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PERTURBATION RESULTS FOR LINEAR OPERATORS AND APPLICATION TO THE TRANSPORT EQUATION

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ABSTRACT. We prove that the components of the Fredholm domains of closed linear operators on Banach spaces remain invariant under additive perturbations belonging to broad classes of perturbing operators. Although our approach is somewhat different than the standard one used to discuss the stability of essential spectra of such operators, our results provide a natural extension of many known ones in the literature and, in particular, of those obtained in the works [**18, 19**]. Of particular interest is the case of polynomially compact operators which furnishes the convenient setting to describe the essential spectra of multi-dimensional neutron transport operators on L_1 spaces which is the topic of the last section.

1. Introduction. Let X and Y be two infinite dimensional complex Banach spaces, and let $\mathcal{C}(X,Y)$, respectively $\mathcal{L}(X,Y)$, denote the set of all closed, densely defined, respectively bounded, linear operators from X into Y. The subset of all compact, respectively finite rank, operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$, respectively $\mathcal{F}_0(X,Y)$. If $A \in \mathcal{C}(X,Y)$, we write $N(A) \subseteq X$ and $R(A) \subseteq Y$ for the null space and range of A. We set $\alpha := \dim N(A)$, $\beta := \operatorname{codim} R(A)$. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed is in } Y\},$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y) : \beta(A) < \infty \quad (\text{and } R(A) \text{ is closed in } Y)\}.$$

Operators in $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ are called semi-Fredholm operators from X on Y while $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denotes the set of Fredholm operators from X on Y. For

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$A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,X)$, $\mathcal{K}(X,X)$, $\mathcal{F}_0(X,X)$, $\mathcal{C}(X,X)$, $\Phi_+(X,X)$, $\Phi_\pm(X,X)$, and $\Phi(X,X)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{F}_0(X)$, $\mathcal{C}(X)$, $\Phi_+(X)$, $\Phi_\pm(X)$ and $\Phi(X)$.

Let $A \in \mathcal{C}(X)$; the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in $\Phi_{+A}, \Phi_{-A}, \Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\pm(X)$ or $\Phi(X)$ respectively. For the properties of these sets, we refer to [**5, 7, 13**].

Recall that when dealing with operators in $\mathcal{C}(X)$ where X is a Banach space, various notions of essential spectrum, generally non equivalent, appear in the applications of spectral theory (see, for instance, [**4, 8, 9, 13**]). Most are enlargement of the continuous spectrum. In this paper we deal with the most familiar of them. They may be ordered as follows

$$\sigma_{e3}(A) := \sigma_{e1}(A) \cap \sigma_{e2}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A),$$

where $\sigma_{ei}(A) = \mathbf{C}\backslash\rho_i(A)$ with $\rho_1(A) := \Phi_{+A}$, $\rho_2(A) := \Phi_{-A}$, $\rho_3(A) := \Phi_{\pm A}$, $\rho_4(A) := \Phi_A$, $\rho_5(A) := \{\lambda \in \rho_4(A), i(\lambda - A) = 0\}$ and $\rho_6(A)$ denotes the set of those $\lambda \in \rho_5(A)$ such that scalars near λ are in $\rho(A)$. The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidmann essential spectra [**9**]. $\sigma_{e3}(.)$ is the Kato essential spectrum [**13**]. $\sigma_{e4}(.)$ is the Fredholm essential (or simply essential) spectrum [**9, 11, 22**]. $\sigma_{e5}(.)$ is the Weyl essential spectrum [**9, 23**], and $\sigma_{e6}(.)$ denotes the Browder essential spectrum [**9, 22**]. Note that all these sets are closed and if X is a Hilbert space and A is self-adjoint, then all these sets coincide (see, for example, [**11**]).

One of the interesting problems in the study of essential spectra of linear operators on Banach spaces is the invariance of the different essential spectrums under (additive) perturbations. The mathematical literature devoted to this subject is considerable, we refer, for example, to the works [**4, 9, 11, 15, 17, 18, 19, 22, 23**] and the references therein. Motivated by a fine description of the spectrum of the transport operator, the behavior of essential spectra of operators in $\mathcal{C}(X)$ subjected to additive perturbations on L_p spaces was discussed in [**17**]. The analysis uses the concept of strictly singular operators which possess some nice properties on these spaces [**25**]. In [**15**] this analysis was extended to operators on Banach spaces which possess the Dunford-Pettis property by means of weakly compact perturbations. In fact, if X has the Dunford-Pettis property, the set of weakly compact operators behaves like that of strictly singular ones on L_p spaces [**21**]. In the work [**18**], the results obtained in [**15, 17**] were extended to general Banach spaces by using the concept of Fredholm perturbations. The case of relatively bounded perturbations was discussed in [**19**]. In particular, it is proved that if $A \in \mathcal{C}(X)$, the various essential spectra of A remain unchanged under additive A-bounded perturbations J (operators J such that $\mathcal{D}(A) \subseteq \mathcal{D}(J)$ and the restriction of J to $\mathcal{D}(A)$ belongs to $\mathcal{L}(\mathcal{D}(A), X)$ where $\mathcal{D}(A)$ is equipped with the graph norm) satisfying $J(\lambda - A)^{-1} \in \mathcal{J}(X)$ where $\lambda \in \rho(A)$ and $\mathcal{J}(X)$ is any proper closed two-sided ideal of $\mathcal{L}(X)$ contained in the ideal of Fredholm perturbation.

Note that in applications, cf. [**2, 4, 11, 13, 14, 16**], $J(\lambda - A)^{-1}$ does not satisfy necessarily the condition $J(\lambda - A)^{-1} \in \mathcal{J}(X)$ for any closed ideal $\mathcal{J}(X)$ of $\mathcal{L}(X)$ contained in $\mathcal{F}(X)$. However, it may happen that $p(J(\lambda - A)^{-1}) \in \mathcal{J}(X)$ where $p(.)$ is a nonzero complex polynomial $p(z) \neq z$, cf. [**15, 20**]. So the approach used in [**14, 16, 17**] does not apply, and the question concerning the stability of the essential spectra of A under such perturbations seems to be open. A typical example, which motivates this work, is provided by the multidimensional neutron transport operator on L_1 spaces, see Section 6. We know from Theorem 4.4 in [**20**] that, under reasonable hypotheses on the collision operator K, $(K(\lambda - T)^{-1})^4$ is compact with $\operatorname{Re} \lambda > s(T)$ where $s(T)$ denotes the spectral bound of T (for the notations, see Section 6). So, for $\lambda \in \rho(T+K) \cap \rho(T)$, $(\lambda - T - K)^{-1} - (\lambda - T)^{-1}$ is not compact nor weakly compact. This example lies outside the scope of the perturbation results developed in [**18, 19**] and therefore the question concerning the determination of the precise picture of the essential spectra of the multi-dimensional neutron transport operator on L_1 spaces remains an open problem, cf. [**14**, Remark 4.3]. Our objective here is to establish some results connected to this problem.

The remainder of this paper is organized as follows. In Section 2 we state the main results of this paper. In Section 3 we establish some perturbation results for both Fredholm and semi-Fredholm operators (Theorem 3.1) and we discuss their incidence on the behavior of essential spectra of operators belonging to $\mathcal{C}(X)$. Some auxiliary results required in the proofs of the results of the second section are established in Section 4. The main result of this section is Theorem 4.1 which provides, in a neighborhood of any point of the Fredholm domain, a construction of particular Fredholm inverses which are uniformly bounded. Section 5 is devoted to the proof of Theorem 2.1. Finally, in the last section we apply Theorem 2.1 to describe the essential spectra of multidimensional neutron transport operators on L_1 spaces.

2. Preliminaries and statement of results. An operator $F \in \mathcal{L}(X)$ is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$. F is called an upper, respectively lower, semi-Fredholm perturbation if $F + U \in \Phi_+(X)$, respectively $\Phi_-(X)$, whenever $U \in \Phi_+(X)$, respectively $\Phi_-(X)$. The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$, respectively.

Let $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$ denote the sets $\Phi(X)\cap\mathcal{L}(X)$, $\Phi_+(X)\cap \mathcal{L}(X)$ and $\Phi_-(X) \cap \mathcal{L}(X)$, respectively. If in the definition of the sets $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ we replace $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ by $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$ we obtain the sets $\mathcal{F}^b(X)$, $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b_-(X)$. These classes of operators were introduced and investigated in [**6**]. In particular, $\mathcal{F}^b_-(X)$, $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b(X)$ are closed two-sided ideals of $\mathcal{L}(X)$ [**23**].

The properties of the sets $\mathcal{F}_-(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}(X)$ were discussed in [**18**, Section 2]. For our own use we recall that $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are closed in $\mathcal{L}(X)$ and $\mathcal{F}(X) = \mathcal{F}^b(X)$. This shows, in particular, that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. It is the largest closed two-sided ideal contained in the set of Riesz operators [**23**, page 222]. Note that the question whether or not $\mathcal{F}_+(X)$, respectively $\mathcal{F}_-(X)$, is equal to $\mathcal{F}^b_+(X)$, respectively $\mathcal{F}^b_-(X)$, seems to be open. It is worth noticing that, in general, the ideal structure of $\mathcal{L}(X)$ is extremely complicated [**1**, Chapter 4]. Most of the results on ideal structure deal with well-known closed ideals which have arisen from applied work with operators. We can quote, for example, compact operators, weakly compact operators $\mathcal{W}(X)$, strictly singular operators $\mathcal{S}(X)$ [**7**], strictly cosingular operators $C\mathcal{S}(X)$ [**21, 24**] and Fredholm perturbations $\mathcal{F}(X)$. On the other hand, Lemma 2.2 in [**18**] and Theorem 2.1

in [**7**, page 117] imply that $\mathcal{K}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{F}_+(X) \subseteq \mathcal{F}(X)$ and $\mathcal{K}(X) \subseteq C\mathcal{S}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{F}(X)$.

Let $A \in \mathcal{C}(X)$. It follows from the closedness of A that $\mathcal{D}(A)$ (the domain of A) endowed with the graph norm $\|.\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$, we have $\|Ax\| \leq \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Let J be a linear operator on X. If $\mathcal{D}(A) \subseteq \mathcal{D}(J)$, then J will be called A-defined. If J is an A-defined operator, we will denote by $\widehat{J}$ the restriction of J to $\mathcal{D}(A)$. Moreover, if $\widehat{J} \in \mathcal{L}(X_A, X)$, we say that J is A-bounded. One checks easily that if J is closed (or closable), see [**13**, Remark 1.5, page 191], then J is A-bounded.

Let $F \in \mathcal{L}(X)$. We write $F \in P\mathcal{J}(X)$ if there is a nonzero complex polynomial $p(.)$ such that the operator $p(F)$ belongs to $\mathcal{J}(X)$. If $F \in P\mathcal{J}(X)$, the nonzero polynomial $p(.)$, of least degree and leading coefficient 1 such that $p(F) \in \mathcal{J}(X)$, will be called the minimal polynomial of F. We denote by $\mathcal{P}_{\mathcal{J}}(X)$ the subset of $P\mathcal{J}(X)$ defined by

$$\mathcal{P}_{\mathcal{J}}(X) := \Big\{F \in P\mathcal{J}(X) \text{ such that the minimal polynomial of } F, \\ p(.), \text{ satisfies } p(-1) \neq 0\Big\}.$$

Let $\mathcal{J}(X)$ denote an arbitrary *nonzero closed two-sided ideal* of $\mathcal{L}(X)$, and assume that the following hypothesis holds

$$(\mathcal{H}) \qquad \mathcal{J}(X) \subseteq \mathcal{F}(X).$$

Obviously, under the condition $(\mathcal{H})$, Proposition 4 in [**6**, page 70] implies that $\overline{\mathcal{F}_0(X)} \subseteq \mathcal{J}(X) \subseteq \mathcal{F}(X)$.

The goal of this paper is to prove the following result.

Theorem 2.1. *Let $A \in \mathcal{C}(X)$, let J be an A-bounded operator on X, and let $U \neq \varnothing$ be an open subset of Φ_A. Let $\mathcal{J}(X)$ be a proper closed two-sided ideal of $\mathcal{L}(X)$ satisfying $(\mathcal{H})$ and assume that, for each $\lambda \in U$, there exists a Fredholm inverse A_λ of $\lambda - A$ such that $JA_\lambda \in \mathcal{P}_{\mathcal{J}}(X)$. Then*

$$(\Phi_A)_U = (\Phi_{A+J})_U,$$

where $(\Phi_A)_U$, respectively $(\Phi_{A+J})_U$, denotes the union of all connected components of Φ_A, respectively Φ_{A+J}, meeting U. Moreover, if $\lambda \in (\Phi_A)_U$, then

$$i(\lambda - A) = i(\lambda - A - J).$$

As a straightforward consequence of Theorem 2.1, we have

Corollary 2.1. *Assume that the hypotheses of Theorem* 2.1 *are satisfied. If, further, Φ_A is connected, then Φ_A is a component of Φ_{A+J} and, for any $\lambda \in \Phi_A$, we have $i(\lambda - A) = i(\lambda - A - J)$.*

It should be observed that, in general, the sets $\mathcal{J}(X)$ and $\mathcal{P}_{\mathcal{J}}(X)$ do not coincide (actually $\mathcal{J}(X)$ is strictly contained in $\mathcal{P}_{\mathcal{J}}(X)$). Indeed, if $p(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$ is the minimal polynomial of $F \in \mathcal{P}_{\mathcal{J}}(X)$, then the spectrum of F consists of countably many points with $\{\lambda_1, \dots, \lambda_k\}$ as only possible limit points and such that all but possibly $\{\lambda_1, \dots, \lambda_k\}$ are eigenvalues with finite dimensional generalized eigenspaces. This together with the fact that the operators belonging to $\mathcal{J}(X) \subset \mathcal{F}(X)$ satisfy the Riesz-Schauder theory of compact operators, see [**1**], shows that $\mathcal{F}(X) \neq \mathcal{P}_{\mathcal{J}}(X)$. (Evidently, if $\mathcal{J}(X) \subset \mathcal{F}(X)$, then $\mathcal{J}(X) \subset_{\neq} \mathcal{P}_{\mathcal{J}}(X) \neq \mathcal{F}(X)$ and $\mathcal{F}(X) \subset_{\neq} \mathcal{P}_{\mathcal{F}}(X)$.) Thus Theorem 1. 1 (and Corollary 1.1) may be regarded as an extension of the results obtained in [**18**, Section 3] and [**19**, Section 2] to wide classes of perturbing operators $\mathcal{P}_{\mathcal{J}}(X)$ where $\mathcal{J}(X)$ is an arbitrary closed two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $(\mathcal{H})$.

We close this section by indicating, for some particular Banach spaces, the largest classes of perturbing operators for which Theorem 2.1 holds true. Indeed, note that even though the description of the ideal structure of $\mathcal{L}(X)$ is a complex task, cf. [**1**], there exist some Banach spaces for which $\mathcal{L}(X)$ has only one proper nonzero closed two-sided ideal. Actually, following Calkin [**1, 6**], if X is a separable Hilbert space, then $\mathcal{K}(X)$ is the unique proper closed two-sided ideal of $\mathcal{L}(X)$. This result holds also true for the spaces l_p, $1 \leq p < \infty$ and c_0 [**6, 10**]. Hence, if X is one of these spaces, then $\mathcal{K}(X) = \mathcal{F}(X)$, and therefore $\mathcal{P}_{\mathcal{F}}(X) = \mathcal{P}_{\mathcal{K}}(X)$. On the other hand, if X is isomorphic to an L_p-space or to $C(\Omega)$ (the Banach space of continuous scalarvalued functions on Ω with the supremum norm where Ω is a compact Hausdorff space), then $\mathcal{S}(X) = \mathcal{F}(X)$, cf. [**17**, equations (2.9), (2.10)]. So, for these spaces we have $\mathcal{P}_{\mathcal{F}}(X) = \mathcal{P}_{\mathcal{S}}(X)$. A Banach space X is an h-space if each closed infinite-dimensional subspace of X contains a complemented subspace isomorphic to X. Any Banach space isomorphic to an h-space is an h-space; c, c_0 and l_p, $1 \leq p < \infty$, are h-spaces. Let X be an h-space, according to [**26**, Theorem 6.2], $\mathcal{S}(X)$ is the greatest proper ideal of $\mathcal{L}(X)$, i.e., $\mathcal{S}(X) = \mathcal{F}(X)$. So, for this family of Banach spaces we have again $\mathcal{P}_{\mathcal{F}}(X) = \mathcal{P}_{\mathcal{S}}(X)$.

3. Some perturbation results.

In this section we present some perturbation results which are intended for use in our subsequent purposes.

Lemma 3.1. *If $F \in \mathcal{P}_{\mathcal{J}}(X)$, then $I + F \in \Phi(X)$ and $i(I + F) = 0$.*

Proof. Let $p(.)$ be the minimal polynomial of F. Since $p(F) \in \mathcal{J}(X)$, then $\sigma_{e6}(p(F)) = \{0\}$. But $p(-1) \neq 0$, then $p(-1) \notin \sigma_{e6}(p(F))$. Next, making use of the spectral mapping theorem of the Browder essential spectrum [**8**] we conclude that $-1 \in \rho_6(F)$. This ends the proof. $\square$

An operator $A \in \mathcal{C}(X, Y)$ is said to have a left Fredholm inverse if and only if there are maps $R_l \in \mathcal{L}(Y, X)$ and $K \in \mathcal{K}(X)$ such that $I_X + K$ extends $R_l A$. Similarly, A has a right Fredholm inverse if and only if there is a map $R_r \in \mathcal{L}(Y, X)$ such that $R_r(Y) \subseteq D(A)$ and $AR_r - I_Y \in \mathcal{K}(Y)$. The operators R_l and R_r are called left and right Fredholm inverses of A, respectively. We shall refer to a map which is both a left and a right Fredholm inverse of an operator A as a Fredholm inverse of A. We know by the classical theory of Fredholm operators, see, for example, [**1, 5, 13**], that A belongs to $\Phi_+(X)$, $\Phi_-(X)$ or $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

Lemma 3.2. *Let A and B be two operators in $\mathcal{L}(X, Y)$. Then*:

(i) *If $A \in \Phi(X, Y)$ and there is a Fredholm inverse of A, $R \in \mathcal{L}(Y, X)$, such that $RB \in \mathcal{P}_{\mathcal{J}}(X)$, then $A+B \in \Phi(X, Y)$ and $i(A+B) = i(A)$.*

(ii) *If* $A \in \Phi_+(X,Y)$ *and there is a left Fredholm inverse of* A, $R_l \in \mathcal{L}(Y,X)$, *such that* $BR_l \in \mathcal{P}_{\mathcal{J}}(Y)$, *then* $A+B \in \Phi_+(X,Y)$.

(iii) *If* $A \in \Phi_-(X,Y)$ *and there is a right Fredholm inverse of* A, $R_r \in \mathcal{L}(Y,X)$, *such that* $R_rB \in \mathcal{P}_{\mathcal{J}}(X)$, *then* $A+B \in \Phi_-(X,Y)$.

Proof. (i) R is a Fredholm inverse of A. Then there exists $F \in \mathcal{F}_0(Y)$ such that $AR = I_Y - F$ on Y. Now the operator $A+B$ can be written in the form

$$A+B = A + (AR+F)B = A(I_X + RB) + FB. \tag{3.1}$$

Since $RB \in \mathcal{P}_{\mathcal{J}}(X)$, using Lemma 3.1 one sees that $I_X + RB$ is a Fredholm operator of index zero. Now applying Atkinson's theorem, we obtain

$$A(I_X+RB) \in \Phi(X,Y) \quad \text{and} \quad i(A(I_X+RB)) = i(A)$$

Next, taking into account (3.1) and remembering that $FB \in \mathcal{F}_0(X,Y)$, we infer that $A+B$ is also a Fredholm operator and $i(A+B) = i(A)$.

(ii) If R_l is a left Fredholm inverse of A, then there exists $F \in \mathcal{F}_0(Y)$ such that $R_lA = I_X - F$ on X. The operator $A+B$ can be written in the form

$$A+B = A + B(R_lA+F) = (BR_l + I_Y)A + BF.$$

Clearly, since $BR_l \in \mathcal{P}_{\mathcal{J}}(X)$, applying Lemma 3.1 we get $(BR_l + I_Y) \in \Phi(Y)$. Now using the fact that $A \in \Phi_+(X,Y)$, $BF \in \mathcal{K}(X,Y)$ and Corollaries 1.33 and 1.37 in [**1**], we infer that $A+B \in \Phi_+(X,Y)$.

(iii) Assume that R_r is a right Fredholm inverse of A. Then there exists an $F \in \mathcal{F}_0(Y)$ such that $AR_r = I_Y - F$ and consequently

$$A+B = A + B(AR_r + F) = A(I_X + R_rB) + FB.$$

Now arguing as in (ii) we get $A+B \in \Phi_-(X,Y)$. $\square$

Let A and B be two operators in $\mathcal{L}(X,Y)$. We denote by $F^{\pm}_{AB}(Y,X)$ the set of left or right Fredholm inverses $R_\pm$ of A satisfying $BR_\pm \in$

$\mathcal{P}_{\mathcal{J}}(X)$ or $R_{\pm}B \in \mathcal{P}_{\mathcal{J}}(X)$ following that $A \in \Phi_{+}(X,Y)$ or $A \in \Phi_{-}(X,Y)$.

Lemma 3.3. *Let A and B be two operators in $\mathcal{L}(X,Y)$. If $A \in \Phi_{\pm}(X,Y)$ and $F^{\pm}_{AB}(Y,X) \neq \varnothing$, then $A+B \in \Phi_{\pm}(X,Y)$.*

Proof. It follows from Lemma 3.2 (ii) and (iii). □

Now we are ready to prove

Theorem 3.1. *Let $A \in \mathcal{C}(X)$, and assume that J is an A-bounded operator on X. Then the following statements hold.*

(i) *If $A \in \Phi(X)$ and there is a Fredholm inverse of A, $R \in \mathcal{L}(X)$, such that $RJ \in \mathcal{P}_{\mathcal{J}}(X)$, then $A+J \in \Phi(X)$ and $i(A+J) = i(A)$.*

(ii) *If $A \in \Phi_{+}(X)$ and there is a left Fredholm inverse of A, $R_l \in \mathcal{L}(X)$ such that $JR_l \in \mathcal{P}_{\mathcal{J}}(X)$, then $A+J \in \Phi_{+}(X)$.*

(iii) *If $A \in \Phi_{-}(X)$ and there is a right Fredholm inverse of A, $R_r \in \mathcal{L}(X)$ such that $R_rJ \in \mathcal{P}_{\mathcal{J}}(X)$, then $A+J \in \Phi_{-}(X)$.*

(iv) *If $A \in \Phi_{\pm}(X)$ and $F^{\pm}_{AJ}(X) \neq \varnothing$, then $A+J \in \Phi_{\pm}(X)$.*

Proof. As noted in the last section, the closedness of A implies that X_A is a Banach space. Let $\widehat{A}$ and $\widehat{J}$ denote, respectively, the restrictions of A and J to $\mathcal{D}(A)$. Clearly, $\widehat{A} \in \mathcal{L}(X_A,X)$ and $\widehat{J} \in \mathcal{L}(X_A,X)$. Moreover, it is not difficult to see that

$$\begin{cases} \alpha(\widehat{A}) = \alpha(A), \quad \beta(\widehat{A}) = \beta(A), \quad R(\widehat{A}) = R(A), \\ \alpha(\widehat{A}+\widehat{J}) = \alpha(A+J), \\ \beta(\widehat{A}+\widehat{J}) = \beta(A+J) \text{ and } R(\widehat{A}+\widehat{J}) = R(A+J). \end{cases} \tag{3.2}$$

Hence, if A belongs to $\Phi(X)$, respectively $\Phi_{+}(X)$, then $\widehat{A} \in \Phi(X_A,X)$, respectively $\Phi_{+}(X_A,X)$. Now the assertions (i) and (ii) follow from (3.2) and Lemma 3.2 (i), (ii).

(iii) Let $A \in \Phi_{-}(X)$. Then by (3.2) one sees that $\widehat{A} \in \Phi_{-}(X_A,X)$. By hypothesis, there exists an $F \in \mathcal{F}_0(X)$ such that $\widehat{A}R_r = (I-F)$. On the other hand,

$$\|R_rx\|_{X_A} = \|R_rx\|_X + \|AR_rx\|_X \leq \{\|R_r\| + \|(I-F)\|\}\,\|x\|.$$

This shows that $R_r \in \mathcal{L}(X, X_A)$ and therefore $R_r\widehat{J} \in \mathcal{L}(X_A)$. Now writing $\widehat{A} + \widehat{B} = \widehat{A}(I_{X_A} + R_r\widehat{J}) + F\widehat{J}$ we conclude the desired result. The last statement is an immediate consequence of the items (ii) and (iii). □

Corollary 3.1. *Suppose that A and J are as in Theorem* 3.1. *Then the following assertions hold.*

(i) *Assume that, for each $\lambda \in \Phi_A$, there exists a Fredholm inverse A_λ of $\lambda - A$ such that $A_\lambda J \in \mathcal{P}_{\mathcal{J}}(X)$, then*

$$\Phi_A \subseteq \Phi_{A+J} \quad \textit{and} \quad i(\lambda - A - J) = i(\lambda - A) \quad \textit{for each } \lambda \in \Phi_A.$$

(ii) *Assume that, for each $\lambda \in \Phi_{+A}$, there exists a left Fredholm inverse $A_{\lambda l}$ of $\lambda - A$ such that $JA_{\lambda l} \in \mathcal{P}_{\mathcal{J}}(X)$. Then*

$$\Phi_{+A} \subseteq \Phi_{+(A+J)}.$$

(iii) *Assume that, for each $\lambda \in \Phi_{-A}$, there exists a right Fredholm inverse $A_{\lambda r}$ of $\lambda - A$ such that $A_{\lambda r}J \in \mathcal{P}_{\mathcal{J}}(X_A)$. Then*

$$\Phi_{-A} \subseteq \Phi_{-(A+J)}.$$

(iv) *Assume that, for each $\lambda \in \Phi_{\pm A}$, the set $F^{\pm}_{(\lambda - A)J}(X_A) \neq \varnothing$. Then*

$$\Phi_{\pm A} \subseteq \Phi_{\pm(A+J)}.$$

Proof. Apply Theorem 3.1 to the operators $\lambda - A$ and $\lambda - A - J$. □

Corollary 3.1 translates in terms of essential spectra as:

Corollary 3.2. *Let A and J be as in Theorem* 3.1. *Then*:

(i) *Assume that for each $\lambda \in \Phi_A$, there exists a Fredholm inverse A_λ of $\lambda - A$ such that $A_\lambda J \in \mathcal{P}_{\mathcal{J}}(X)$. Then*

$$\sigma_{e4}(A + J) \subseteq \sigma_{e4}(A) \quad \textit{and} \quad \sigma_{e5}(A + J) \subseteq \sigma_{e5}(A).$$

(ii) *Assume that for each* $\lambda \in \Phi_{+A}$, *there exists a left Fredholm inverse* $A_{\lambda l}$ *of* $\lambda - A$ *such that* $JA_{\lambda l} \in \mathcal{P}_{\mathcal{J}}(X)$. *Then*

$$\sigma_{e1}(A+J) \subseteq \sigma_{e1}(A).$$

(iii) *Assume that for each* $\lambda \in \Phi_{-A}$, *there exists a right Fredholm inverse* $A_{\lambda r}$ *of* $\lambda - A$ *such that* $A_{\lambda r}J \in \mathcal{P}_{\mathcal{J}}(X_A)$. *Then*

$$\sigma_{e2}(A+J) \subseteq \sigma_{e2}(A).$$

(iv) *Assume that for each* $\lambda \in \Phi_{\pm A}$, *the set* $F_{(\lambda-A)J}(X_A) \neq \varnothing$. *Then*

$$\sigma_{e3}(A+J) \subseteq \sigma_{e3}(A).$$

We end this section by noticing that Theorem 3.1, respectively Corollaries 3.1 and 3.2, extends Lemma 3.1, respectively Theorem 2.1, in [**19**] to the family of perturbing operators $\mathcal{P}_{\mathcal{J}}(X)$.

4. Auxiliary results. The goal of this section is to establish Theorem 4.1 which contains one of the main steps in the proof of Theorem 2.1. We begin with the following elementary statement which must be surely well known. Its proof is omitted.

Lemma 4.1. *Let* $A \in \mathcal{C}(X)$, *and let* λ *and* μ *be two elements in* $\Phi(A)$. *If* A_λ, *respectively* A_μ, *denotes a Fredholm inverses of* $\lambda - A$, *respectively* $\mu - A$, *then*

$$\begin{aligned} A_\lambda - A_\mu - (\mu - \lambda)A_\lambda A_\mu &\in \mathcal{F}_0(X),\\ A_\lambda A_\mu - A_\mu A_\lambda &\in \mathcal{F}_0(X). \end{aligned}$$

A consequence of Lemma 4.1 is that, if $\lambda \in \Phi_A$ and A_λ and A'_λ are two Fredholm inverses of $\lambda - A$, then $A_\lambda - A'_\lambda \in \mathcal{F}_0(X)$.

Theorem 4.1. *Let $A \in \mathcal{C}(X)$. If $\lambda_0 \in \Phi_A$, then there exists $\varepsilon > 0$ and $M(\lambda_0) > 0$ such that $B(\lambda_0, \varepsilon)$, the open ball with radius ε centered at λ_0, is contained in Φ_A. Moreover, for all $\lambda \in B(\lambda_0, \varepsilon)$, there exists A_λ, a Fredholm inverse of $\lambda - A$, such that $\|A_\lambda\| \leq M(\lambda_0)$.*

The proof of this theorem is based on a construction of particular Fredholm inverses and requires some preparation.

Lemma 4.2. *Assume that $X = X_1 \oplus X_2$ and X_2 is finite dimensional. If there exists $T \in \mathcal{L}(X)$ such that $Tx = x$ for all $x \in X_1$, then $T - I \in \mathcal{F}_0(X)$.*

Proof. Since X_2 is complemented, then there exists a bounded projection P from X onto X_1. Thus, for all $x \in X$, we have $x = Px + (I-P)x$ so that $(T-I)x = T(I-P)x - (I-P)x = (T-I)(I-P)x$. Now the result follows from the boundedness of $I - P$ and the fact that $\dim X_2 < \infty$. $\square$

Let $\lambda_0 \in \Phi_A$ and denote by X_0, respectively Y_0, a complementary subspace of $N(\lambda_0 - A)$, respectively $R(\lambda - A)$, in X. So X_0 is of finite codimension and Y_0 is of finite dimension. Note that $X = N(\lambda_0 - A) \oplus X_0$ and $N(\lambda_0 - A) \subset \mathcal{D}(A)$. Hence, $\mathcal{D}(A) = N(\lambda_0 - A) \oplus (\mathcal{D}(A) \cap X_0)$, $X = R(\lambda_0 - A) \oplus Y_0$ and $\mathcal{D}(A) \cap X_0$ is dense in X_0. Since $N(\lambda_0 - A)$ is a closed subspace of X_A of finite dimensional, then $\widetilde{X_0} := \mathcal{D}(A) \cap X_0$ is closed in X_A. Let B_{λ_0} be the operator defined by

$$\begin{aligned} B_{\lambda_0} : \widetilde{X_0} \times Y_0 &\longrightarrow R(\lambda_0 - A) \oplus Y_0, \\ (x_0, y_0) &\longrightarrow (\lambda_0 - A)x_0 + y_0. \end{aligned}$$

It follows from the estimate $\|(\lambda_0 - A)x_0 + y_0\| \leq \max(1, |\lambda_0|)(\|x_0\|_{X_A} + \|y_0\|_{Y_0})$ that the operator B_{λ_0} is bounded and $\|B_{\lambda_0}\| \leq \max(1, |\lambda_0|)$. On the other hand, since $X_A := N(\lambda_0 - A) \oplus \widetilde{X_0}$, then the operator

$$(\lambda_0 - A)_{|\widetilde{X_0}} : \widetilde{X_0} \to R(\lambda_0 - A)$$

is bijective and therefore B_{λ_0} is invertible. Next, putting

$$A_{\lambda_0} = I_0\, P_{\widetilde{X_0}}\, B_{\lambda_0}^{-1}$$

where $P_{\widetilde{X_0}}$ denotes the projection from $\widetilde{X_0} \times Y_0$ onto $\widetilde{X_0}$ and I_0 is the embedding of $\widetilde{X_0}$ into X_A.

Lemma 4.3. *The operator A_{λ_0} is a Fredholm inverse of $\lambda_0 - A$.*

Proof. If $x \in \widetilde{X_0}$, then $B_{\lambda_0}^{-1}(\lambda_0 - A)x = (x, 0)$ and $I_0\, P_{\widetilde{X_0}}(x, 0) = x$. Accordingly, $A_{\lambda_0}(\lambda_0 - A)x = x$ for all $x \in \widetilde{X_0}$. Since $\widetilde{X_0}$ is of finite codimension in X_A, then Lemma 4.2 gives $A_{\lambda_0}(\lambda_0 - A) - I \in \mathcal{F}_0(X_A)$. If $y_0 \in R(\lambda_0 - A)$, then there exists a unique $x_0 \in \widetilde{X_0}$ such that $y_0 = (\lambda_0 - A)x_0$, so $B_{\lambda_0}^{-1} y_0 = (x_0, 0)$ and $A_{\lambda_0} y = x_0$ and therefore $(\lambda_0 - A)\, A_{\lambda_0} y_0 = (\lambda_0 - A)x_0 = y_0$. Now applying Lemma 4.2 we infer that $(\lambda_0 - A)\, A_{\lambda_0} - I \in \mathcal{F}_0(X)$. $\square$

Let $\lambda \in \Phi_A$, and let B_λ be the operator defined by

$$B_\lambda : \widetilde{X_0} \times Y_0 \to R(\lambda - A) \oplus Y_0, \qquad (x_0, y_0) \to (\lambda - A)x_0 + y_0.$$

Lemma 4.4. *For any λ satisfying $|\lambda - \lambda_0| < 1/\|B_{\lambda_0}^{-1}\|$, the operator B_λ is invertible and $A_\lambda = I_0\, P_{\widetilde{X_0}}\, B_\lambda^{-1}$ is a Fredholm inverse of $\lambda - A$.*

Proof. Let $x_0 \in \widetilde{X_0}$. Using the estimate $\|(\lambda - A)x_0 - (\lambda_0 - A)x_0\| \leq |\lambda - \lambda_0|\, \|x_0\|_{X_A}$, we conclude that $\|(\lambda - A) - (\lambda_0 - A)\|_{\mathcal{L}(X, X_A)} \leq |\lambda - \lambda_0|$. On the other hand, for all $(x_0, y_0) \in \widetilde{X_0} \times Y_0$, we have

$$\begin{aligned} \|(B_\lambda - B_{\lambda_0})(x_0, y_0)\|_X &= \|(\lambda - A)x_0 - (\lambda_0 - A)x_0\|_X \\ &\leq |\lambda - \lambda_0|\, \|x_0\|_{X_A} + |\lambda - \lambda_0|\, \|y_0\|_{Y_0} \\ &\leq |\lambda - \lambda_0|\, \|(x_0, y_0)\|_{\widetilde{X_0} \times Y_0}. \end{aligned}$$

This proves that $\|B_\lambda - B_{\lambda_0}\| \leq |\lambda - \lambda_0|$. Next, note that, for any λ close to λ_0 and $(x_0, y_0) \in \widetilde{X_0} \times Y_0$, we have $(B_\lambda - B_{\lambda_0})(x_0, y_0) = (\lambda - \lambda_0)x_0$. Making use of the invertibility of B_{λ_0}, we obtain $B_{\lambda_0}^{-1} B_\lambda - I = (\lambda - \lambda_0) B_{\lambda_0}^{-1}$. So, if $|\lambda - \lambda_0|\, \|B_{\lambda_0}^{-1}\| < 1$, then the operator $B_{\lambda_0}^{-1} B_\lambda$ is invertible and

$$(B_{\lambda_0}^{-1} B_\lambda)^{-1} = \sum_{n \geq 0} (-1)^n (\lambda - \lambda_0)^n\, (B_{\lambda_0}^{-1})^n.$$

Accordingly,

$$B_\lambda^{-1} = \sum_{n\geq 0}(-1)^n(\lambda-\lambda_0)^n\,(B_{\lambda_0}^{-1})^{n+1}. \tag{4.1}$$

Finally, applying Lemma 4.3, we conclude the desired result. □

Proof of Theorem 4.1. Let λ be such that $|\lambda-\lambda_0| < 1/(2\|B_{\lambda_0}^{-1}\|)$. It follows from Lemma 4.4 that B_λ is invertible and A_λ is a Fredholm inverse of $\lambda - A$. On the other hand, using (4.1), we get

$$\|B_\lambda^{-1}\| \leq \sum_{n\geq 0}\left(\frac{1}{2\|B_{\lambda_0}^{-1}\|}\right)^n \|B_{\lambda_0}^{-1}\|^{n+1} = 2\,\|B_{\lambda_0}^{-1}\|.$$

This leads to $\|A_\lambda\|_{\mathcal{L}(X,\widetilde{X_0})} \leq 2\,\|P_{\widetilde{X_0}}\|\,\|B_{\lambda_0}^{-1}\|$ which completes the proof. □

5. Proof of Theorem 2.1. Let $\mathcal{J}(X)$ be a nonzero closed two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $(\mathcal{H})$. We denote by $\mathcal{Q}_{\mathcal{J}}(X)$ the Banach algebra $\mathcal{L}(X)/\mathcal{J}(X)$ equipped with the norm $\|\widetilde{Q}\|_{\mathcal{J}} := \inf\{\|Q+H\|,\ H\in\mathcal{J}(X)\}$, where $\widetilde{Q}$ stands for the residual class in $\mathcal{Q}_{\mathcal{J}}(X)$ which contains $Q\in\mathcal{L}(X)$. As a matter of convenience, let us recall the following result due to Yood, see [**6**, Theorem 3.2], which we will use below.

Proposition 5.1. *Let $\mathcal{J}(X)$ be a nonzero closed two-sided ideal of $\mathcal{L}(X)$ satisfying $(\mathcal{H})$. Then $Q\in\mathcal{L}(X)$ is a Fredholm operator if and only if $\widetilde{Q}$ is invertible in $\mathcal{Q}_{\mathcal{J}}(X)$.*

Let $A\in\mathcal{C}(X)$ be such that $\Phi_A\neq\varnothing$. By Theorem 4.1 we can define an application

$$\zeta : \Phi_A \longrightarrow \mathcal{Q}_{\mathcal{J}}(X), \quad \lambda \longrightarrow \pi(A_\lambda),$$

where $A_\lambda\in\mathcal{L}(X)$ is a Fredholm inverse of $\lambda - A$ and $\pi(.)$ is the canonical surjection from $\mathcal{L}(X)$ onto $\mathcal{Q}_{\mathcal{J}}(X)$. It is well known that $\pi(A_\lambda)$ does not depend on the choice of the Fredholm inverse A_λ, hence $\zeta(.)$ is well defined. In the same way, if J is an A-bounded operator, we can define a function ζ_J on Φ_A by

$$\zeta_J : \Phi_A \longrightarrow \mathcal{Q}_{\mathcal{J}}(X), \quad \lambda \longrightarrow \pi(JA_\lambda).$$

Similarly, if $\Phi_{A+J} \neq \varnothing$, we define the function η_J on Φ_{A+J} by

$$\eta_J : \Phi_{A+J} \longrightarrow \mathcal{Q}_{\mathcal{J}}(X), \quad \lambda \longrightarrow \pi(J(A+J)_\lambda)$$

where $(A+J)_\lambda$ is a Fredholm inverse of $\lambda - A - J$.

Lemma 5.1. *If $A \in \mathcal{C}(X)$, then $\zeta(.)$ is analytic on Φ_A. Moreover, if J is A-bounded, then $\zeta_J(.)$, respectively $\eta_J(.)$, is analytic on Φ_A, respectively Φ_{A+J}.*

Proof. Let λ and λ_0 be two elements of Φ_A. Since J is A-bounded, then $\pi(JA_\lambda) - \pi(JA_{\lambda_0}) = \pi(J(A_\lambda - A_{\lambda_0}))$. By Lemma 4.1 we have $\pi(JA_\lambda) - \pi(JA_{\lambda_0}) = (\lambda - \lambda_0)\pi(JA_{\lambda_0})\,\pi(A_\lambda)$ and therefore $\|\zeta_J(\lambda) - \zeta_J(\lambda_0)\|_{\mathcal{J}} = |\lambda - \lambda_0|\,\|\zeta_J(\lambda_0)\|_{\mathcal{J}}\,\|\zeta(\lambda)\|_{\mathcal{J}}$. It follows from the definition of $\|.\|_{\mathcal{J}}$ that, for any Fredholm inverse A_λ of $\lambda - A$, we have $\|\zeta(\lambda)\|_{\mathcal{J}} \leq \|A_\lambda\|_{\mathcal{L}(X)}$. On the other hand, by Theorem 4.1, there exist $\varepsilon > 0$ and $M(\lambda_0) > 0$ such that, for any λ satisfying $|\lambda - \lambda_0| < \varepsilon$, we have $\lambda \in \Phi_A$ and $\|\zeta(\lambda)\|_{\mathcal{J}} \leq M(\lambda_0)$. Hence, $\|\zeta_J(\lambda) - \zeta_J(\lambda_0)\|_{\mathcal{J}} \leq M(\lambda_0)\,|\lambda - \lambda_0|\,\|\zeta_J(\lambda_0)\|_{\mathcal{J}}$ which implies the continuity of $\zeta_J(.)$ on Φ_A. Similar calculations show the continuity of $\zeta(.)$ on Φ_A. Consider now $\lambda \in \Phi_A$, and let h be a complex number satisfying $0 < |h| < d(\lambda, \sigma_{e4}(A))$. Writing $(1/h)(\zeta_J(\lambda + h) - \zeta_J(\lambda)) = -\zeta(\lambda + h)\,\zeta_J(\lambda))$ and using the continuity of $\zeta(.)$, we find

$$\left\| \frac{1}{h}[\zeta_J(\lambda + h) - \zeta_J(\lambda)] + \zeta(\lambda + h)\,\zeta_J(\lambda)) \right\| \longrightarrow 0 \quad \text{as } h \to 0.$$

Thus, ζ_J is differentiable at any λ in Φ_A and $\zeta_J'(\lambda) = -\zeta_J(\lambda)\zeta(\lambda) = -\pi(JA_\lambda)\,\pi(A_\lambda)$. In the same way we prove also that $\zeta(.)$ is differentiable on Φ_A and $\zeta'(\lambda) = -[\zeta(\lambda)]^2 = -[\pi(A_\lambda)]^2$. The proof of the analyticity of $\eta_J(.)$ follows the same lines as that of $\zeta_J(.)$. □

Lemma 5.2. *Let f be an entire function, and let U be an open subset of Φ_A. If J is an A-bounded operator and $f(JA_\lambda) \in \mathcal{J}(X)$*

for all $\lambda \in U$, then $f(JA_\lambda) \in \mathcal{J}(X)$ for all $\lambda \in (\Phi_A)_U$ where $(\Phi_A)_U$ denotes the union of all connected components of Φ_A meeting U. In particular, if Φ_A is connected, then $f(JA_\lambda) \in \mathcal{J}(X)$ for every $\lambda \in \Phi_A$.

Proof. Let us first observe that $\pi(f(JA_\lambda)) = f(\zeta_J(\lambda))$. Further, Lemma 5.1 shows that $f \circ \zeta_J : \Phi_A \to \mathcal{Q}_J(X)$ is analytic. Since $f(JA_\lambda) \in \mathcal{J}(X)$ for all $\lambda \in U$, then $f \circ \zeta_J = 0$ on U. Therefore, the analytic continuation theorem implies that $f \circ \zeta_J = 0$ on $(\Phi_A)_U$, i.e., $f(JA_\lambda) \in \mathcal{J}(X)$ for all $\lambda \in (\Phi_A)_U$. $\square$

Define the functions $\varphi(.)$ and $\psi(.)$ by

$$\varphi : \mathbf{C}\backslash\{1\} \longrightarrow \mathbf{C}\backslash\{-1\}, \qquad \varphi(z) = \frac{z}{1-z},$$
$$\psi : \mathbf{C}\backslash\{-1\} \to \mathbf{C}\backslash\{1\}, \qquad \psi(z) = \frac{z}{1+z}.$$

Clearly φ and ψ are meromorphic functions and $\psi^{-1} = \varphi$.

Proposition 5.2. *Let U be an open subset of $\Phi_A \cap \Phi_{A+J}$. Then $1 \in \Phi_{JA_\lambda}$ if and only if $-1 \in \Phi_{J(A+J)_\lambda}$. In this case we have $\eta_J(\lambda) = \varphi(\zeta_J(\lambda))$ and $\zeta_J(\lambda) = \psi(\eta_J(\lambda))$ for all $\lambda \in U$.*

Proof. Let $\lambda \in U$, and assume that $1 \in \Phi_{JA_\lambda}$. Then, by Proposition 5.1, $I - \zeta_J(\lambda)$ in invertible in $\mathcal{Q}_{\mathcal{J}}(X)$, so there exist $F_{0,\lambda} \in \mathcal{J}(X)$ and a Fredholm inverse $(I-JA_\lambda)^*$ of $I-JA_\lambda$ such that $(I-JA_\lambda)^*(I-JA_\lambda) = I - F_{0,\lambda}$. Since A_λ is a Fredholm inverse of $\lambda - A$, applying again Proposition 5.1 one sees

$$\begin{aligned}(I - JA_\lambda)^* JA_\lambda(\lambda - A - J) &= (I - JA_\lambda)^*(J(I + F_{1,\lambda}) - JA_\lambda J)\\ &= (I - JA_\lambda)^*(I - JA_\lambda)J + F_{2,\lambda}\\ &= (I + F_{0,\lambda})J + F_{2,\lambda}\\ &= J + F_{3,\lambda},\end{aligned}$$

where $F_{1,\lambda}$, $F_{2,\lambda}$ and $F_{3,\lambda}$ belong to $\mathcal{J}(X)$. Since $\lambda \in \Phi_{A+J}$, if $(A+J)_\lambda$ is a Fredholm inverse of $(\lambda-A-J)$, then $(I-JA_\lambda)^* JA_\lambda(\lambda-A-J)(A+J)_\lambda = J(A+J)_\lambda + F_{4,\lambda}$ which writes in the form $(I - JA_\lambda)^* JA_\lambda = J(A+J)_\lambda + F_{5,\lambda}$ where $F_{4,\lambda}$ and $F_{5,\lambda}$ belong to $\mathcal{J}(X)$. This implies

$\pi[(I - J_A\lambda)^* JA_\lambda] = \pi(J(A+J)_\lambda)$ or $(I_{\mathcal{Q}(X)} - \pi(JA_\lambda))^{-1}\pi(JA_\lambda) = \pi(J(A+J)_\lambda)$. Hence, we get

$$\eta_J(\lambda) = \varphi(\zeta_J(\lambda)). \tag{5.1}$$

Now the spectral mapping theorem gives $\sigma(\eta_J(\lambda)) = \varphi(\sigma(\zeta_J(\lambda)))$. Since $-1 \notin \varphi(\mathbf{C}\backslash\{1\})$, then $-1 \in \Phi_{J(A+J)_\lambda}$. The converse follows the same lines as above; it suffices to replace in the computations JA_λ by $J(A+J)_\lambda$. To complete the proof, let $\lambda \in U$ be such that $1 \in \Phi_{JA_\lambda}$. It follows from (5.1) that $\eta_J(\lambda) = \varphi(\zeta_J(\lambda))$. Since $-1 \notin \sigma(\eta_J(\lambda))$, applying the classical functional calculus in $\mathcal{Q}_{\mathcal{J}}(X)$, we find $\psi(\eta_J(\lambda)) = (\psi \circ \varphi)(\zeta_J(\lambda))$. Now the fact that $(\psi \circ \varphi)(z) = z$ in a neighborhood of $\sigma(\zeta(\lambda))$, because $1 \notin \sigma(\zeta(\lambda))$, gives $\zeta_J(\lambda) = \psi(\eta_J(\lambda))$, which ends the proof. $\square$

Proposition 5.3. *Let U be an open subset of $\Phi_A \cap \Phi_{A+J}$. If for each $\lambda \in U$ there exist a Fredholm inverse A_λ of $\lambda - A$ and a polynomial $p(.) \neq 0$ such that $p(-1) \neq 0$ and $p(JA_\lambda) \in \mathcal{J}(X)$, then there exists a polynomial $q(.) \neq 0$ satisfying $q(1) \neq 0$ and, for each $\lambda \in U$, there exists a Fredholm inverse $(A+J)_\lambda$ of $\lambda - A - J$ such that $q((A+J)_\lambda) \in \mathcal{J}(X)$.*

Proof. Let Z_0 be the set of zeros of $p(.)$. We first note that, since $p(JA_\lambda) \in \mathcal{J}(X)$, then $\sigma_{e4}(JA_\lambda) = \sigma_{e5}(JA_\lambda) = \sigma_{e6}(JA_\lambda) = Z_0$. Moreover, it follows from Proposition 5.2 that $\zeta_J(\lambda) = \psi(\eta_J(\lambda))$. Also, according to the fact that $p(JA_\lambda) \in \mathcal{J}(X)$ we have $p(\zeta_J(\lambda)) = 0$ in $\mathcal{A}_J(X)$ and, consequently, $(p \circ \psi)(\eta_J(\lambda)) = 0$ in $\mathcal{Q}_J(X)$ for every $\lambda \in U$. Note that $(p \circ \psi)(z) = p(z/(1+z))$ for $z \neq -1$, i.e., $(p \circ \psi)(z) = q(z)/(1+z)^n$ for a polynomial $q(.) \neq 0$ and a certain integer $n \geq 1$. On the other hand, it follows from $p(-1) \neq 0$ that $-1 \in \Phi_{JA_\lambda}$. Applying Proposition 5.2 we infer that $1 \in \Phi_{J(A+J)_\lambda}$ and, consequently, $q(\eta_J(\lambda)) = 0$ in $\mathcal{Q}_{\mathcal{J}}(X)$ for all $\lambda \in U$. This shows that there exists a polynomial $q(.) \neq 0$ such that $q(J(A+J)_\lambda) \in \mathcal{J}(X)$ for all $\lambda \in U$. $\square$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. It follows from Lemma 5.2 and Corollary 3.1 (i) that $(\Phi_A)_U \subset \Phi_{A+J}$ and $i(\lambda - A - J) = i(\lambda - A)$ for all $\lambda \in (\Phi_A)_U$. In particular, $U \subset \Phi_{A+J}$. Applying Proposition 5.3 we infer that there exists a polynomial $q(.) \neq 0$ such that for each $\lambda \in U$ there exists a Fredholm inverse $(A+J)_\lambda$ of $\lambda - A - J$ satisfying $q(J(A+J)_\lambda) \in \mathcal{J}(X)$. Now using Proposition 5.1 we conclude that $q(J(A+J)_\lambda) \in \mathcal{J}(X)$ for all $\lambda \in (\Phi_{A+J})_U$ (all connected components of Φ_{A+J} meeting U). Let Ω be a component of $(\Phi_{A+J})_U$, and let $\lambda \in \Omega$. Clearly the operator $\lambda - A$ may be written in the form $(I + J(A+J)_\lambda)(\lambda - A - J) - JF$ for some $F \in \mathcal{J}(X)$. Since $q(-1) \neq 0$, then Lemma 3.1 implies that $I + J(A+J)_\lambda \in \Phi(X)$ and $i(I + J(A+J)_\lambda) = 0$. Now, applying Atkinson's theorem (see, for example, [**23**, Theorem 5.7]) we conclude that $\lambda - A$ is a Fredholm operator and $i(\lambda - A) = i(\lambda - A - J)$. This shows that $\Omega \subset \Omega'$ where Ω' is a connected component of Φ_A. Hence $\Omega = \Omega'$ and consequently $(\Phi_A)_U = (\Phi_{A+J})_U$. This achieves the proof. $\square$

6. Application to transport equations. In this section we will apply Corollary 2.1 to collect information about essential spectra of neutron transport operators on L_1 spaces. More precisely, we are concerned with the following integro-differential operator

$$\begin{aligned} A\psi(x,\xi) &= -v.\nabla_x \psi(x,v) - \sigma(v)\psi(x,v) + \int_V \kappa(x,v,v')\psi(x,v')\,d\mu(v') \\ &= T\psi + K\psi \end{aligned}$$

where $(x, v) \in D \times V$. Here D is a smooth open subset of $\mathbf{R}^n$, $d\mu(.)$ is a positive Radon measure on $\mathbf{R}^n$ satisfying $d\mu(0) = 0$. We denote by V the support of $d\mu(.)$, and we refer to V as the velocity space. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D. The function $\psi(x,v)$ represents the number (or probability) density of gas particles having the position x and the velocity v. The functions $\sigma(.,.)$ and $\kappa(.,.,.)$ are called, respectively, the collision frequency and the scattering kernel, cf. [**2, 12, 20**]. We deal with abstract velocity measures $d\mu(.)$, hence our analysis works for continuous models (Lebesgue measure on open subsets of $\mathbf{R}^n$), multigroup models (surface Lebesgue measures on spheres) as well as discrete ones (finite sum of Dirac measures). Let Γ_- denote the set

$$\Gamma_- = \{(x,v) \in \partial D \times V,\ v.\nu_x \leq 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial D$. We introduce the following partial Sobolev space

$$\mathcal{W} = \{\psi \in \mathcal{X} \text{ such that } v.\nabla_x \psi \in \mathcal{X}\},$$

where

$$\mathcal{X} := L_1(D \times V;\, dx\, d\mu(v)).$$

Recall that a very precise theory of traces is available for functions belonging to kinetic spaces such as $\mathcal{W}$, see, for example, [**2**]. However, when dealing with vacuum boundary conditions, i.e., $\psi_{|\Gamma_-} = 0$, the situation is very easy. (By $\psi_{|\Gamma_-} = 0$ we means that, for all compact $K \subset \Gamma_-$, $\psi_{|K} = 0$.) Actually, by [**2**, Theorem 1, page 1085], functions belonging to $\mathcal{W}$ possess traces in $L^1_{\text{loc}}(\Gamma_-; |v.\nu_x|\, d\gamma\, d\mu(v))$ where $d\gamma$ stands for the Lebesgue measure on ∂D. This suffices to define rigorously the domain of the transport operator with vacuum boundary conditions.

The streaming operator T is defined by

$$\begin{cases} T\psi(x,v) = -v.\nabla_x \psi(x,v) - \sigma(x,v)\psi(x,v) \\ \mathcal{D}(T) = \{\psi \in \mathcal{W} \text{ such that } \psi_{\Gamma_-} = 0\}, \end{cases}$$

where the collision frequency satisfies $\sigma(.,.) \in L^\infty(D \times V)$, and ψ_{Γ_-} denotes the trace of ψ on Γ_-. Set

$$\lambda^* := d\mu \text{ ess-}\inf_{(x,v)\in D\times V} \sigma(x,v).$$

It is well known that

$$\sigma(T) = \big\{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda^*\big\}$$

(see, for instance, Corollary 12.11 in [**12**, page 272]). In fact, we can easily show that $\sigma(T)$ is reduced to $\sigma C(T)$, the continuous spectrum of T, that is,

$$\sigma(T) = \sigma C(T) = \big\{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda^*\big\}. \tag{6.1}$$

On the other hand, if $\lambda \in \sigma C(T)$, then $R(\lambda - T)$ (the range of $\lambda - T$) is not closed (otherwise $\lambda \in \rho(T)$). So, $\lambda \in \sigma_{ei}\,(T)$, $i = 1, \dots, 6$. This implies that $\sigma C(T) \subseteq \cap_{i=1}^6 \sigma_{ei}\,(T)$. Thus, according to (6.1), we have

$$\sigma_{ei}\,(T) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda^*\} \quad \text{for } i = 1, 2, \dots, 6. \tag{6.2}$$

The transport operator A is defined as a bounded perturbation of T, i.e., $A = T + K$ where K is a bounded operator on $\mathcal{X}$ given by

$$K : \mathcal{X} \longrightarrow \mathcal{X}, \quad \psi \longrightarrow \int_V \kappa(x, v, v')\ \psi(x, v')\, d\mu(v'),$$

where the scattering kernel $\kappa :\ D \times V \times V \to \mathbf{R}$ is assumed to be measurable. So, $\mathcal{D}(A) = \mathcal{D}(T)$. Note that the operator K is local in x so it can be viewed as a mapping

$$K(.) : x \in D \longrightarrow K(x) \in \mathcal{L}(L_1(V;\, d\mu)).$$

We assume that $K(.)$ is strongly measurable, i.e.,

$$x \in D \longrightarrow K(x)f \in L_1(V;\, d\mu) \text{ is measurable for any } f \in L_1(V;\, d\mu)$$

and bounded, i.e., $\text{ess-}\sup_{x\in D} \|K(x)\|_{\mathcal{L}(L_1(V;\, d\mu))} < \infty$. Hence, K defines a bounded operator on the space $\mathcal{X}$ according to the rule

$$\varphi \in \mathcal{X} \longrightarrow K(x)\varphi(x) \in \mathcal{X}.$$

Thus,

$$\|K(x)\|_{\mathcal{L}(\mathcal{X})} \le \text{ess-}\sup_{x\in D} \|K(x)\|_{\mathcal{L}(L_1(V;\, d\mu))}.$$

Next we will use the class of regular collision operators introduced in [**20**].

Definition 6.1. Let $\mathcal{K}(L_1(V;\, d\mu))$ be the subspace of compact operators of $\mathcal{L}(L_1(V;\, d\mu))$. A collision operator

$$K : x \in D \to K(x) \in \mathcal{L}(L_1(V;\, d\mu))$$

is regular if $K(x) \in \mathcal{K}(L_1(V;\, d\mu))$ almost everywhere, $x \in D \to K(x) \in \mathcal{K}(L_1(V;\, d\mu))$ is measurable and

$$\{K(x) :\ x \in D\} \text{ is relatively compact in } \mathcal{L}(L_1(V;\, d\mu)).$$

From now on we will assume that the measure $d\mu$ satisfies the following geometrical property

$$\int_{\alpha_1 \le |x| \le \alpha_2} d\mu(x) \int_0^{\alpha_3} \chi_B(tx)\, dt \longrightarrow 0 \text{ as } |B| \to 0 \tag{6.3}$$

for every $\alpha_1 < \alpha_2 < \infty$ and $\alpha_3 < \infty$, where $|B|$ is the Lebesgue measure of $B \subset \mathbf{R}^n$ and $\chi_B(.)$ denotes its characteristic function.

We recall also the following result proved in [**20**, Theorem 4.4] which is needed below.

Proposition 6.1. *Let K be a regular collision operator and assume that the measure $d\mu$ satisfies* (6.3)*, then $K(\lambda - T)^{-1}K$ is weakly compact on $\mathcal{X}$.*

Now we are ready to prove

Theorem 6.1. *Let D be a bounded subset of $\mathbf{R}^n$ and let $d\mu(.)$ be a positive Radon measure on $\mathbf{R}^n$. If K is a regular collision operator and $d\mu(.)$ satisfies the condition* (6.3)*, then Φ_T is a connected component of Φ_A. Moreover, for all $\lambda \in \Phi_T$, we have $i(\lambda - A) = 0$.*

Obviously, Theorem 6.1 and equation (6.2) imply

$\sigma_{e4}(A) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leq -\lambda^*\}$ and $\sigma_{e5}(A) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leq -\lambda^*\}$.

Since $\Phi_T = \rho(T)$, see (6.1), the last inclusion shows that $\sigma(A) \cap \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > -\lambda^*\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicity. So,

$$\sigma_{e6}(A) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leq -\lambda^*\}.$$

Evidently this inclusion together with the inclusions between the various essential spectra implies

$$\sigma_{ei}(A) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leq -\lambda^*\}, \quad i = 1, \ldots, 6.$$

This shows, in particular, that the asymptotic spectrum of A consists of, at most, isolated eigenvalues with finite algebraic multiplicities. In practice, this is sufficient to describe the time asymptotic behavior of the solution (when it exists) to the Cauchy problem associated to A.

Proof of Theorem 6.1. Clearly, it follows from equations (6.1) and (6.2) that

$$\Phi_T = \rho(T) = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > -\lambda^*\}.$$

Since T generates a strongly continuous semi-group on $\mathcal{X}$ [**2, 20**], then it follows from the semi-group theory that $\lim_{\mathrm{Re}\,\lambda\to\infty}\|(\lambda-T)^{-1}\|=0$. So there exists $\tau>-\lambda^*$ such that for $\mathrm{Re}\,\lambda>\tau$ we have $r_\sigma(K(\lambda-T)^{-1})<1$ ($r_\sigma(.)$ denotes the spectral radius). Accordingly, the open half plane $U=\{\lambda\in\mathbf{C}:\ \mathrm{Re}\,\lambda>\tau\}$ is contained in $\Phi_T\cap\Phi_A$, so $\Phi_T\cap\Phi_A\neq\varnothing$. On the other hand, Proposition 6.1 together with [**3**, Corollary 13, page 510] shows that $[K(\lambda-T)^{-1}]^4$ is compact on $\mathcal{X}$ for all $\lambda\in\Phi_T$. Since Φ_T is connected and $\Phi_T\cap\Phi_{T+K}\neq\varnothing$, applying Corollary 2.1 we conclude that Φ_T is a component of Φ_A and, for any $\lambda\in\Phi_T$, we have $i(\lambda-A)=0$. This concludes the proof. □

Notice that essential spectra of the one-dimensional transport operator with general boundary conditions on L_p spaces with $p\in[1,\infty)$ were described in detail in [**18**]. It was shown that, if K is a regular collision operator, then

$$\sigma_{ei}(A)=\{\lambda\in\mathbf{C}:\ \mathrm{Re}\,\lambda\leq-\lambda^*\},\quad i=1,\dots,6.$$

The possibility of such a result for the case $p=1$ is due to the fact that, in slab geometry, if K is regular, then $(\lambda-T)^{-1}K$ is weakly compact, cf. [**17**, Proposition 3.2 (i)]. Similar results were obtained for the multi-dimensional transport operator with abstract boundary conditions in bounded geometry on L_p spaces with $1<p<\infty$, i.e., $\sigma_{ei}(A)=\{\lambda\in\mathbf{C}:\ \mathrm{Re}\,\lambda\leq-\lambda^*\}$ for $i=1,\dots,6$, [**16**]. Here again the fact that $(\lambda-T)^{-1}K$ is compact on $L_p(D\times V)$, $1<p<\infty$, for regular collision operators plays a crucial role in the proofs. Unfortunately, the operators $(\lambda-T)^{-1}K$ and $K(\lambda-T)^{-1}$ are not compact nor weakly compact on $L_1(D\times V)$. The analysis performed above is based on analytic continuation arguments, so only partial results were obtained. Therefore, the precise description of the various essential spectra of A on $L_1(D\times V)$ seems to be a delicate matter and certainly requires another approach rather than that used here and that used in the works [**18, 19**].

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