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GEODESICS AND CURVATURE OF MÖBIUS INVARIANT METRICS

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ABSTRACT. We confirm that certain circular arcs are geodesics for both the Ferrand and Kulkarni-Pinkall metrics. We demonstrate that 'most' Kulkarni-Pinkall isometries are Möbius transformations. We analyze the generalized Gaussian curvatures of these metrics. We exhibit numerous illustrative examples.

1. Introduction. This article is a continuation of [**5, 6**] wherein we studied a Möbius invariant metric $\mu_\Omega(z)|dz|$ introduced by Kulkarni and Pinkhall [**8**] as a canonical metric for Möbius structures on n-dimensional manifolds. In [**5**] we employed the definition given below, see subsection 2.E, and corroborated various properties of this metric using classical function theory. In [**6**] we established pointwise and uniform estimates between the Kulkarni-Pinkall metric and the hyperbolic and quasi-hyperbolic metrics.

Here we examine both the Kulkarni-Pinkall metric and a related metric first studied by Ferrand in [**3**]. We show that certain curves are always geodesics for these metrics, confirm that many Kulkarni-Pinkall isometries are Möbius transformation, and investigate the generalized Gaussian curvatures of both metrics. We also prove a number of basic facts concerning the Kulkarni-Pinkall metric.

Throughout this paper Ω is a region on the Riemann sphere $\widehat{\mathbf{C}}$ with at least two boundary points. Circular geodesics are one of the central objects of our study: we call Γ a *circular geodesic* in Ω if there exists a disk D in $\widehat{\mathbf{C}}$ with $D \subset \Omega$ and such that Γ is a hyperbolic geodesic line in D with endpoints in $\partial D \cap \partial\Omega$. (See below for all definitions.)

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While it is rarely true that a circular geodesic in Ω is also a hyperbolic geodesic in Ω, we will see that circular geodesics are actually geodesics for many natural metrics. In particular, we have the following.

Theorem A. *Circular geodesics are both Kulkarni-Pinkall and Ferrand geodesics.*

We can describe the Kulkarni-Pinkall isometries for many regions.

Theorem B. *Every Kulkarni-Pinkall isometry between nonsimply nondoubly connected regions is a Möbius transformation.*

In fact this conclusion also holds for most simply and doubly connected regions; the above is just an easily stated consequence of Theorem 4.10 in conjunction with Theorem 4.11.

Both the Ferrand and Kulkarni-Pinkall metrics are negatively curved, and we have additional information in any region which is the Möbius image of a convex domain.

Theorem C. *The generalized Gaussian curvatures of the Kulkarni-Pinkall metric lie in the interval $[-1,0]$. For a Möbius convex region this improves to $[-1,-1/2]$. The upper estimates also hold for the Ferrand metric, but not necessarily the lower bound.*

We mention that the above result is sharp in several ways. First, there are simply connected regions in which the Gaussian curvature takes on both values 0 and -1, e.g., any concave infinite sector; in fact, this even holds for 'nearly' convex regions which are infinite sectors with an angle opening just larger than π. Next, for a punctured disk the Gaussian curvature assumes all values in $(-1,0)$, and for an infinite strip it takes on every value in $(-1,-1/2]$. See subsections 3.B and 3.C.

Finally, the Gaussian curvature of these metrics can be constant only in special cases.

Theorem D. *If the Gaussian curvature of the Ferrand or the Kulkarni-Pinkall metric is a constant k in some quasi-hyperbolic re-*

gion Ω*, then either* $k = 0$ *and* Ω *is a twice punctured sphere, or* $k = -1$ *and* Ω *is a disk on* $\widehat{\mathbf{C}}$*. In the former case the metrics are both the corresponding Möbius quasi-hyperbolic metric; in the latter case they are the hyperbolic metric.*

This document is organized as follows. Section 2 contains preliminary information including definitions and terminology as well as basic and/or well-known facts. We exhibit fundamental examples in Section 3, but other examples appear throughout the article. A Euclidean interpretation for the Ferrand and Kulkarni-Pinkall metrics is presented in subsection 4.A and the proofs of Theorems A, B, C and D appear in subsections 4.B, 4.C and 4.D, respectively.

We remark that the quantities $1/\delta$, τ_{ab} and φ can be defined as below for appropriate regions in Euclidean n-space (or on the n-sphere). Also, the hyperbolic metric can be defined for balls, half-spaces, and the exterior of closed balls. Thus, the quasi-hyperbolic, Ferrand, and Kulkarni-Pinkall metrics can be defined for these regions. Many of our results, such as Propositions 2.7 and 4.4, continue to hold in this setting although sometimes dimensional constants must be included.

2. Preliminaries.

2A. General information. Our notation is relatively standard. We work in the complex plane $\mathbf{C}$; stated results are valid for the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ in terms of local coordinates as the reader may verify. Everywhere Ω is a domain, i.e., an open connected set, and $\partial\Omega$, Ω^c denote the boundary, complement, respectively, of Ω with respect to $\widehat{\mathbf{C}}$. The Euclidean disk centered at the point $a \in \mathbf{C}$ of radius r is denoted by $D(a; r)$ and $\mathbf{D} := D(0; 1)$ is the *unit disk*. We also let $\mathbf{H}$ denote the *right half-plane*, $\mathbf{H} := \{\Re(z) > 0\}$, and we define

$$\mathbf{C}^* := \mathbf{C} \setminus \{0\}, \quad \mathbf{C}_{ab} := \mathbf{C} \setminus \{a, b\}, \quad \widehat{\mathbf{C}}_{ab} := \widehat{\mathbf{C}} \setminus \{a, b\};$$

the latter two definitions are for distinct points a, b in $\mathbf{C}$ or in $\widehat{\mathbf{C}}$, respectively.

The quantity $\delta(z) = \delta_\Omega(z) := \mathrm{dist}\,(z, \partial\Omega) = \mathrm{dist}\,(z, \partial\Omega \cap \mathbf{C})$ is the Euclidean distance from $z \in \mathbf{C}$ to the boundary of Ω, and $1/\delta$ is the density for the so-called *quasi-hyperbolic* metric $|dz|/\delta(z)$ when $\Omega \subset \mathbf{C}$. We call $\Omega \subset \widehat{\mathbf{C}}$ a *quasi-hyperbolic* domain provided $\widehat{\mathbf{C}} \setminus \Omega$ contains at least two points (one of which may be the point at infinity). We make frequent use of the notation

$$D(z) = D_\Omega(z) := D(z;\delta(z)) = D(z;\delta_\Omega(z))$$

for the *maximal* disk in Ω *centered* at z. The reader should take care not to confuse the two disks $D(z)$ and $\Delta(z)$ (the latter is defined below in subsection 2.B) each associated with a point $z \in \Omega$.

As our notation above suggests, we do not include Ω whenever the region is clear from context. Often, if there are two regions in consideration, say Ω and Ω', we will use a prime, i.e., a $'$, to indicate quantities associated with Ω'. For example, $\delta'(z) = \delta_{\Omega'}(z)$. We hope no confusion arises!

2B. Conformal metrics and geodesics. Recall that a conformal metric on a region $\Omega \subset \mathbf{C}$ has the form $\rho(z)|dz|$ where ρ is some positive continuous function defined on Ω. Here we consider several such metrics. We remind the reader that whenever $f : \Omega \to \Omega'$ is a (locally univalent) holomorphic map, every conformal metric $\sigma(w)|dw|$ on Ω' determines a metric, the so-called *pullback*, $\rho(z)|dz| := f^*[\sigma(w)|dw|]$ on Ω where

$$\rho(z)|dz| = \sigma(f(z))|f'(z)||dz|.$$

We often abuse notation and abbreviate this by writing $\rho = f^*[\sigma]$.

A *geodesic* in a metric space X is an isometric embedding $\gamma : I \to X$ where $I \subset \mathbf{R}$ is an interval; we use the adjectives *segment, ray*, or *line*, respectively, to indicate that I is bounded, semi-infinite, or all of $\mathbf{R}$. We let $|\gamma| := \gamma(I)$ denote the image of γ. A metric space is *geodesic* if each pair of points can be joined by a geodesic segment.

Given a conformal metric $\rho(z)|dz|$ on Ω, there is an associated distance function d_ρ obtained in the usual way by letting

$$l_\rho(\gamma) = \int_\gamma \rho(z)\,|dz|$$

denote the ρ-length of a rectifiable curve γ, and then defining the ρ-distance between two points a, b as

$$d_\rho(a,b) = \inf\{l_\rho(\gamma) : \gamma \text{ a rectifiable curve joining } a, b \text{ in } \Omega\}.$$

In this way we always obtain a length space (Ω, d_ρ) which is often a geodesic space.

In our setting we have the associated hyperbolic, quasi-hyperbolic, Kulkarni-Pinkall, and Ferrand distances d_λ, $d_{1/\delta}$, d_μ, d_φ, respectively. Since these distance functions all yield complete locally compact metric spaces, the Hopf-Rinow theorem ensures we get geodesic spaces: all points can be joined by geodesics. While we do not study these distances per se, we are interested in their geodesics which we call, respectively, *hyperbolic, quasi-hyperbolic, Kulkarni-Pinkall and Ferrand* geodesics.

2C. Quasi-hyperbolic metrics. As alluded to above, for proper subdomains $\Omega \subsetneq \mathbf{C}$, the so-called *quasi-hyperbolic metric* is given by $|dz|/\delta(z)$. This metric has proven useful in many areas of modern analysis. We mention only the interesting articles by Martin and Osgood [**9**] and Koskela [**7**]; see these for additional references.

The quasi-hyperbolic metric in the punctured plane $\mathbf{C}^*$ is simply $|dz|/|z|$, which classically was called the logarithmic metric. Employing an auxiliary Möbius transformation, we can define a Möbius invariant analog of this metric in the region $\widehat{\mathbf{C}}_{ab}$ by

$$\tau_{ab}(z) = \frac{|a-b|}{|z-a||z-b|},$$

with the standard interpretation if one of a or b is the point at infinity (in which case τ_{ab} reduces to the quasi-hyperbolic metric on the appropriate punctured plane).

The density for the *Ferrand metric* $\varphi_\Omega(z)|dz|$, introduced in [**3**], is given for $z \in \Omega \cap \mathbf{C}$ by

$$\varphi(z) = \varphi_\Omega(z) := \sup_{a,b \in \Omega^c} \tau_{ab}(z).$$

In fact, there exist points $a, b \in \partial\Omega$ such that $\varphi(z) = \tau_{ab}(z)$; cf. Proposition 4.4.

2D. Hyperbolic metric. When $\Omega \subset \widehat{\mathbf{C}}$ has at least three boundary points, usually dubbed a *hyperbolic domain*, there exists a universal covering projection $p : \mathbf{D} \to \Omega$ and the density $\lambda = \lambda_\Omega$ of the *Poincaré hyperbolic metric* $\lambda_\Omega(z)|dz|$ is determined from

$$\lambda(z) = \lambda_\Omega(z) = \lambda(p(\zeta)) := 2(1 - |\zeta|^2)^{-1}|p'(\zeta)|^{-1};$$

of course, this is only valid at points $z \in \Omega \cap \mathbf{C}$. Alternatively, $\lambda(z)|dz|$ is the unique metric on Ω which enjoys the property that its pullback $p^*[\lambda(z)|dz|]$ is the hyperbolic metric on $\mathbf{D}$. Yet another description is that $\lambda(z)|dz|$ is the maximal constant curvature -1 metric on Ω.

We remind the reader that the hyperbolic geodesic lines in Euclidean disks and half planes are subarcs of circles and lines orthogonal to the domain's boundary.

Here is a perhaps surprising extension of Schwarz's lemma. A proof can be modeled on the argument for [**10**, Theorem 1].

2.1. Fact. *Let Ω and Ω' be hyperbolic regions. Assume that f is holomorphic in some neighborhood of $a \in \Omega$ and takes values in Ω'. Suppose that for all z near a, $f^*[\lambda'](z) = \lambda'(f(z))|f'(z)| \le \lambda(z)$ with equality holding at $z = a$. Then $f : \Omega \to \Omega'$ is a holomorphic covering projection; in particular, $f^*[\lambda'] = \lambda$.*

We require the following 'folklore' information. See, for example, [**2**, Theorem 4.1, page 163] and note that any holomorphic covering $\mathbf{C} \to \Omega \subset \widehat{\mathbf{C}}$ must be the exponential followed by a Möbius transformation. See subsection 2.F for the definition of Gaussian curvature.

2.2. Fact. *Let $\rho(z)|dz|$ be a conformal metric on some quasi-hyperbolic region Ω on $\widehat{\mathbf{C}}$. Suppose that $\rho(z)|dz|$ is complete and has constant Gaussian curvature $\mathcal{K}_\rho = k$ throughout Ω. Then either $k = 0$ and Ω is a twice punctured sphere $\widehat{\mathbf{C}}_{ab}$ with $\rho = c\,\tau_{ab}$ for some constant $c > 0$, or $k < 0$, Ω is a hyperbolic region, and $\rho = (-k)^{-1/2}\lambda$.*

The following hyperbolic geometric information will be useful.

2.3. Lemma. *The hyperbolic geodesic line in $\mathbf{D}$ with Euclidean midpoint $x \in [0, 1)$ has endpoints*

$$e^{-i\theta},\ e^{i\theta} = \frac{x+i}{1+ix} = \frac{2x+i(1-x^2)}{1+x^2}$$

and

$$1-x \le |x-e^{i\theta}| \le \sqrt{2}\,(1-x).$$

Consequently, if Γ *is a hyperbolic geodesic line in some disk* D*,* $z \in |\Gamma|$*, and* ζ *is an endpoint of* Γ *closest to* z*, then*

$$|z-\zeta| \le \sqrt{2}\,\delta(z), \quad \textit{where } \delta(z) = \operatorname{dist}(z, \partial D).$$

Proof. The Möbius transformation $w = T(z) = (z+x)/(1+xz)$ is a hyperbolic isometry of $\mathbf{D}$ which maps the Euclidean segment $(-i, i)$ to the hyperbolic geodesic line in $\mathbf{D}$ with Euclidean midpoint x, so $e^{i\theta} = T(i)$. An easy calculation reveals that

$$|x-e^{i\theta}| = (1-x)f(x) \quad \text{where } f(x) = \frac{1+x}{\sqrt{1+x^2}}.$$

The distance inequalities now follow since f is increasing with $f(0) = 1$ and $f(1) = \sqrt{2}$.

It remains to confirm the last assertion. Suppose $z \in |\Gamma|$ with ζ an endpoint of the hyperbolic geodesic line Γ closest to z. Consider the hyperbolic geodesic line with Euclidean midpoint z, and let η be one of its endpoints. According to what was just proved,

$$|z-\zeta| \le |z-\eta| \le \sqrt{2}\,\delta(z)$$

as desired. $\square$

We require the following estimate concerning the values of certain hyperbolic metric-densities.

2.4. Lemma. *Given* $x \ge 0$*, choose* $r = r(x) > x$ *so that the hyperbolic geodesic line in* $D(0;r)$ *with Euclidean midpoint* x *has endpoints* $\xi, \bar{\xi}$ *which satisfy* $|x-\xi| = 1$*. Then as* x *increases, the values of* $\lambda_{D(0;r)}(x)$ *decrease from* 2 *to* $\sqrt{2}$*.*

Proof. Using Lemma 2.3 we find that

$$\xi = re^{i\theta} = r\,\frac{2(x/r) + i(1-(x/r)^2)}{1+(x/r)^2} = r\,\frac{2xr + i(r^2 - x^2)}{r^2+x^2};$$

the requirement $|x - \xi| = 1$ yields $r^2 + x^2 = |x + ir|(r^2 - x^2)$, whence $r^2 + x^2 = (r^2 - x^2)^2$. From this we calculate $r^2 = x^2 + 1/2 + (2x^2 + (1/4))^{1/2}$, so $(r/x)^2 = f(1/2x^2)$ where $f(t) = 1 + t + \sqrt{4t + t^2}$. Note that f monotonically increases from $f(0) = 1$ to ∞ as $t \to \infty$. Finally, $\lambda_{D(0;r)}(x) = 2r/(r^2 - x^2) = 2r[r^2 + x^2]^{-1/2} = 2[1 + (x/r)^2]^{-1/2} = 2[1 + f(1/2x^2)^{-1}]^{-1/2}$. $\square$

2E. Kulkarni-Pinkall metric. The density for the *Kulkarni-Pinkall metric* $\mu_\Omega(z)|dz|$ can be defined for points $z \in \Omega \cap \mathbf{C}$ as

$$\mu(z) = \mu_\Omega(z) := \inf\left\{\lambda_D(z) : z \in D \subset \Omega, D \text{ is a disk on } \widehat{\mathbf{C}}\right\}.$$

We follow the standard convention that a disk in $\widehat{\mathbf{C}}$ is either an open Euclidean disk, an Euclidean half-plane, or the complement of a closed Euclidean disk together with the point at infinity. Clearly, $\mu_\Omega(z)|dz|$ is defined (and positive) for any quasi-hyperbolic domain $\Omega \subset \widehat{\mathbf{C}}$. The 'infimum' in this definition can be replaced by 'minimum,' see [**5**]. This metric enjoys the usual domain monotonicity property, is Möbius invariant and complete and bilipschitz equivalent to the quasi-hyperbolic metric. For precise statements of these results, along with various other useful facts, we refer the interested reader to [**5, 6**] and/or [**8**]; but, see below as well.

Notice that for the twice punctured sphere $\widehat{\mathbf{C}}_{ab}$ we have

$$\mu(z) = \varphi(z) = \tau_{ab}(z) \quad \text{for all } z \in \widehat{\mathbf{C}}_{ab}.$$

For each $z \in \Omega$ (a quasihyperbolic domain in $\widehat{\mathbf{C}}$) there is an associated unique *extremal disk* $\Delta = \Delta(z) = \Delta_\Omega(z) \subset \Omega$ with the property that

$$\mu_\Omega(z) = \lambda_\Delta(z).$$

The extremal disk $\Delta = \Delta(z)$ is either an open Euclidean disk, an open half-plane, or the exterior of a closed disk, and is characterized by the property that $K = K(z) = \partial\Delta \cap \partial\Omega$ contains two or more points and z belongs to the so-called *hyperbolically convex hull* $\widehat{K}$ (of K in Δ) defined by

$$\widehat{K} = \bigcap \{H : H \subset \Delta,\ \text{the spherical closure of } H \text{ contains } K\};$$

here H is a closed (relative to hyperbolic geometry on Δ) hyperbolic half-plane in Δ. (As examples: (i) If $K = \{a, b\}$, $\widehat{K}$ is the hyperbolic geodesic in Δ ending at a and b. (ii) If $K = \{a, b, c\}$, $\widehat{K}$ is the closed ideal hyperbolic triangle in Δ with vertices a, b, c. (iii) If K contains n points, $n \geq 3$, then $\widehat{K}$ is the closed ideal hyperbolic n-gon in Δ with vertices at the points of K. (iv) When $K = \partial\Delta$, we let $\widehat{K} = \Delta$.) Moreover, it turns out that such a disk Δ is the extremal disk for each point of $\widehat{K}$.

Given an extremal disk Δ and $K = \partial\Delta \cap \partial\Omega$, we write $\mathrm{Bd}\,(\widehat{K}) = \partial\widehat{K} \cap \Omega$. Thus, $\mathrm{Bd}\,(\widehat{K})$ is a union of hyperbolic geodesic lines in Δ (therefore circular geodesics in Ω) having endpoints in K; of course, there may be such a geodesic in $\mathrm{Int}\,(\widehat{K})$ which is not in $\mathrm{Bd}\,(\widehat{K})$. Note that when K consists of precisely two points, $\mathrm{Bd}\,(\widehat{K}) = \widehat{K}$ is precisely the circular geodesic in Ω with these endpoints.

For convenience, below we collect some useful information regarding extremal disks. We call a point of $\partial\Omega$ an *extremal boundary point* if it lies on the boundary of some extremal disk in Ω. Each $z \in \Omega$ has at least two associated extremal boundary points, namely the points of $K(z) = \partial\Delta(z) \cap \partial\Omega$. Note too that the endpoints of each circular geodesic are extremal boundary points. For detailed information and proofs regarding extremal disks, we refer to [**5**]. In particular, Theorems 3.4 and 4.2 therein provide explicit descriptions for the extremal disks (and formulae for the Kulkarni-Pinkall metric thereof) in the regions obtained by puncturing the Riemann sphere at two and three points, respectively.

2.5. Proposition. *Let Ω be a quasi-hyperbolic domain in $\widehat{\mathbf{C}}$. Then*:

(a) *For all $z \in \Omega$ there is a unique extremal disk $\Delta = \Delta(z) = \Delta_\Omega(z) \subset \Omega$ with the property that $\mu_\Omega(z) = \lambda_\Delta(z)$, $K = K(z) = \partial\Delta \cap \partial\Omega$ contains two or more points, and $z \in \widehat{K}$.*

(b) *Each disk* $\Delta \subset \Omega$ *with* $K = \partial\Delta \cap \partial\Omega$ *containing at least two points is the extremal disk for every point* $z \in \widehat{K}$.

(c) *Suppose* $\Delta = \Delta(z) \subset \mathbf{C}$, *and let* Γ *be any hyperbolic geodesic line in* Δ *with Euclidean midpoint* z (Γ *is unique unless* z *is the center of* Δ). *Then either both endpoints of* Γ *belong to* $K = K(z)$, *or* K *contains a point in each of the components of* $\partial\Delta \setminus |\overline{\Gamma}|$.

(d) *If* $\Delta(z)$ *is the Euclidean disk* $D(c; r)$, *then* $r \geq \delta(z)$.

(e) *The extremal boundary points for* Ω *are dense in* $\partial\Omega$.

Proof. Parts (a) and (b) can be found in [**5**, Theorems 3.5, 4.1, 4.6]; (c) is a consequence of $z \in \widehat{K}$. To see that (d) holds, we note that

$$\frac{2}{\delta(z)} \geq \mu(z) = \lambda_\Delta(z) \geq \lambda_\Delta(c) = \frac{2}{r}.$$

It remains to verify (e).

We start with an arbitrary point $\eta \in \partial\Omega$. Since Möbius transformations preserve extremal boundary points, we may suppose $\eta \in \partial\Omega \cap \mathbf{C}$. There are 'closest boundary points' arbitrarily close to η; if one of these is an extremal boundary point, then we are done. Thus, we assume $\zeta \in \partial\Omega$ is near η with ζ nonextremal, and ζ a closest boundary point for some $z \in \Omega$, i.e., $\delta(z) = |z - \zeta|$. Making an affine change of variables, we further assume that $z = 0$ and $\zeta = 1$.

We claim that for all $a \in [0, 1)$ there is an $x \in [a, 1)$ such that x lies on a circular geodesic in Ω. If $a \in \operatorname{Bd}\widehat{K}(a)$, we can just take $x = a$; otherwise, since $\zeta = 1 \notin K(a)$, we must have $(a, 1) \cap \operatorname{Bd}\widehat{K}(a) \neq \varnothing$.

Now we confirm that there are extremal boundary points arbitrarily close to ζ. Let $\varepsilon > 0$, and select $x \in [1 - \varepsilon/3, 1)$ with $x \in |\Gamma|$ for some circular geodesic Γ in Ω. Let ξ be an endpoint of Γ closest to x. Since $\Delta = \Delta(x) \subset \Omega$, $\delta_\Delta(x) \leq \delta(x) = |x - \zeta| = 1 - x \leq \varepsilon/3$. Thanks to Lemma 2.3, we now obtain

$$|x - \xi| \leq \sqrt{2}\,\delta_\Delta(x) \leq (\sqrt{2}/3)\varepsilon,$$

and thus $|\xi - \zeta| \leq |x - \xi| + |x - \zeta| < \varepsilon$. $\square$

Note that one consequence of (a) and (b) above is that for all points $z_1, z_2 \in \Omega$, the sets $K_i = K(z_i)$, and also $\widehat{K}_i$, are either disjoint or identical. In particular, the sets $\widehat{K}$ form a partition of Ω into disjoint relatively closed 2-cells and 1-cells. A similar statement about extremal disks is false; however, we can say something about how the extremal circles intersect.

2.6. Proposition. *Let Ω be a quasi-hyperbolic domain in $\widehat{\mathbf{C}}$, fix a point $a \in \Omega$, and put $\Delta = \Delta(a)$, $K = K(a)$. (Then each component of $\partial\Delta \setminus K$ is an open arc whose closure is the 'outer' boundary of a corresponding component of $\Delta \setminus \widehat{K}$.) Let U be a component of $\Delta \setminus \widehat{K}$. For all $z \in U$, the arc $A = \partial\Delta(z) \cap \Delta$ separates z from $\partial\Delta \setminus \overline{U}$ in Δ and has endpoints on $\partial\Delta \cap \overline{U}$.*

Proof. Using a preliminary Möbius transformation, we assume $a = 0$, $\Delta = \mathbf{D}$, $\pm i \in K$, and $U = \mathbf{D} \cap \mathbf{H}$. Since $\partial\mathbf{D}$ and $\partial\Delta(z)$ are distinct circles, they have 0, 1, or 2 points of intersection. Since $z \in \mathbf{D}$ and $\pm i \in \partial\Omega$, it is not difficult to see that neither of the first two cases can arise, so $\partial\mathbf{D} \cap \partial\Delta(z) = \{\xi, \eta\}$ for some $\xi \neq \eta$.

Let us first check that $A = \partial\Delta(z) \cap \mathbf{D}$ (the subarc of $\partial\Delta(z)$ between ξ, η and inside $\mathbf{D}$) separates z from $C = \partial\mathbf{D} \setminus \overline{U}$ in $\mathbf{D}$. Suppose that A does not separate z, C in $\mathbf{D}$. Now, if one or both of ξ, η lies in $\mathbf{H}$, then we see that one of $\pm i$ lies in $\Delta(z)$ which cannot happen. Thus, both points ξ, η must lie in the closed left half-plane. We claim that this contradicts $z \in \widehat{K}(z)$. Indeed, a careful examination of the hyperbolic geodesics joining ξ and η, one in $\mathbf{D}$ and one in $\Delta(z)$, reveals that the $\mathbf{D}$ hyperbolic geodesic separates z from the $\Delta(z)$ hyperbolic geodesic. In particular, we see that the hyperbolic half-plane H in $\Delta(z)$, determined by the hyperbolic geodesic joining ξ and η, and not containing z, enjoys the property that $H \supset K(z)$. But then $\widehat{K}(z) \subset \overline{H}$, yet $z \notin \overline{H}$.

We conclude that A does separate z from C in $\mathbf{D}$. Once again, we see that if both ξ, η fail to lie in $\overline{\mathbf{H}}$, then one of $\pm i$ belongs to $\Delta(z)$ which cannot happen. Thus, both endpoints ξ, η of A must lie on $\partial\mathbf{D} \cap \overline{\mathbf{H}}$ as asserted. □

Next we record the following easy proof that the metric $\mu(z)|dz|$ is smooth. Kulkarni and Pinkhall assert that the metric is $\mathcal{C}^{1,1}$; see [**7**, page 105].

2.7. Proposition. *The Kulkarni-Pinkhall metric is continuously differentiable. Moreover, given $a \in \Omega$, $\Delta = \Delta(a)$, $K = K(a)$, we have for all $z \in \widehat{K} \cap \mathbf{C}$:*

$$\mu(z) = \lambda_\Delta(z) \quad \textit{and} \quad D\mu(z) = D\lambda_\Delta(z),$$

where $D\mu$ denotes the derivative of μ.

Proof. It suffices to examine the metric near a point $a \in \Omega \cap \Delta$. If $a \in \operatorname{Int}(\widehat{K})$ ($K = K(a) = \partial\Delta \cap \partial\Omega$, $\Delta = \Delta(a)$) then $\mu(z) = \lambda_\Delta(z)$ for all z in an open neighborhood of a; therefore, μ is in fact real-analytic near such points.

Thus, we may assume that $a \in \operatorname{Bd}(\widehat{K})$. In this situation, a lies on one of the circular geodesics forming the boundary of $\widehat{K}$, so by employing a preliminary Möbius transformation, we may further assume that $a = 0$, $\Delta = \mathbf{D}$ and $\pm 1 \in K$. Then $\mathbf{D} \subset \Omega \subset \widehat{\mathbf{C}}_{-11}$, so by domain monotonicity we obtain $2/|z^2 - 1| \le \mu(z) \le 2/(1 - |z|^2)$, which holds for all points $z \in \mathbf{D}$ with equality for $z = x \in (-1, 1)$. Continuity at $z = 0$ is now evident.

In fact we have $g(z) = 4/|z^2 - 1|^2 \le \mu(z)^2 \le 4/(1 - |z|^2)^2 = h(z)$. It is easy to check that the tangent planes for both g and h at $z = 0$ are the horizontal plane at height 4. This says that μ^2, and hence μ, is differentiable at $z = 0$.

Notice that g, h–which are real analytic–are symmetric about $(-1, 1)$; thus, $\partial g/\partial y = 0 = \partial h/\partial y$ along $(-1, 1)$. Since $g \le \mu^2 \le h$ everywhere in $\mathbf{D}$ with equality along $(-1, 1)$, we deduce that μ is differentiable at each point of $(-1, 1)$ and $Dg = D(\mu^2) = Dh$ on $(-1, 1)$. Our formula for $D\mu$, valid in $\widehat{K}$, now follows.

It remains to see that $D\mu$ is continuous, at the point a. It is clear that $D\mu(z) \to D\mu(a)$ when $z \to a$ with $z \in \operatorname{Int}(\widehat{K})$. Using our formula for $D\mu$ at points $z \notin \operatorname{Int}(\widehat{K})$, and the fact that the extremal disks $\Delta(z)$ vary continuously, see Proposition 2.9, we conclude again that $D\mu(z) \to D\mu(a)$ as $z \to a$. $\square$

There is an invariant way to understand the above. Suppose $a \in |\Gamma|$ with Γ a circular geodesic in Ω (so, a hyperbolic geodesic in some extremal disk Δ) and having endpoints $\xi, \eta \in \partial\Omega$. Then $\Delta \subset \Omega \subset \widehat{\mathbf{C}}_{\xi\eta}$, so

$$\tau_{\xi\eta}(z) \le \mu(z) \le \lambda_\Delta(z) \text{ for all } z \in \Delta \text{ with equality on } \quad \Gamma.$$

Thus, the tangential derivatives of $\tau_{\xi\eta}$, μ, λ_Δ along Γ all exist and are equal. Since $\tau_{\xi\eta}$ and λ_Δ are symmetric about Γ, the normal derivatives, along Γ, of these three metrics also exist and vanish there.

It is known that $\mu(z) \le 2/\delta(z)$ with equality if and only if $\Delta(z) = D(z)$. Moreover, there is even a geometric characterization for when the supremum of $\mu\delta$ is strictly less than 2. See [**6**, Theorems 2.1, 2.2]. Here we provide quantitative estimates demonstrating that when $\mu(z)\delta(z)$ is close to 2, the extremal disk $\Delta(z)$ is close to the maximal disk $D(z)$.

2.8. Proposition. *Let Ω be a quasi-hyperbolic region on $\widehat{\mathbf{C}}$. For each $\varepsilon > 0$ there exists $\vartheta \in (0,1)$ such that for all $z \in \Omega \cap \mathbf{C}$: if $\mu(z)\delta(z) \ge 2\vartheta$, then $\Delta(z) = D(c;r)$ is an Euclidean disk with $|z-c| \le \varepsilon\,\delta(z)$ and $\delta(z) \le r \le (1+\varepsilon)\delta(z)$.*

Proof. According to [**6**, Theorem 2.1], we know that $\Delta(z)$ must be an Euclidean disk whenever $\mu(z)\delta(z) > \sqrt{2}$. Recall from Lemma 2.4 that $\lambda_{D(0;t)}(x)$ decreases from 2 to $\sqrt{2}$ as x increases: here $t > x \ge 0$ with $|\xi| = t$, $|x-\xi| = 1$ and x is the Euclidean midpoint of the hyperbolic geodesic line in $D(0;t)$ with endpoints $\xi, \bar{\xi}$. Let $\varepsilon > 0$ be given and select $\tau \in (0,1)$ so that $\lambda_{D(0;t)}(x) \ge 2\tau$ implies $0 \le x < \varepsilon$.

Put $\vartheta = \max\{(1+\varepsilon)^{-1}, \tau, 0.8\}$ and suppose $a \in \Omega \cap \mathbf{C}$ satisfies $\mu(a)\delta(a) \ge 2\vartheta$. Since $2\vartheta \ge 1.6 > \sqrt{2}$, $\Delta(a)$ is an Euclidean disk. Using the affine change of variables $w = (z-a)/\delta(a)$, followed by a rotation if necessary, we may assume that $a = 0$, $D(0) = \mathbf{D}$, and $\Delta = \Delta(0) = D(c;r)$ with $c \ge 0$. Of course, $r \ge c$. By Proposition 2.5(d), $r \ge \delta(0) = 1$. Also, $2\vartheta \le \mu(0) = \lambda_\Delta(0) \le 2/(r-c)$, so $r \le (1/\vartheta) + c \le 1 + \varepsilon + c$. Thus $1 \le r \le 1 + \varepsilon + c$, and therefore it remains to check that $c \le \varepsilon$.

We claim that there is a point $x \in \Delta$ with $x \le 0$ and such that the hyperbolic geodesic line in Δ with Euclidean midpoint x has endpoints $\xi, \bar{\xi} \in \partial\Delta$ satisfying $|x-\xi| = 1$. Assuming this, and noting that 0 lies between x and c, we deduce that

$$\lambda_\Delta(x) \geq \lambda_\Delta(0) = \mu(0) = \mu(0)\delta(0) \geq 2\vartheta \geq 2\tau.$$

Because of our choice of τ, this means that $c \leq c - x \leq \varepsilon$ as desired.

To see that our claim holds, consider the hyperbolic geodesic line Γ in Δ with Euclidean midpoint 0. According to Proposition 2.5(c), either both endpoints of Γ belong to $K = \partial\Delta \cap \partial\Omega$ or K contains a point in each of the components of $\partial\Delta \setminus |\overline{\Gamma}|$. Since $\mathbf{D} \subset \Omega$, it follows that the endpoints of Γ must lie outside $\mathbf{D}$ (although possibly on $\partial\mathbf{D}$). The existence of the point x as described above should now be transparent. $\square$

Next we confirm that extremal disks vary continuously. Note that for points $a \in \operatorname{Int}(\widehat{K})$, $K = \partial\Delta \cap \partial\Omega$ for some extremal disk $\Delta \subset \Omega$, we have $\Delta(z) = \Delta$ for all z near a, so we actually only need to examine points $a \in \operatorname{Bd}(\widehat{K})$.

2.9. Proposition. *Let Ω be a quasi-hyperbolic region on $\widehat{\mathbf{C}}$. Then as $z \to a \in \Omega$, $\Delta(z) \to \Delta(a)$. (The latter convergence means $\operatorname{dist}_{\mathcal{H}}(\bar{\Delta}(z), \bar{\Delta}(a)) \to 0$ where $\operatorname{dist}_{\mathcal{H}}$ denotes spherical Hausdorff distance for closed subsets of $\widehat{\mathbf{C}}$.)*

Proof. Fix a point $a \in \Omega$. By performing a preliminary Möbius transformation, we may assume that $a = 0$ and $\Delta(0) = \mathbf{D}$. Thus, $\mu(0)\delta(0) = \mu(0) = \lambda_{\mathbf{D}}(0) = 2$. Now as $z \to a = 0$ we know that $\delta(z) \to \delta(0) = 1$ and $\mu(z) \to \mu(0) = 2$, so $\mu(z)\delta(z) \to 2$. According to Proposition 2.8, for such z we have $\Delta(z) = D(c(z); r(z))$ with $c(z) \to 0$ and $r(z) \to 1$. $\square$

Armed with the information from above, we now examine how the sets $K(z)$ of extremal boundary points vary. Elementary examples reveal that it is not true that $K(z) \to K(a)$ as $z \to a$. However, it is easy to see that the points of $K(z)$ will accumulate at points of $K(a)$.

2.10. Lemma. *Let $\langle z_n \rangle$ be a sequence of points in a quasi-hyperbolic region $\Omega \subset \widehat{\mathbf{C}}$ converging to $a \in \Omega$. Then any sequence of points $\zeta_n \in K(z_n)$ subconverges to some point of $K(a)$, and every subsequential limit point belongs to $K(a)$.*

Proof. Since $\partial\Omega$ is a closed subset of the compact $\widehat{\mathbf{C}}$, any sequence $\langle\zeta_n\rangle$ of points in $\partial\Omega$ subconverges to some point of $\partial\Omega$. When $\zeta_n \in \partial\Delta(z_n) \to \partial\Delta(a)$, any such limit point must also belong to $\partial\Delta(a)$, hence to $K(a)$. (Alternatively, we may assume that $a = 0$, $\Delta(a) = \mathbf{D}$ and $\Delta(z_n) = D(c_n; r_n)$ with $c_n \to 0$ and $r_n \to 1$. Thus, if $\zeta_n \to \zeta$, then $\zeta \in \partial\Omega$ and also $|\zeta| = \lim|\zeta_n - c_n| = \lim r_n = 1$.) □

We continue our analysis of how the sets $K(z)$ vary. Obviously, when $a \in \operatorname{Int}(\widehat{K}(a))$, then (by Proposition 2.5(b) we have) $K(z) = K(a)$ for all z close to a. Thus, we need only examine what happens for z near a point $a \in \operatorname{Bd}(\widehat{K}(a))$.

2.11. Proposition. *Let Ω be a quasi-hyperbolic region on $\widehat{\mathbf{C}}$. Fix $a \in \Omega$, and put $\Delta = \Delta(a)$, $K = K(a)$. Suppose that $a \in |\Gamma| \subset Bd(\widehat{K})$ where Γ is a circular geodesic with endpoints $\xi, \eta \in K$. Then, given $\varepsilon > 0$, there is a $\delta > 0$ such that for $z \in D(a;\delta) \setminus \widehat{K}(a)$, $K(z) \subset D(\xi;\varepsilon) \cup D(\eta;\varepsilon)$ and $K(z) \cap D(\xi;\varepsilon) \neq \varnothing \neq K(z) \cap D(\eta;\varepsilon)$.*

Proof. We assume $a = 0$, $\Delta(a) = \mathbf{D}$, $|\Gamma| = (-i, i)$, i.e., $\xi = -i$ and $\eta = i$, and that $\widehat{K} \subset \mathbf{D} \setminus \mathbf{H}$. Recall from Proposition 2.9 that as $z \to 0$ we get $\Delta(z) = D(c; r)$ with $c \to 0$ and $r \to 1$. Let $\varepsilon > 0$ be given.

Since $\widehat{K} \subset \mathbf{D} \setminus \mathbf{H}$, $\partial\mathbf{D} \cap \mathbf{H} \subset \Omega$. Thus, $T = \{e^{i\theta} : |\theta| \le (\pi/2) - \varepsilon\}$ is a compact subset of Ω, so $d = \operatorname{dist}(T, \partial\Omega) > 0$. Now $\partial\Delta(z) \to \partial\mathbf{D}$ (as $z \to 0$), so we can select $\delta > 0$ so that for all $|z| < \delta$, $\partial\Delta(z) \subset \{w : 1 \le |w| < 1 + t\}$ where $t = \min\{\varepsilon, d\}$. Note however that no points of $K(z) = \partial\Delta(z) \cap \partial\Omega$ can lie in the t-neighborhood of T.

Now consider a point z with $|z| < \delta$ and $z \in \mathbf{H}$. According to Proposition 2.6, the arc $A = \partial\Delta(z) \cap \mathbf{D}$ separates z from $\partial\mathbf{D} \setminus \mathbf{H}$ in $\mathbf{D}$ and moreover has endpoints in $\mathbf{H}$. Thus, $\partial\Delta(z) \setminus \overline{\mathbf{H}} \subset \mathbf{D} \subset \Omega$, and therefore $K(z) = \partial\Delta(z) \cap \partial\Omega \subset \overline{\mathbf{H}}$. Combining this with the facts from the prior paragraph (that $K(z)$ lies in $1 \le |w| < t$ but has no points in the t-neighborhood of T) we see that $K(z) \subset D(-i;\varepsilon) \cup D(i;\varepsilon)$ as desired. In this setting, with $z \in \mathbf{H}$, $K(z)$ must contain points in each of the small disks because $z \in \widehat{K}(z)$.

It is possible that $K = K(a) = \{i, -i\}$ in which case $\widehat{K} = |\Gamma|$. In this situation we must also consider points z with $|z| < \delta$ and $\Re(z) < 0$.

Here we have $\partial\mathbf{D} \setminus \{i, -i\} \subset \Omega$, so we can argue as above replacing T with the compact set $T \cup T' \subset \Omega$ where $T' = \{-\zeta : \zeta \in T\}$. $\square$

One might conjecture that more can be said about the sets $K(z)$ when z is near a point a as above: For example, could it be that each such $K(z)$ contains precisely two extremal boundary points? The following example illustrates that, at least from a cardinality perspective, such conjectures are false. We start by putting $a_n = 1/2^n$, $r_n = 1 + 1/10^n$, and $\Delta_n = D(a_n; r_n)$; here we take $n = 4, 5, \dots$. Then $1 < r_n < |a_n \pm i|$ and $r_n \to 1$. Straightforward calculations reveal that $a_n \pm i\, r_n \notin \bar{\Delta}_{n+1} \cup \bar{\Delta}_{n-1}$. Hence, there are subarcs α_n, $\beta_n = \bar{\alpha}_n$ of $\partial\Delta_n \setminus (\bar{\Delta}_{n+1} \cup \bar{\Delta}_{n-1})$; e.g., α_n can be the component which joins $\partial\Delta_{n+1}$ to $\partial\Delta_{n-1}$ and contains $a_n + i\, r_n$ and β_n its reflection across the real axis. Finally, we let $\Omega = \mathbf{C} \setminus [\{x \pm i : x \leq 0\} \cup \cup_{n \geq 4}(\alpha_n \cup \beta_n)]$. Then $a = 0$ has $\widehat{K}(a) = \mathbf{D} \setminus \mathbf{H}$, $K(a) = \{i, -i\}$, and there exist points $z_n \to a$ with $z_n > 0$ and such that $K(z_n) = \alpha_n \cup \beta_n$.

2F. Gaussian curvature. Recall that the *Gaussian curvature* of a $\mathcal{C}^2$ conformal metric $\rho(z)|dz|$ can be calculated via

$$\mathcal{K}_\rho(z) = -\rho^{-2}(z)\Delta \log \rho(z),$$

where Δ is the usual Laplacian operator. When u is $\mathcal{C}^2$ in a neighborhood of a,

$$\Delta u(a) = \lim_{r\to 0} \frac{4}{r^2}\left[\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta - u(a)\right].$$

Heins [**4**] defined the *upper and lower Gaussian curvatures* of a continuous, or upper semi-continuous metric, $\rho(z)|dz|$ by

$$\overline{\mathcal{K}}_\rho(a) = -\rho^{-2}(a)\liminf_{r\to 0} \frac{4}{r^2}\left[\frac{1}{2\pi}\int_0^{2\pi} \log \rho(a + re^{i\theta})\, d\theta - \log \rho(a)\right]$$

and

$$\overline{\mathcal{K}}_\rho(a) = -\rho^{-2}(a)\limsup_{r\to 0} \frac{4}{r^2}\left[\frac{1}{2\pi}\int_0^{2\pi} \log \rho(a + re^{i\theta})\, d\theta - \log \rho(a)\right].$$

When ρ is $\mathcal{C}^2$ in a neighborhood of a, $\overline{\mathcal{K}}_\rho(a) = \mathcal{K}_\rho(a) = \underline{\mathcal{K}}_\rho(a)$ is just the (ordinary) Gaussian curvature of $\rho(z)|dz|$ at $z = a$. In general, $\underline{\mathcal{K}}_\rho(z) \le \overline{\mathcal{K}}_\rho(z)$.

As a simple example, we mention that $\mathcal{K}_{\tau_{ab}} = 0$. Additional examples are presented in Section 3.

Next we prove a comparison lemma.

2.12. Lemma. *Let ρ and σ be metric densities on Ω. Suppose $\rho \le \sigma$ in a neighborhood of some point $a \in \Omega \cap \mathbf{C}$ with $\rho(a) = \sigma(a)$. Then*

$$\overline{\mathcal{K}}_\sigma(a) \le \overline{\mathcal{K}}_\rho(a) \quad \textit{and} \quad \underline{\mathcal{K}}_\sigma(a) \le \underline{\mathcal{K}}_\rho(a).$$

Proof. For all sufficiently small $r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\rho(a+re^{i\theta})\,d\theta - \log\rho(a) \le \frac{1}{2\pi}\int_0^{2\pi} \log\sigma(a+re^{i\theta})\,d\theta - \log\sigma(a),$$

so

$$\liminf_{r\to 0} \frac{2}{\pi r^2}\left[\int_0^{2\pi} \log\rho(a+re^{i\theta})\,d\theta - \log\rho(a)\right] \le \liminf_{r\to 0} \frac{2}{\pi r^2}\left[\int_0^{2\pi} \log\sigma(a+re^{i\theta})\,d\theta - \log\sigma(a)\right].$$

Therefore, $-\overline{\mathcal{K}}_\rho(a) \le -\overline{\mathcal{K}}_\sigma(a)$. The proof of the second inequality is similar. $\square$

3. Examples. Here we mention a few special examples where one can calculate the hyperbolic, Kulkarni-Pinkall and Ferrand metrics as well as their curvatures. We also compute associated universal holomorphic covering maps (from the right half-plane $\mathbf{H}$ to the region) and indicate certain special hyperbolic geodesics. The reader should note that these examples reveal information about all Möbius images of these special regions.

Of course for any disk on $\widehat{\mathbf{C}}$ we know that these metrics agree with the hyperbolic metric, the geodesics are subarcs of circles orthogonal to the boundary, and any covering map is a Möbius transformation. Our first four examples share the property that $\varphi = \mu$.

It is also of interest to consider the Gaussian curvature of the quasi-hyperbolic metric in these examples. For the infinite strip and infinite sectors this is identically -1 except along the 'center-line' (the so-called centered points which have two or more closest boundary points) where it is identically $-\infty$, see [**9**, Corollary 3.12]. For the punctured disk it is 0 in $0 < |z| < 1/2$, $-\infty$ on $|z| = 1/2$, and $-1/|z|$ for $1/2 < |z| < 1$.

3A. The strip $\mathcal{S}_0$. For the infinite strip $\mathcal{S}_0 := \{x + iy : |y| < \pi/2\}$,

$$\lambda(x+iy) = \sec(y) \quad \text{and} \quad \mu(x+iy) = \frac{\pi}{(\pi/2)^2 - y^2}.$$

A conformal map $f : \mathbf{H} \to \mathcal{S}_0$ is given by $f(\zeta) = \operatorname{Log}(\zeta)$ (the principal branch of the logarithm). Semi-circles centered at the origin in $\mathbf{H}$ are mapped to vertical segments in $\mathcal{S}_0$, so these are (circular) hyperbolic geodesics in $\mathcal{S}_0$.

A straightforward calculation reveals that $\mathcal{K}_\mu(x+iy) = -2(y/\pi)^2 - 1/2$. From this we see that $-1 < \mathcal{K}_\mu(x+iy) \le -1/2$ with equality for $y = 0$ and with $\mathcal{K}_\mu(x+iy) \to -1$ as $|y| \to \pi/2$.

3B. Infinite sectors S_t. Next, we consider the infinite sectors

$$\mathcal{S}_t := \{re^{i\theta} : r > 0, |\theta| < \alpha\} \quad \text{where } \alpha = \pi t/2 \text{ and } 0 < t \le 2.$$

Here $f : \mathbf{H} \to \mathcal{S}_t$, $f(\zeta) = \zeta^t$, is conformal and we have

$$\lambda(re^{i\theta}) = \frac{\sec(\theta/t)}{tr} \quad \text{and} \quad \mu(re^{i\theta}) = \frac{\sin(\alpha)}{r[\cos(\theta) - \cos(\alpha)]}.$$

The conformal change of variables $z = f(\zeta)$ gives the first formula, which holds for all $0 < t \le 2$. The *righthand formula is only valid for convex sectors*, i.e., for $0 < t \le 1$, and can be established using the fact that $re^{i\theta}$ lies on the hyperbolic geodesic line through the points $re^{\pm i\alpha}$. (See below for concave sectors.) Now semi-circles centered at the origin in $\mathbf{H}$ are mapped to circular arcs centered at the origin in $\mathcal{S}_t$; these are (circular) hyperbolic geodesics in $\mathcal{S}_t$.

To compute the Gaussian curvature of the Kulkarni-Pinkall metric we naturally employ polar coordinates and so recall that $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$. Since $\log r$ is harmonic, we obtain

$$\mathcal{K}_\mu(re^{i\theta}) = \frac{\cos(\alpha)\cos(\theta) - 1}{\sin^2(\alpha)}.$$

For $|\theta| < \alpha < \pi/2$, $\cos(\alpha) - 1 \geq \cos(\alpha)\cos(\theta) - 1 > \cos^2(\alpha) - 1 = -\sin^2(\alpha)$, and thus

$$-1 < \mathcal{K}_\mu(re^{i\theta}) = \frac{\cos(\alpha)\cos(\theta) - 1}{\sin^2(\alpha)} \leq \frac{-1}{1+\cos(\alpha)} < -\frac{1}{2}.$$

Of course $\mathcal{K}_\mu(re^{i\theta}) = -1$ when $\alpha = \pi/2$ (indeed, $\mathcal{S}_1 = \mathbf{H}$) and, as $\alpha \to 0$, the above upper bound tends to $-1/2$ (the upper bound for the infinite strip $\mathcal{S}_0$).

For a concave sector $\mathcal{S}_t$, with $1 \leq t \leq 2$, we easily see that $\mu = 1/\delta$, e.g., the extremal disk associated with each point is actually an Euclidean half-plane, cf. [**6**, Theorem 2.1(c)]. A straightforward calculation, see [**9**, Proposition 3.8], reveals that

$$\mathcal{K}_\mu(re^{i\theta}) = \mathcal{K}_{1/\delta}(re^{i\theta}) = \begin{cases} -1 & \text{for } \pi(t-1)/2 < |\theta| < \pi t/2, \\ -1/2 & \text{for } |\theta| = \pi(t-1)/2, \\ 0 & \text{for } |\theta| < \pi(t-1)/2. \end{cases}$$

3C. Punctured disk $\mathbf{D}^*$. Another simple, although important, example is the punctured unit disk $\mathbf{D}^* := \mathbf{D} \setminus \{0\}$ for which

$$\lambda(z) = \frac{1}{|z||\log|z||} \quad \text{and} \quad \mu(z) = \frac{1}{|z|(1-|z|)}.$$

In this case a holomorphic covering $f : \mathbf{H} \to \mathbf{D}^*$ is provided by $f(\zeta) = e^{-\zeta}$. We see that horizontal rays in $\mathbf{H}$ are mapped to radial segments in $\mathbf{D}^*$; of course, these are (circular) hyperbolic geodesics in $\mathbf{D}^*$.

The curvature can again be calculated using polar coordinates, and we readily find that $\mathcal{K}_\mu(re^{i\theta}) = -r$.

3D. Annuli $A(R)$. Finally, we take a look at the annulus $A(R) := \{z : 1/R < |z| < R\}$. A holomorphic covering $f : \mathbf{H} \to A(R)$ is obtained via $f(\zeta) = \zeta^{it} = e^{it\mathrm{Log}\,\zeta}$ where $\log R = \pi t/2$. Then a routine computation produces

$$\lambda_{A(R)}(z) = \frac{\pi/2}{|z|\log R}\sec\left(\frac{\pi}{2}\frac{\log|z|}{\log R}\right)$$

and

$$\mu_{A(R)}(z) = \frac{R-1/R}{(R-|z|)(|z|-1/R)}.$$

Now semi-circles centered at the origin in $\mathbf{H}$ are mapped to radial segments in $A(R)$, so these are (circular) hyperbolic geodesics.

3E. Thrice punctured sphere. In all of the above examples $\varphi = \mu$. This is also true for any twice punctured sphere (in which case both metrics reduce to a Möbius quasi-hyperbolic metric and hence are flat). For the three-times punctured sphere $\widehat{\mathbf{C}}_{-11} = \widehat{\mathbf{C}} \setminus \{-1, 1\}$, we find that for each $z \in \widehat{\mathbf{C}}_{-11}$,

$$\mu(z) = \begin{cases} 2/|z^2-1| & \text{if } z \in \overline{\mathbf{D}}, \\ 1/|z-1| & \text{if } \Re(z) \geq 1, \\ 1/|z+1| & \text{if } \Re(z) \leq -1, \\ 1/\Im(z) & \text{if } z \in \Delta, \\ -1/\Im(z) & \text{if } z \in \Delta^*, \end{cases}$$

and

$$\varphi(z) = \begin{cases} 2/|z^2-1| & \text{if } z \in \Lambda, \\ 1/|z-1| & \text{if } \Re(z) \geq 0 \text{ and } z \notin \Lambda, \\ 1/|z+1| & \text{if } \Re(z) \leq 0 \text{ and } z \notin \Lambda. \end{cases}$$

Here $\Delta = \{z : |\Re(z)| < 1,\ \Im(z) > 0, \text{and} |z| > 1\}$, Δ^* is the reflection of Δ over the real axis, and $\Lambda = \{z : |z-1| < 2 \text{ and } |z+1| < 2\}$. The calculations for μ were given in [**5**, Theorem 4.2]; the interested reader can readily verify the formula for φ.

We note that $\mathcal{K}_\mu$ is almost everywhere identically -1 or identically 0; in particular, $\mathcal{K}_\mu$ is not continuous. On the other hand, $\mathcal{K}_\varphi$ is almost everywhere identically 0, but according to [**9**, Corollary 3.12], $\mathcal{K}_\varphi = -\infty$ on the rays $\Re(z) = 0$, $|\Im(z)| \geq 1$ and also on $\partial\Lambda$ by Möbius invariance.

4. Proofs of main results. We first provide an Euclidean interpretation for the Ferrand and Kulkarni-Pinkall metrics. Then we study their geodesics, isometries and Gaussian curvatures.

4A. Euclidean eyes. Here we present a method for calculating the Ferrand and Kulkarni-Pinkall metrics based on Euclidean diameters and circumdiameters.

Recall that, for any compact set $A \subset \mathbf{C}$, there is a unique smallest closed disk $\mathcal{D}_A$ which contains A; we call $\mathcal{D}_A$ the *circumdisk* about A. Jung's theorem, see [**1**, 11.5.8, page 357], provides the following information about circumdisks.

4.1. Fact. *Let $\mathcal{D}_A = D(a; r)$ be the circumdisk about a compact set $A \subset \mathbf{C}$. Then*:

(a) *The center a of $\mathcal{D}_A$ belongs to the convex hull of $A \cap \partial\mathcal{D}_A$.*

(b) *For all subarcs $\alpha \subset \partial\mathcal{D}_A \setminus A$, $l(\alpha) \leq \pi r$.*

(c) *There exist points $b, c \in A \cap \partial\mathcal{D}_A$ such that the shorter subarc β of $\partial\mathcal{D}_A$ joining b, c has $(2\pi/3)r \leq l(\beta) \leq \pi r$.*

(d) $\operatorname{diam}(A) \leq \operatorname{diam}(\mathcal{D}_A) \leq (2/\sqrt{3})\operatorname{diam}(A)$.

It is convenient to introduce the following notation. For $z \in \mathbf{C}$, let J_z be the inversion

$$J_z(\zeta) = \frac{1}{\zeta - z}.$$

4.2. Lemma. *Fix any set $E \subset \widehat{\mathbf{C}}$ and a point $z \in \mathbf{C} \setminus E$. Then for any Möbius transformation T with $T(z) \in \mathbf{C}$,*

$$\operatorname{diam} J_z(E) = |T'(z)| \operatorname{diam} J_{T(z)}(TE).$$

Proof. Fix a point $a \in \mathbf{C} \setminus E$, and let $b = T(a)$, $E' = T(E)$. Consider $F = J_b \circ T \circ J_a^{-1}$, which is a Möbius transformation fixing the point at infinity, hence a complex linear map of the form $F(\zeta) = A\zeta + B$ for some $A, B \in \mathbf{C}$. Clearly,

$$\operatorname{diam} J_{T(a)}(TE) = \operatorname{diam} J_b(E') = \operatorname{diam} F[J_a(E)] = |A| \operatorname{diam} J_a(E);$$

therefore, it suffices to confirm that $|A| = 1/|T'(a)|$.

Writing $w = T(z)$ and $\zeta = J_a(z)$, we have

$$\zeta(w-b) = \frac{w-b}{z-a} = \frac{T(z)-T(a)}{z-a} \longrightarrow T'(a) \quad \text{as } z \to a,$$

and thus

$$A = \lim_{\zeta \to \infty} F'(\zeta) = \lim_{z \to a} \frac{T'(z)}{[\zeta(w-b)]^2} = \frac{1}{T'(a)},$$

as desired. □

4.3. Corollary. *For any disk D on $\widehat{\mathbf{C}}$, $\lambda_D(z) = \operatorname{diam} J_z(\partial D)$ for all $z \in D \cap \mathbf{C}$.*

Proof. Put $E = \partial D$, and choose a Möbius transformation T with $T(z) = 0$ and $T(D) = \mathbf{D}$. Since $J_0(\zeta) = 1/\zeta$, $J_{T(z)}(TE) = J_0(\partial \mathbf{D}) = \partial \mathbf{D}$, and thus

$$\lambda_D(z) = \lambda_{\mathbf{D}}(0)|T'(z)| = |T'(z)| \operatorname{diam} J_{T(z)}(TE) = \operatorname{diam} J_z(E),$$

as asserted. □

Now we explain how to calculate the Ferrand and Kulkarni-Pinkall metrics in Euclidean terms. Recall that $\mathcal{D}_A$ denotes the circumdisk about A.

4.4. Proposition. *Assume Ω is a quasi-hyperbolic region in $\widehat{\mathbf{C}}$. For each $z \in \Omega \cap \mathbf{C}$, let $\Omega_z = J_z(\Omega)$, $B_z = \Omega_z^c$ and $D_z = \mathcal{D}_{B_z}$. Then, for such z,*

$$\varphi(z) = \operatorname{diam}(B_z) \quad \text{and} \quad \mu(z) = \operatorname{diam}(D_z).$$

Proof. First, $w = J_z(\zeta) \in B_z = J_z(\Omega^c)$ if and only if $\zeta \in \Omega^c$. Thus, for $w_i = J_z(\zeta_i) \in B_z$,

$$|w_1 - w_2| = \frac{|\zeta_1 - \zeta_2|}{|z - \zeta_1||z - \zeta_2|} \leq \varphi(z) \quad \text{(by definition of } \varphi).$$

Since equality does hold for some pair of points ζ_1 and ζ_2, $\varphi(z) = \operatorname{diam}(B_z)$.

Next, we claim that $D = J_z^{-1}(D_z^c)$ is the extremal disk $\Delta(z)$ in Ω containing z. Clearly $z \in D \subset \Omega$, D is a disk in Ω and by Corollary 4.3 we have $\lambda_D(z) = \operatorname{diam}(D_z)$. Thus, it remains to verify that $D = \Delta(z)$.

According to Proposition 2.5, to corroborate this claim it suffices to check that $K = \partial D \cap \partial\Omega$ contains two points and $z \in \widehat{K}$. The former condition holds because $\partial D_z \cap \partial\Omega_z$ must contain two points by Fact 4.1. The latter condition is equivalent to having the point at infinity belong to $J_z(\widehat{K})$, and this is also a simple consequence of Fact 4.1. □

An immediate consequence of the above is an easy method to determine extremal disks.

4.5. Corollary. *Let Ω be a quasi-hyperbolic region in $\widehat{\mathbf{C}}$. The extremal disk $\Delta(z)$ associated with a point $z \in \Omega \cap \mathbf{C}$ is given by $\Delta(z) = T^{-1}(D^c)$ where T is any Möbius transformation mapping z to the point at infinity and $D = \mathcal{D}_{T(\Omega^c)}$.*

Another easy corollary of the above yields some of the following inequalities. For more information of this nature, we refer the interested reader to [**6**].

4.6. Corollary. *For all $z \in \Omega \cap \mathbf{C}$,*

$$\lambda(z) \leq \varphi(z) \leq \mu(z) \leq \frac{2}{\delta(z)} \quad \textit{and} \quad \mu(z) \leq \frac{2}{\sqrt{3}}\,\varphi(z);$$

the leftmost inequality requires Ω to be a hyperbolic region but the other *inequalities hold for quasi-hyperbolic regions. In addition, we note that*

$$\lambda(z) \geq \frac{1}{\delta(z)} \quad \textit{when } \Omega \subset \mathbf{C} \textit{ is a disk or half-plane},$$
$$\varphi(z) \geq \frac{1}{\delta(z)} \quad \textit{when } \Omega \subsetneq \mathbf{C},$$
$$\mu(z) \geq \frac{1}{\delta(z)} \quad \textit{when } \Delta(z) \subset \mathbf{C}.$$

Proof. The inequality $\lambda \leq \varphi$ was established by Solynin who also observed that equality holds at a point if and only if the region is a disk on $\widehat{\mathbf{C}}$, see [**11**, Theorem 3]. Using domain monotonicity, it is straightforward to check that $\varphi \leq \mu$, but possibly Proposition 4.4 provides an illuminating interpretation. The inequality $\mu \leq 2/\delta$ was mentioned in [**5**, Theorem 3.3], see [**6**, Theorem 2.1(a)] for a discussion of when equality can hold; it also follows from Proposition 4.4 by observing that $B_z = \Omega_z^c = J_z(\Omega)^c$ lies inside the disk $\{|w| \leq 1/\delta(z)\}$. Finally, $\mu \leq (2/\sqrt{3})\varphi$ is a consequence of Proposition 4.4 and Fact 4.1(d). The lower estimates involving $1/\delta$ are well known. $\square$

It is natural to inquire about equality between the Ferrand and Kulkarni-Pinkall metrics. Let us call z an FKP-*point* if $\varphi(z) = \mu(z)$. Notice that the examples $\mathcal{S}_0$, $\mathcal{S}_t$, $\mathbf{D}^*$, $A(R)$ presented in Section 3 each have the property that all points are FKP points. It is worthwhile to mention the following.

4.7. Lemma. *For points z in a quasi-hyperbolic region $\Omega \subset \widehat{\mathbf{C}}$, these are equivalent.*

(a) *z is an* FKP *point.*

(b) *z lies on some circular geodesic.*

(c) $\mu(z) = \tau_{ab}(z)$ *for some $a, b \in \partial\Omega$.*

Proof. If (a) holds, then by Proposition 4.4, $\operatorname{diam}(B_z) = \varphi(z) = \mu(z) = \operatorname{diam}(D_z)$, and so (b) holds for the circular geodesic which is the J_z preimage of the complement of some diameter of D_z. Assuming (b), we have $z \in |\Gamma|$ where Γ is a hyperbolic geodesic line in some disk $D \subset \Omega$ with endpoints $a, b \in \partial\Omega$. According to Proposition 2.5(b), $D = \Delta(z)$ and $\mu(z) = \lambda_D(z) = \tau_{ab}(z)$. Finally, if (c) is true, then $\varphi(z) \le \mu(z) = \tau_{ab}(z) \le \varphi(z)$. $\square$

A notable byproduct of the above is that there does not exists a region Ω with $\varphi < \mu$ everywhere in Ω. Also, we point out that in the above situation, with Γ a circular geodesic, say a hyperbolic geodesic in a disk $D \subset \Omega$, in Ω having endpoints $a, b \in \partial D \cap \partial\Omega$, we have

$$\varphi_\Omega(z) = \mu_\Omega(z) = \lambda_D(z) = \tau_{ab}(z) = \frac{|a-b|}{|z-a||z-b|}$$

for every point $z \in |\Gamma|$. In any event we see that, given an extremal disk Δ and $K = \partial\Delta \cap \partial\Omega$, every point of $\mathrm{Bd}\,(\widehat{K})$ is an FKP-point; indeed, $\mathrm{Bd}\,(\widehat{K})$ is a union of circular geodesics. However, there are simple examples with FKP points which satisfy $z \in \mathrm{Int}\,(\widehat{K}(z))$. Indeed, the origin is such a point for the domain $\mathbf{C} \setminus \{1, i, -1, -i\}$.

We also see that

$$\varphi = \mu \iff \Omega \quad \text{is foliated by its circular geodesics.}$$

The examples $\mathcal{S}_0$, $\mathcal{S}_t$, $\mathbf{D}^*$, $A(R)$ from Section 3 are foliated by their circular geodesics and moreover enjoy the property that each extremal disk meets the boundary in exactly two points. Of course, this latter property describes the regions which satisfy the condition $\#K(z) = 2$ for all $z \in \Omega$, but as the example just above illustrates, it is not necessarily true for regions with $\varphi = \mu$.

In fact we shall see below, see Theorem 4.11, that the condition $\#K(z) = 2$ for all $z \in \Omega$ implies that Ω must be simply or doubly connected. On the other hand, it is easy to construct domains of arbitrary connectivity with $\varphi = \mu$. Indeed, given $n \in \mathbf{N} \cup \{\infty\}$, let $\Omega = \mathbf{C} \setminus ((-\infty, 0] \cup \{k \in \mathbf{N} : 1 \le k < n\})$. Then Ω is n-connected and, since Ω is foliated by its circular geodesics we have $\varphi = \mu$.

4B. Ferrand and Kulkarni-Pinkall geodesics. Here we prove that every circular geodesic is both a Kulkarni-Pinkall and a Ferrand geodesic line. Then we provide additional information concerning certain Kulkarni-Pinkall geodesic segments.

Proof of Theorem A. Let Γ be a circular geodesic with endpoints $\xi, \eta \in \partial D \cap \partial\Omega$ where $D \subset \Omega$ is a disk (and Γ is a hyperbolic geodesic in D). Fix points $a, b \in |\Gamma|$, and let $\alpha = \Gamma[a, b]$. According to Corollary 4.6 and Lemma 4.7, $\varphi \leq \mu$ with equality along Γ, so

$$d_\varphi(a, b) \leq d_\mu(a, b) \leq l_\mu(\alpha) = l_\varphi(\alpha).$$

Thus, to see that Γ is both a Ferrand and a Kulkarni-Pinkall geodesic line, it suffices to confirm that $l_\varphi(\alpha) \leq d_\varphi(a, b)$.

Select a Möbius transformation which maps D to $\mathbf{H}$, ξ to 0, η to the point at infinity and say $\Omega, \Gamma, \alpha, a, b$ to $\Omega', \Gamma', \alpha', a', b'$, respectively. Then α' is a subarc lying on the positive real axis (which is just Γ'). Let β be an arbitrary rectifiable curve in Ω' joining a', b', and consider the curve β' defined via $\beta'(t) = |\beta(t)|$. Note that $|\alpha'| \subset |\beta'| \subset |\Gamma'|$. Also, for each point $w \in |\beta|$ we have a point $|w| \in |\beta'|$ with

$$\varphi'(w) \geq \frac{1}{\delta'(w)} \geq \frac{1}{|w|} = \varphi'(|w|),$$

and thus

$$\int_\beta \varphi'(w)\,|dw| \geq \int_{\beta'} \varphi'(|w|)\,d|w| \geq l_{\varphi'}(\alpha'),$$

as desired. $\square$

A careful look at the above proof reveals that we have established a slightly stronger result. We call a conformal metric $\rho(z)|dz|$ an FKP *metric* if it satisfies $\varphi \leq \rho \leq \mu$. Notice that such metrics are complete, since $\varphi(z)|dz|$ is complete, and, by Lemma 4.7, agree with $\varphi = \mu$ along every circular geodesic.

4.8. Corollary. *Circular geodesics are geodesics for any* FKP *metric.*

In certain cases we can describe all of the *local* Kulkarni-Pinkall geodesic segments.

4.9. Proposition. *Suppose $a \in \Omega \cap \mathbf{C}$ belongs to* Int $(\widehat{K}(a))$. *Then there is an Euclidean disk $D \subset \Omega$ containing a with the property that* *any compact curve γ, with $a \in |\gamma| \subset D$, is a Kulkarni-Pinkall geodesic segment if and only if γ is a hyperbolic geodesic segment in $\Delta(a)$.*

Proof. Since a belongs to the interior of $\widehat{K}(a)$, we can simply let D be a hyperbolic disk in $\Delta(a)$ with center a and sufficiently small radius. The desired assertion now follows from the fact that $\mu_\Omega = \lambda_{\Delta(a)}$ in $\mathrm{Int}\,(\widehat{K}(a))$. □

4C. Kulkarni-Pinkall isometries. The following, one of the main results in this paper, is the key ingredient in our proof of Theorem B. Because of its importance, we present two proofs for this crucial fact. It is convenient to introduce the notation

$$N = N(\Omega) := \sup_{z\in\Omega} \#K(z).$$

4.10. Theorem. *Every Kulkarni-Pinkall isometry between two regions, one of which has $N > 2$, is* (*the restriction of*) *a Möbius transformation.*

Proof. Assume that $f : \Omega \to \Omega'$ is an orientation preserving Kulkarni-Pinkall isometry; thus, f is a conformal homeomorphism. Suppose $N(\Omega') > 2$; this ensures the existence of a point $b = f(a)$ with $b \in G' = \mathrm{Int}\,(\widehat{K}')$ where $K' = K'(b) = \Delta' \cap \partial\Omega'$ and $\Delta' = \Delta(b) = \Delta_{\Omega'}(b)$. Let $G = f^{-1}(G')$ and $\Delta = \Delta(a)$.

Notice that in $G \cap \Delta$ we have

$$f^*(\lambda_{\Delta'}) = f^*(\mu') = \mu \le \lambda_\Delta$$

with equality holding at the point $z = a$. Indeed, the first equality holds because $\mu' = \lambda_{\Delta'}$ in G', the second equality holds because f is a Kulkarni-Pinkall isometry, and the inequality holds by the very definition of μ. Appealing to Fact 2.1 we can now assert that f maps all of Δ conformally onto Δ'. Therefore, f must be a Möbius transformation.

In lieu of the above argument, we can also finish our proof as follows. We have $a \in \widehat{K}$ where $K = K(a) = \partial\Delta \cap \partial\Omega$. Thus, either $a \in \mathrm{Int}\,(\widehat{K})$

or a lies on one of the circular geodesics which form $\mathrm{Bd}\,(\widehat{K})$. In either case we can find an arc α with the properties that $a \in |\alpha|$, α and $\alpha' = f \circ \alpha$ are hyperbolic geodesic segments in Δ and Δ', respectively, $|\alpha'| \subset G'$, and such that f is a hyperbolic isometry along α (with respect to the hyperbolic metrics in Δ and Δ'). (Proposition 4.9 and Lemma 4.7 are useful here.) An easy lemma now confirms that f must be a Möbius transformation along α and hence (the restriction of) a Möbius transformation. □

Because of the above result, it is worthwhile to understand which regions have $N = 2$; of course, these domains are foliated by their circular geodesics and described precisely by the condition that each extremal disk meets the boundary in exactly two points. See also the discussion at the end of subsection 4.A.

4.11. Theorem. *Every region with $N = 2$ is either simply connected or doubly connected.*

Proof. Let Ω be such a region. According to Lemma 2.10(c), the sets $K(z)$ (of extremal boundary points associated to each $z \in \Omega$) can be described locally by pairs of continuous functions. Fix a point $a \in \Omega$, let $K(a) = \{b, c\}$, and select components X, Y of $\partial\Omega$ (possibly $X = Y$) such that $b \in X$ and $c \in Y$.

We claim that for all $z \in \Omega$, $K(z) \subset X \cup Y$. Assuming this, we employ the fact (Proposition 2.5(e)) that the extremal boundary points are dense in $\partial\Omega$ to deduce that

$$\partial\Omega = \overline{\bigcup_{z\in\Omega} K(z)} \subset \overline{X \cup Y} = X \cup Y \subset \partial\Omega.$$

It therefore follows that $\partial\Omega = X \cup Y$, as desired.

It remains to confirm the assertion $K(z) \subset X \cup Y$; so, let $z \in \Omega$. There is a continuous path $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = a$ and $\gamma(1) = z$. Using Lemma 2.10(c) we construct continuous $\xi, \eta : [0, 1] \to \partial\Omega$ with the property that for all $t \in [0, 1]$, $K(\gamma(t)) = \{\xi(t), \eta(t)\}$. Since $[0, 1]$ is connected, so are its images under ξ, η and hence $K(z) = \{\xi(1), \eta(1)\} \subset X \cup Y$. □

Proof of Theorem B. Use Theorems 4.10 and 4.11.

4D. Ferrand and Kulkarni-Pinkall curvatures. Here we establish curvature inequalities for both the Kulkarni-Pinkall and Ferrand metrics and then we investigate when these curvatures can be constant. Our arguments are based on Lemma 2.12 along with knowledge of the curvatures in the special examples from Section 3.

Proof of Theorem C. First we consider the Kulkarni-Pinkall metric. Fix a point $a \in \Omega \cap \mathbf{C}$, and let $\Delta = \Delta(a)$, $K = K(a)$. Then $\mu \leq \lambda_\Delta$ in Δ with $\mu(a) = \lambda_\Delta(a)$, by so Lemma 2.12 we always have $\underline{\mathcal{K}}_\mu(a) \geq \mathcal{K}_{\lambda_\Delta}(a) = -1$.

If $a \in \operatorname{Int}(\widehat{K})$, then as $\mu = \lambda_\Delta$ in $\operatorname{Int}(\widehat{K})$, $\mathcal{K}\mu(a) = -1$. Assume $a \in \operatorname{Bd}(\widehat{K})$; say, $a \in |\Gamma|$ for some circular geodesic Γ with endpoints $\xi, \eta \in \partial\Omega$. Then $\Omega \subset \Omega' = \widehat{\mathbf{C}}_{\xi\eta}$ and $\mu \geq \mu' = \tau_{\xi\eta}$. Since $\mu(a) = \lambda_D(a) = \mu'(a)$ and $\mu' = \tau_{\xi\eta}$ is flat, an appeal to Lemma 2.12 yields $\overline{\mathcal{K}}_\mu(a) \leq \mathcal{K}_{\mu'}(a) = 0$.

Next suppose Ω is convex. As above, we may assume that $a \in \operatorname{Bd}(\widehat{K})$ with say $a \in |\Gamma|$ for some circular geodesic Γ having endpoints $\xi, \eta \in \partial\Omega$. Let Ω' be the region formed by the intersection of supporting half-planes for Ω at each of ξ, η; thus, Ω' is either an infinite strip or a convex sector, i.e., Ω is affine equivalent to some $\mathcal{S}_t$ with $0 \leq t \leq 1$. Since $\Omega \subset \Omega'$ with $\mu(a) = \mu'(a)$, Lemma 2.12 again produces $\overline{\mathcal{K}}_\mu(a) \leq \mathcal{K}_{\mu'}(a) \leq -1/2$, where the latter inequality holds because of Examples 3.A and 3.B.

Now for the Ferrand metric: fix $a \in \Omega$ and $\xi, \eta \in \partial\Omega$ with $\varphi(a) = \tau_{\xi\eta}(a)$. Since $\Omega \subset \Omega' = \widehat{\mathbf{C}}_{\xi\eta}$ and $\varphi' = \tau_{\xi\eta}$ is flat, $\overline{\mathcal{K}}_\varphi(a) \leq \mathcal{K}_{\varphi'}(a) = 0$. Notice that as $\varphi \leq \mu$, Lemma 2.12 ensures that $\underline{\mathcal{K}}_\varphi(a) \geq \underline{\mathcal{K}}_\mu(a) \geq -1$ at each FKP-point a. However, in general, there is no lower bound on the curvature of the Ferrand metric; indeed, this may be $-\infty$ as the thrice-punctured sphere example shows, cf. subsection 3.E.

Finally, suppose Ω is convex. Let Ω' be the intersection of two open supporting half-planes for Ω at each of ξ, η; thus, Ω' is either an infinite strip or a convex sector, i.e., Ω' is affine equivalent to some $\mathcal{S}_t$ with $0 \leq t \leq 1$. Since $\Omega \subset \Omega'$ with $\varphi(a) = \varphi'(a)$, Lemma 2.12 once again produces $\overline{\mathcal{K}}_\varphi(a) \leq \mathcal{K}_{\varphi'}(a) = \mathcal{K}_{\mu'}(a) \leq -1/2$.

Proof of Theorem D. This is an immediate consequence of the following.

4.12. Theorem. *Let $\rho(z)|dz|$ be an* FKP *metric on a quasi-hyperbolic region Ω on $\widehat{\mathbf{C}}$. Suppose that $\rho(z)|dz|$ has constant Gaussian curvature $\mathcal{K}_\rho = k$ in Ω. Then either $k = 0$ and Ω is a twice punctured sphere, or $k = -1$ and Ω is a disk on $\widehat{\mathbf{C}}$.*

Proof. Since every FKP metric is complete, the asserted claim follows directly from Fact 2.2 if $k = 0$. Thus, by Fact 2.2, we may assume that $k < 0$, Ω is a hyperbolic region and $\rho = t\,\lambda$ where $t = 1/\sqrt{-k}$. Since $\lambda \le \varphi \le \rho = t\,\lambda$, $t \ge 1$ (and so $-1 \le k < 0$).

Fix a circular geodesic Γ in Ω (there are lots of these :-)). Using an auxiliary Möbius transformation, if necessary, we can assume that Γ is the positive real axis $\mathbf{R}_+$ (and so the associated disk in Ω is the right half-plane $\mathbf{H}$). According to Corollary 4.8, Γ is a geodesic for $\rho(z)|dz|$, hence also a hyperbolic geodesic in Ω.

Now let $f : \mathbf{H} \to \Omega$ be a holomorphic covering projection. Since $\mathbf{H}$ is a simply connected subdomain of Ω, there exists a single-valued branch g of f^{-1} defined in $\mathbf{H}$. Then $\gamma = g \circ \Gamma$ is a hyperbolic geodesic in $\mathbf{H}$. Employing another auxiliary Möbius transformation, if necessary, we can further assume that γ is also the positive real axis, that f is increasing along $\gamma = \mathbf{R}_+$, and that $f(1) = 1$.

Recall that the hyperbolic distance between points $1, x \in \mathbf{R}_+$ in $\mathbf{H}$ is $|\log x|$. Along $\Gamma = \mathbf{R}_+$ we have $\lambda_{\mathbf{H}} = \mu = \rho = t\,\lambda$, and $f^*[\rho] = t\,f^*[\lambda] = t\,\lambda_{\mathbf{H}}$, so we find that

$$|\log f(x)| = \int_{[f(1),f(x)]} \rho(z)\,|dz| = t \int_{[f(1),f(x)]} \lambda(z)\,|dz| = t\,|\log x|.$$

We conclude that $f(x) = x^t$ for all $x \in \mathbf{R}_+$, and hence $f(\zeta) = \zeta^t$ for all $\zeta \in \mathbf{H}$.

If $t > 2$ were true, then we would have $\Omega = f(\mathbf{H}) = \mathbf{C}^*$ which would contradict Ω being a hyperbolic region; thus, $1 \le t \le 2$. Therefore, $\Omega = f(\mathbf{H}) = \mathcal{S}_t$ is a concave sector. Clearly, $\mathcal{S}_t$ is foliated by circular geodesics (each of which is an actual Euclidean ray), and in fact $\varphi = \mu = 1/\delta$ in $\mathcal{S}_t$. It follows that $\rho = 1/\delta$, but then (see the end of subsection 3B) $\mathcal{K}_\rho = \mathcal{K}_{1/\delta} = k$ can only be constant when $t = 1$, $k = -1$ and $\Omega = \mathbf{H}$. $\square$

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