{M}_s(C)$ has decomposition.*

(4) *$\Gamma(C)$ has decomposition.*

(5) *$\{x\} \in \Gamma(C)$ for each $x \in X$.*

(6) *X is independent.*

Proof. (1) $\Rightarrow$ (2). Let Y be a finite independent subset of V such that $C = \langle Y \rangle$. It suffices to show that the homomorphism $\Phi : \mathcal{M}(C) \to (\mathbf{R}^+)^Y$ from Lemma 2.1 is surjective. Given an element $(r_x)_{x \in Y} \in (\mathbf{R}^+)^Y$, we can define a function $f : C \to \mathbf{R}^+$ by $f(y) = \sum_{x \in Y} a_x r_x$ where $y = \sum_{x \in Y} a_x x$ is the unique representation of y as a convex combination of elements of Y. It is easy to check that $f \in \mathcal{M}(C)$ and that $\Phi(f) = (r_x)_{x \in Y}$. Thus, Φ is surjective.

(2) $\Rightarrow$ (3). Since $(\mathbf{R}^+)^n$ has decomposition, this is clear.

(3) $\Rightarrow$ (4). Let $A \subseteq B_1 \cup B_2$ in $\Gamma(C)$, where $A = \mathrm{supp}_X(f)$, $B_1 = \mathrm{supp}_X(g_1)$ and $B_2 = \mathrm{supp}_X(g_2)$ for $f, g_1, g_2 \in \mathcal{M}(C)$. To show that there are $A_1, A_2 \in \Gamma(C)$ such that $A_1 \subseteq B_1$, $A_2 \subseteq B_2$ and $A = A_1 \cup A_2$, observe that for all $x \in \mathrm{supp}_X(f)$ we have $g_1(x) + g_2(x) > 0$. Let

$$r = \max\{f(x)/(g_1(x) + g_2(x)) \mid x \in \mathrm{supp}_X(f)\}.$$

Then, by Lemma 2.1, $f \leq r(g_1 + g_2)$. Since $\mathcal{M}(C)$ has decomposition by hypothesis, $f = f_1 + f_2$ with $f_i \in \mathcal{M}(C)$ and $f_i \leq r g_i$ for $i = 1, 2$. Let $A_1 = \mathrm{supp}_X(f_1)$ and $A_2 = \mathrm{supp}_X(f_2)$. Then the claims are immediate.

(4) $\Rightarrow$ (5). Let Y be minimal among elements of $\Gamma(C)$ which contain x. We will show that $Y = \{x\}$. Suppose to the contrary that $x \neq y \in Y$. By Lemma 2.2, $A = X \setminus \{x\}$ and $B = X \setminus \{y\}$ are in $\Gamma(C)$. Since $B \subseteq A \cup Y = X$, there are $B_1, B_2 \in \Gamma(C)$ such that $B_1 \subseteq A$, $B_2 \subseteq Y$ and $B = B_1 \cup B_2$. Since $x \in B$ but $x \notin A$, we must have $x \in B_2$. Since $B_2 \subseteq Y$, the minimality of Y then implies $Y = B_2$. In particular, $y \in Y = B_2 \subseteq B$. This contradicts the definition of B.

(5) $\Rightarrow$ (6). By assumption, for each $x \in X$ there is $f_x \in \mathcal{M}(C)$ such that $\mathrm{supp}_X(f_x) = \{x\}$. If $z = \sum_{x \in X} a_x x \in C$, then for each $x \in X$, $f_x(z) = a_x f_x(x)$. Thus, a_x is uniquely determined by the equation $a_x = f_x(z)/f_x(x)$.

(6) $\Rightarrow$ (1). Definition.

It remains to show that $(3')$ is equivalent to the other properties. This equivalence is a special case of a general result in [**11**] which we discuss next. □

In [**11**, Theorem 11.4], a general relationship between Choquet simplices and interpolation groups is proved. For the definitions of these concepts, see [**11**, Chapters 2, 10]. All we need about interpolation groups is that a partially ordered group $(G, \leq)$ is an interpolation group if and only the positive cone G^+ of G has decomposition. (See [**11**, pages 22–23], especially [**11**, Proposition 2.1].) The only fact that we require about Choquet simplices is that, if C is a subset of a finite dimensional vector space V, then C is a Choquet simplex if and only if C is a classical simplex [**11**, Theorem 10.16]. The equivalence (3) $\Leftrightarrow$ $(3')$

of Proposition 2.3 now follows from the following theorem. (Actually we only need the implication $(3') \Rightarrow (3)$ in what follows, and this only uses $(c) \Rightarrow (b)$ of Theorem 2.4, for which the proof given in [**11**] is elementary and self contained.)

Theorem 2.4 [**11**, Theorem 11.4]. *Let C be a compact convex subset of a locally convex Hausdorff topological real vector space. The following are equivalent*:

(a) *C is a Choquet simplex.*

(b) $(\mathrm{Aff}\,(C), \leq)$ *is an interpolation group.*

(c) $(\mathrm{Aff}\,(C), \leqslant)$ *is an interpolation group.*

3. Examples. The following result gives a method of obtaining generalized power series antimatter domains which are not pre-Schreier.

Theorem 3.1. *Let C be a convex polytope that is not a simplex, let $M = \mathcal{M}_s(C)$, and let K be a field. Then $K[[M, \leqslant]]$ is an antimatter domain that is not pre-Schreier.*

Proof. To see that $M^* = M \setminus \{0\} = \{f \in \mathrm{Aff}\,(C) \mid f \gg 0\}$ is strictly downward directed, we begin by letting X be the set of extreme points of C. Let $\Phi : (\mathcal{M}(C)) \to (\mathbf{R}^+)^X$ be the restriction map as defined in Lemma 2.1. Now if f, $g \in M^*$, let $w = \min\{f(x), g(x) \mid x \in X\}$. The constant function $w/2$ is in $\mathcal{M}_s(C)$ and it follows by Lemma 2.1 that $w/2 \ll f$ and $w/2 \ll g$. Thus, $M^* = M \setminus \{0\}$ is strictly downward directed. Therefore, by Proposition 1.2, $K[[M, \leqslant]]$ is an antimatter domain. Since C is not a simplex, Proposition 2.3 implies that $\mathcal{M}_s(C)$ does not have decomposition. Therefore by Proposition 1.3, $K[[M, \leqslant]]$ is not pre-Schreier. □

Example 3.2. It is well known that 'most' convex polytopes are not simplices. The easiest example is the unit square $C = \langle X \rangle \subseteq \mathbf{R}^2$ where $X = \{(0,0), (1,0), (0,1), (1,1)\}$. The set X is not independent since the center of the square $(1/2, 1/2)$ does not have a unique representation as a convex combination of the elements of $X : (1/2, 1/2) = 1/2(0,0) +$

$1/2(1,1) = 1/2(1,0) + 1/2(0,1)$. Thus, C is not a simplex, and so $K[[\mathcal{M}_s(C), \leqslant]]$ is an antimatter domain that is not pre-Schreier.

Remark 3.3. Although, as shown in the proof of Theorem 3.1, $\mathcal{M}_s(C)$ is strictly downward directed for any polytope C, this is not true of $\mathcal{M}(C)$. For example, let $C = [0,1]$ be a unit interval in the real line, and let $f, g \in \mathcal{M}(C)$ be such that $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$. Then $f, g \in M^*$, but there is no nonzero h such that $h < f$ and $h < g$. Thus, unlike $\mathcal{M}_s(C)$, $\mathcal{M}(C)$ is not strictly downward directed.

At the expense of making the exposition less accessible, we can give the following generalization of Theorem 3.1, where the finite subset X of V is replaced by a compact subset of an $\mathbf{R}$-vector space V. The set of all extreme points of a compact convex set C is called the *extreme boundary* of C and is denoted $\partial_e C$.

Theorem 3.4. *Let C be a compact convex subset of a locally convex Hausdorff topological real vector space V with $\partial_e C$ compact, let $M = \mathcal{M}_s(C)$ and let K be a field. If C is not a Choquet simplex, then $K[[M, \leqslant]]$ is an antimatter domain that is not pre-Schreier.*

Proof. Again, by Proposition 1.2, $K[[M, \leqslant]]$ is an antimatter domain. Since C is not a Choquet simplex, then by Theorem 2.4 and [**11**, Proposition 2.1], M does not have decomposition. Therefore, by Proposition 1.3, $K[[M, \leqslant]]$ is not pre-Schreier. $\square$

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