

CONVEXITY PROPERTIES OF TWISTED ROOT MAPS

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ABSTRACT. The strong spectral order induces a natural partial ordering on the manifold \mathcal{H}_n of monic hyperbolic polynomials of degree n . We prove that twisted root maps associated with linear operators acting on \mathcal{H}_n are Gårding convex on every polynomial pencil and we characterize the class of polynomial pencils of logarithmic derivative type by means of the strong spectral order. Let \mathcal{A}' be the monoid of linear operators that preserve hyperbolicity as well as root sums. We show that any polynomial in \mathcal{H}_n is the global minimum of its \mathcal{A}' -orbit and we conjecture a similar result for complex polynomials.

1. Introduction. An important chapter in the theory of distribution of zeros of polynomials and transcendental entire functions pertains to the study of linear operators that preserve certain prescribed properties (cf., e.g., [8, 13, 16 and references therein]). The following example illustrates the viewpoint adopted in this paper. Denote by $\text{End } \Pi$ the set of linear mappings from the vector space $\Pi := \mathbf{C}[x]$ to itself, and let $\Pi(\Omega)$ be the class of polynomials in Π whose zeros lie in a fixed set $\Omega \subseteq \mathbf{C}$. As noted in [8], the fundamental problem of characterizing all operators $T \in \text{End } \Pi$ such that $T(\Pi(\Omega)) \subseteq \Pi(\Omega)$ is open for all but trivial choices of Ω . Indeed, this question remains unanswered even in the important special cases when Ω is a line or a half-plane.¹ Moreover, in many applications such as stability problems one often needs additional information on the relative geometry of the zeros of $T(P)$ and P for $P \in \Pi(\Omega)$ when $T \in \text{End } \Pi$ preserves $\Pi(\Omega)$. For instance, if $T = D := d/dx$ these questions amount to studying the geometry of zeros and critical points of complex polynomials, which is in itself a vast and intricate subject [16]. In this case the Gauss-Lucas theorem implies that the zero set of $T(P)$ is contained in the convex hull of the zeros of P and thus $T(\Pi(\Omega)) \subseteq \Pi(\Omega)$ whenever Ω

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is convex. However, this result can be substantially refined in various circumstances [15, 16].

In this paper we propose a general setting for studying the relative geometry of zeros of polynomials and their distribution under the action of various classes of linear operators. Let \mathcal{C}_n be the manifold of monic complex polynomials of degree $n \geq 1$. For $P \in \mathcal{C}_n$ let $\mathcal{Z}(P)$ be the unordered n -tuple consisting of the zeros of P , each zero occurring as many times as its multiplicity. Hence $\mathcal{Z}(P) \in \mathbf{C}^n/\Sigma_n$, where Σ_n is the symmetric group on n elements. Denote by $\Re\mathcal{Z}(P)$, respectively $\Im\mathcal{Z}(P)$, the unordered n -tuple whose components are the real, respectively imaginary, parts of the points in $\mathcal{Z}(P)$. One says that P is *hyperbolic* provided that $\Re\mathcal{Z}(P) = \mathcal{Z}(P)$, i.e., if all the zeros of P are real. A hyperbolic polynomial with only simple zeros is called *strictly hyperbolic*. Let $\mathcal{H}_n = \mathcal{C}_n \cap \Pi(\mathbf{R})$ be the real submanifold of \mathcal{C}_n consisting of hyperbolic polynomials. There is a natural set-theoretic identification between \mathcal{C}_n and \mathbf{C}^n/Σ_n by means of the *root map*

$$(1) \quad \begin{aligned} \mathcal{Z} : \mathcal{C}_n &\longrightarrow \mathbf{C}^n/\Sigma_n \\ P &\longmapsto \mathcal{Z}(P) \end{aligned}$$

whose restriction to \mathcal{H}_n obviously induces a bijection between \mathcal{H}_n and \mathbf{R}^n/Σ_n . Let $T \in \text{End } \Pi$ be an operator such that $T(\mathcal{C}_n) \subseteq \mathcal{C}_n$. The composition $\mathcal{Z} \circ T$ is called the *T -twisted root map*. Note that if T also acts on \mathcal{H}_n then the restriction of the T -twisted root map to \mathcal{H}_n has real components. Given a nonempty set $\Omega \subseteq \mathbf{C}$ we define a multiplicative monoid of linear operators by setting

$$(2) \quad \mathcal{A}_n(\Omega) = \{T \in \text{End } \Pi \mid T(\mathcal{C}_n \cap \Pi(\Omega)) \subseteq \mathcal{C}_n \cap \Pi(\Omega)\}.$$

The relevance of twisted root maps in the present context is quite clear. Indeed, for degree-preserving linear operators, the aforementioned questions on the distribution and the relative geometry of zeros of polynomials may be summarized as follows.

Problem 1. Describe the properties of T -twisted root maps for $T \in \mathcal{A}_n(\Omega)$, where n is a fixed positive integer and Ω is an appropriate set of interest.

Below we shall mainly focus on the important special case of Problem 1 when $\Omega = \mathbf{R}$. The following fundamental result from the theory of majorization is a key ingredient in our analysis of twisted root maps.

Theorem 1. *Let $X = (x_1, x_2, \dots, x_n)^t$ and $Y = (y_1, y_2, \dots, y_n)^t$ be two n -tuples of vectors in \mathbf{R}^k . The following conditions are equivalent:*

- (i) *For any convex function $f : \mathbf{R}^k \rightarrow \mathbf{R}$ one has $\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i)$.*
- (ii) *There exists a doubly stochastic $n \times n$ matrix A such that $\tilde{X} = A\tilde{Y}$, where \tilde{X} and \tilde{Y} are $n \times k$ matrices obtained by some (and then any) ordering of the vectors in X and Y .*

If the conditions of Theorem 1 are satisfied we say that X is *majorized* by Y or that X is *less than Y in the spectral order*, and write $X \prec Y$. One can easily check that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ if $X \prec Y$. Theorem 1 is due to Schur as well as to Hardy, Littlewood, and Pólya in the case $k = 1$ [12], and to Sherman in the general case [17]. These cases are also known as (strong) classical and multivariate majorization, respectively. Surprisingly, Sherman’s theorem was long assumed to be an open problem and does not appear in [14], which is the definite reference on majorization theory (see p. 433 in *loc. cit.*). We refer to [3] for a simple new proof of this result. Note that although the spectral order is only a preordering on \mathbf{R}^n , Birkhoff’s theorem [14, Theorem 2.A.2] implies that it actually induces a partial ordering on \mathbf{R}^n/Σ_n . Therefore, Theorem 1 and the root map in (1) allow us to define a poset structure (\mathcal{H}_n, \preceq) by setting $P \preceq Q$ if $P, Q \in \mathcal{H}_n$ and $\mathcal{Z}(P) \prec \mathcal{Z}(Q)$.

In Section 2 we establish a general convexity property for twisted root maps associated with operators in $\mathcal{A}_n(\mathbf{R})$. Namely, we show that the restriction of any such map to arbitrary polynomial pencils in \mathcal{H}_n is Gårding convex (Definition 1 and Theorem 2). This has several interesting consequences for the so-called span (or spread) function and its twisted versions (Corollaries 1–2).

In Section 3 we characterize the class of polynomial pencils of logarithmic derivative type contained in \mathcal{H}_n by means of a local minimum property with respect to the partial ordering \preceq on \mathcal{H}_n (Theorem 4).

Let $\mathcal{A} := \bigcap_{k=0}^\infty \mathcal{A}_k(\mathbf{R})$ and denote by \mathcal{A}' the submonoid of \mathcal{A} whose action on Π preserves the average of the zeros of any polynomial. In Section 4 we show that \mathcal{A} consists of ordinary differential operators of Laguerre-Pólya type (Theorem 7) and that any polynomial in \mathcal{H}_n is the global minimum of its \mathcal{A}' -orbit (Theorem 6). In particular, this implies

that the action of \mathcal{A} on \mathcal{H}_n does not decrease the span of polynomials (Corollary 5) and that the polynomial pencils characterized in Section 2 satisfy in fact a global minimum property with respect to the spectral order.

As we point out in Section 5, a natural question that arises from our study is whether one can describe classical majorization by means of (differential) operators in $\mathcal{A}_n(\mathbf{R})$ acting on polynomials in \mathcal{H}_n (Problem 2). We discuss this question as well as possible complex analogs of Theorem 6 (Conjecture 1) and extensions of this theorem to the Laguerre-Pólya class of functions.

2. Polynomial pencils and Gårding convexity. A fundamental theorem of Gårding asserts that the largest root of a multivariate homogeneous polynomial which is hyperbolic with respect to a given vector is always a convex function [11]. The properties of such polynomials play a significant role in the theory of partial differential equations, convex analysis and matrix theory (see, e.g., [2]). The following definition is motivated by Gårding's result.

Definition 1. Let K be a convex subset of a vector space. A map $f : K \rightarrow \mathbf{R}^n / \Sigma_n$ given by $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$ is called *Gårding convex* if $\mathbf{x} \mapsto \max_{1 \leq i \leq n} f_i(\mathbf{x})$ is a convex function on K .

Gårding's theorem is a rich source of examples of maps that satisfy Definition 1, cf. [2, 11]. For instance, the restriction of the eigenvalue map to the real space of $n \times n$ Hermitian matrices is an important such example. The main result of this section shows that twisted root maps associated to operators in $\mathcal{A}_n(\mathbf{R})$ are Gårding convex when restricted to certain convex subsets of \mathcal{H}_n , as we shall now explain. Recall the notations \mathcal{C}_n , \mathcal{H}_n , $\mathcal{A}_n(\mathbf{R})$ from the introduction and denote by \mathcal{R}_n the (real) submanifold of \mathcal{C}_n consisting of monic real polynomials of degree n . The inclusion $\mathcal{H}_n \subseteq \mathcal{R}_n$ is obviously strict if $n \geq 2$, which we assume henceforth.

Definition 2. Let P_1 and P_2 be distinct polynomials in \mathcal{R}_n . The real line through P_1 and P_2 , i.e., the set $\mathcal{L} = \{(1 - \lambda)P_1 + \lambda P_2 \mid \lambda \in \mathbf{R}\}$, is called a *polynomial pencil* in \mathcal{R}_n . A *basis* of a polynomial pencil

$\mathcal{L} \subset \mathcal{R}_n$ is a pair of real polynomials $\{P, Q\}$ that satisfy the following conditions:

$$(3) \quad \begin{aligned} &P \in \mathcal{R}_n, \quad \text{the dominant coefficient of } Q \text{ equals } n, \\ &\deg Q \leq n - 1, \quad \text{and} \quad \mathcal{L} = \{P - \lambda Q \mid \lambda \in \mathbf{R}\}. \end{aligned}$$

A polynomial pencil \mathcal{L} is said to be of *logarithmic derivative type* or an *LD-pencil* if there exist $Q_1, Q_2, Q_3 \in \mathcal{L}$ such that $Q'_3 = Q_1 - Q_2$.

Remark 1. As defined above, a polynomial pencil in \mathcal{R}_n has only an affine structure. One can produce real polynomial pencils endowed with a natural linear structure by using an appropriate projective version of Definition 2 where a real polynomial pencil is defined instead as a real line in projective space \mathbf{RP}^n identified with the space of all homogeneous degree n real polynomials in two real variables considered up to a constant factor. Such a pencil is called generic if it intersects the standard discriminant $\mathcal{D}_{n+1} \subset \mathbf{RP}^n$ transversally. A topological classification of all generic pencils in \mathbf{RP}^n was obtained in [5].

Clearly, any polynomial pencil has a basis. Moreover, if $\{P, Q\}$ and $\{R, S\}$ are two bases of the same polynomial pencil then $S = Q$ and $R = P - \mu Q$ for some $\mu \in \mathbf{R}$. Note also that a polynomial pencil $\mathcal{L} \subset \mathcal{R}_n$ is an LD-pencil if and only if it has a (necessarily unique) basis of the form $\{P, P'\}$, which we call the *canonical basis* of the LD-pencil \mathcal{L} . The main result of this section is as follows.

Theorem 2. *If \mathcal{L} is an arbitrary polynomial pencil in \mathcal{H}_n and $T \in \mathcal{A}_n(\mathbf{R})$, then the T -twisted root map $\mathcal{Z} \circ T|_{\mathcal{L}}$ is Gårding convex.*

Since we are mainly interested in polynomial pencils contained in \mathcal{H}_n , which we shall alternatively refer to as *hyperbolic* (polynomial) pencils, let us first give a complete description of these pencils.

Theorem 3. *Let P and R be distinct polynomials in \mathcal{R}_n , and set $Q = P - R$ and $\mathcal{L} = \{P - \lambda Q \mid \lambda \in \mathbf{R}\}$. The following statements are equivalent:*

- (i) $\mathcal{L} \subset \mathcal{H}_n$.
- (ii) $\alpha P + \beta R$ is hyperbolic for any $\alpha, \beta \in \mathbf{R}$ such that $\alpha^2 + \beta^2 \neq 0$.

(iii) *The polynomials P and R are in \mathcal{H}_n and have weakly interlacing zeros.*

(iv) *$P \in \mathcal{H}_n$, Q is hyperbolic, $\deg Q = n - 1$, and the zeros of P and Q are weakly interlacing. Equivalently, these conditions hold with R instead of P .*

(v) *The polynomial $P+iR$ or indeed $P+iQ$ or $R+iQ$ has all its zeros either in the closed upper half-plane or in the closed lower half-plane.*

Proof. It is clear that (ii) \Rightarrow (i). Note that if (i) holds then $\alpha P + \beta R \in \mathcal{H}_n$ for any $\alpha, \beta \in \mathbf{R}$ such that $\alpha^2 + \beta^2 \neq 0$ and $\alpha + \beta \neq 0$. By assumption the polynomial Q is not identically zero and in fact $\deg Q \geq 1$, because otherwise $P - \lambda Q \notin \mathcal{H}_n$ for some $|\lambda| \gg 0$. Since $\{Q - k^{-1}P\}_{k=1}^{\infty}$ is a sequence of hyperbolic polynomials which tends to Q uniformly on compact sets, Hurwitz's theorem for analytic functions [16, Theorem 1.6.9] implies that Q is hyperbolic. This proves that (i) \Rightarrow (ii).

The equivalence between (ii) and (iii) is known in the literature as Obreschkoff's theorem [15, 5.2] or the Hermite-Kekeya theorem [16, Theorem 6.3.8] in the generic case when P and R are strictly hyperbolic polynomials with no common zeros. In the general case, this equivalence is due to Dedieu [10, Theorem 4.1]. Actually, the arguments used in *loc. cit.* yield also a proof of (i) \Leftrightarrow (iv). Note that the roles of P and R may be interchanged and that the condition $\deg Q = n - 1$ may alternatively be seen as a consequence of (iii) since P and R are distinct.

The well-known Hermite-Biehler theorem [16, Theorem 6.3.4] asserts that statements (iii) and (v) are equivalent if the words "weakly" and "closed" in these statements are replaced by "strictly" and "open," respectively. For the general case, let us set $\tilde{P} = P/S$ and $\tilde{R} = R/S$, where S denotes the greatest common monic divisor of P and R . Note that $\deg \tilde{P} = \deg \tilde{R} \geq 1$ since P and R are distinct. If (iii) holds then \tilde{P} and \tilde{R} have strictly interlacing zeros, cf. [16, Remark 6.3.3]. By the Hermite-Biehler theorem, the polynomial $\tilde{P} + i\tilde{R}$ must have all its zeros either in the open upper half-plane or in the open lower half-plane. It follows that the polynomial $P + iR = S(\tilde{P} + i\tilde{R})$ has all its zeros either in the closed upper half-plane or in the closed lower half-plane, which proves (v). Conversely, if the latter holds, then $\tilde{P} + i\tilde{R}$ has all its zeros

either in the open upper half-plane or in the open lower half-plane (since any real zero of this polynomial would have to be a common zero of \tilde{P} and \tilde{R}). Thus, \tilde{P} and \tilde{R} have strictly interlacing zeros so that $P = S\tilde{P}$ and $R = S\tilde{R}$ have weakly interlacing zeros, as stated in (iii). The claims in (v) \Leftrightarrow (iv) concerning the polynomials P , Q , and $P + iQ$ can be verified in similar fashion by using the Hermite-Biehler theorem for the pair $\{P, Q\}$. \square

Proof of Theorem 2. Let \mathcal{L} be a polynomial pencil in \mathcal{H}_n and $T \in \mathcal{A}_n(\mathbf{R})$. Then either $T|_{\mathcal{L}}$ is a constant map, in which case the conclusion of the theorem holds trivially, or the image $T(\mathcal{L})$ is again a polynomial pencil in \mathcal{H}_n . Thus, it is enough to prove the theorem for $T = \text{Id}_{\Pi}$, which we assume henceforth. Fix a basis $\{P, Q\}$ of \mathcal{L} as in (3) and denote the zeros of P and Q by $x_1 \leq x_2 \leq \dots \leq x_n$ and $y_1 \leq y_2 \leq \dots \leq y_{n-1}$, respectively. By Theorem 3 we know that

$$(4) \quad x_1 \leq y_1 \leq x_2 \leq \dots \leq x_{n-1} \leq y_{n-1} \leq x_n.$$

Set $R_\lambda = P - \lambda Q$, $\lambda \in \mathbf{R}$, and denote the zeros of R_λ by $x_i(\lambda)$, $1 \leq i \leq n$, which we label so that $x_i(0) = x_i$, $1 \leq i \leq n$. Since $R_\lambda - \mu Q \in \mathcal{H}_n$ for any $\mu \in \mathbf{R}$, Theorem 3 again implies that

$$(5) \quad x_1(\lambda) \leq y_1 \leq x_2(\lambda) \leq \dots \leq x_{n-1}(\lambda) \leq y_{n-1} \leq x_n(\lambda) \quad \text{for } \lambda \in \mathbf{R}.$$

Step 1. P and Q have strictly interlacing zeros. This means that P and Q are strictly hyperbolic and have no common zeros. Clearly, any common zero of R_λ and Q would also have to be a zero of P . It follows that the interlacing properties in (4) and (5) are both strict. In particular, the polynomial R_λ is strictly hyperbolic for any $\lambda \in \mathbf{R}$. We may therefore differentiate the identities $R_\lambda(x_i(\lambda)) = 0$, $1 \leq i \leq n$, with respect to λ to get

$$(6) \quad x'_i(\lambda) = \frac{Q(x_i(\lambda))}{R'_\lambda(x_i(\lambda))} \quad \text{for } \lambda \in \mathbf{R} \quad \text{and } 1 \leq i \leq n,$$

where $R'_\lambda(x) = (\partial/\partial x)R_\lambda(x)$. Note that (6) readily implies that $x'_i(\lambda) > 0$ for all $\lambda \in \mathbf{R}$ and $1 \leq i \leq n$ since both $Q(x_i(\lambda))$ and $R'_\lambda(x_i(\lambda))$ have constant signs while

$$\frac{Q(x_i(0))}{P'(x_i(0))} = \frac{Q(x_i)}{P'(x_i)} = n \prod_{j=1}^{n-1} (x_i - y_j) \prod_{\substack{j=1 \\ j \neq i}}^n (x_i - x_j)^{-1} > 0$$

because of the (strict) inequalities in (4). Thus, all the zeros of the polynomial R_λ are increasing functions of λ . By differentiating (6) with respect to λ we obtain

$$\begin{aligned}
 (7) \quad x_i''(\lambda) &= x_i'(\lambda) \left[\frac{Q'R'_\lambda - QR'_\lambda}{R_\lambda'^2} \right] (x_i(\lambda)) + \left[\frac{QQ'}{R_\lambda'^2} \right] (x_i(\lambda)) \\
 &= 2x_i'(\lambda) \frac{Q'(x_i(\lambda))}{R'_\lambda(x_i(\lambda))} - (x_i'(\lambda))^2 \frac{R_\lambda''(x_i(\lambda))}{R'_\lambda(x_i(\lambda))} \\
 &= (x_i'(\lambda))^2 \left[\frac{2Q'}{Q} - \frac{R_\lambda''}{R'_\lambda} \right] (x_i(\lambda))
 \end{aligned}$$

for $\lambda \in \mathbf{R}$ and $1 \leq i \leq n$. The special case when $i = n$ in (7) yields

$$\begin{aligned}
 x_n''(\lambda) &= (x_n'(\lambda))^2 \left[\sum_{j=1}^{n-1} \frac{2}{x_n(\lambda) - y_j} - \sum_{j=1}^{n-1} \frac{2}{x_n(\lambda) - x_j(\lambda)} \right] \\
 &= 2(x_n'(\lambda))^2 \sum_{j=1}^{n-1} \frac{y_j - x_j(\lambda)}{(x_n(\lambda) - y_j)(x_n(\lambda) - x_j(\lambda))} > 0
 \end{aligned}$$

by (5). This implies that $\mathbf{R} \ni \lambda \mapsto \max \mathcal{Z}(R_\lambda) = x_n(\lambda)$ is a convex function. Thus $\mathcal{Z}|_{\mathcal{L}}$ is a Gårding convex map, which proves the theorem in the generic case.

Step 2. The general case. Let S denote the greatest common monic divisor of P and Q . Step 1 shows that the theorem is true if $S \equiv 1$ and so we may assume that $\deg S \geq 1$. Set $\tilde{P} = P/S$, $\tilde{Q} = Q/S$, and $\tilde{R}_\lambda = \tilde{P} - \lambda\tilde{Q}$, so that $R_\lambda = S\tilde{R}_\lambda$ and thus $\max \mathcal{Z}(R_\lambda) = \max(\alpha, \max \mathcal{Z}(\tilde{R}_\lambda))$ for $\lambda \in \mathbf{R}$, where α is the largest zero of S . If $\tilde{Q} \equiv n$, then $\max \mathcal{Z}(\tilde{R}_\lambda)$ is obviously a linear function of λ . Otherwise \tilde{P} and \tilde{Q} must have strictly interlacing zeros and so by Step 1 the function $\mathbf{R} \ni \lambda \mapsto \max \mathcal{Z}(\tilde{R}_\lambda)$ is convex. In either case it follows that $\mathbf{R} \ni \lambda \mapsto \max \mathcal{Z}(R_\lambda)$ is a convex function so that $\mathcal{Z}|_{\mathcal{L}}$ is a Gårding convex map, which completes the proof. \square

Remark 2. Theorem 2 generalizes the result announced in [6, Theorem 1.7], where the Gårding convexity property was stated only for LD -pencils in \mathcal{H}_n .

Recall that the span (or spread) of a polynomial $P \in \mathcal{H}_n$ is the length of the smallest interval that contains all its zeros, i.e., $\Delta(P) = \max \mathcal{Z}(P) - \min \mathcal{Z}(P)$. A review of the literature on the span of hyperbolic polynomials and related questions may be found in [16, Chapter 6]. Given an operator $T \in \mathcal{A}_n(\mathbf{R})$ we define the T -twisted span function on \mathcal{H}_n to be the composite map $\Delta \circ T$. From Theorem 2 we deduce the following properties for twisted span functions:

Corollary 1. *If \mathcal{L} is an arbitrary polynomial pencil in \mathcal{H}_n and $T \in \mathcal{A}_n(\mathbf{R})$, then the T -twisted span function $\Delta \circ T|_{\mathcal{L}}$ is convex and has a global minimum.*

Proof. As in the proof of Theorem 2, it is enough to consider the case $T = \text{Id}_H$. Let \mathcal{L} be a polynomial pencil in \mathcal{H}_n with a basis $\{P, Q\}$ as in (3). Set $\tilde{P}(x) = (-1)^n P(-x)$ and $\tilde{Q}(x) = (-1)^{n-1} Q(-x)$, so that $\mathcal{Z}(P - \lambda Q) = -\mathcal{Z}(\tilde{P} + \lambda \tilde{Q})$ for all $\lambda \in \mathbf{R}$. By Theorem 2 the function $\mathbf{R} \ni \lambda \mapsto \max \mathcal{Z}(\tilde{P} + \lambda \tilde{Q})$ is convex and thus $\mathbf{R} \ni \lambda \mapsto \min \mathcal{Z}(P - \lambda Q) = -\max \mathcal{Z}(\tilde{P} + \lambda \tilde{Q})$ is a concave function. Hence $\mathbf{R} \ni \lambda \mapsto \Delta(P - \lambda Q)$ is a convex function and therefore also Lipschitz continuous on any compact interval. Since $\Delta(P - \lambda Q) \rightarrow \infty$ as $|\lambda| \rightarrow \infty$ it follows that $\Delta|_{\mathcal{L}}$ has a global minimum, as required. \square

Corollary 1 can be further refined in the case of hyperbolic LD -pencils:

Corollary 2. *If \mathcal{L} is an LD -pencil in \mathcal{H}_n with canonical basis $\{P, P'\}$, then the span function $\Delta|_{\mathcal{L}}$ is convex and has a global minimum at P . In particular, for any $P \in \mathcal{H}_n$ and $\lambda \in \mathbf{R}$, one has $\Delta(P) \leq \Delta(P - \lambda P')$. More generally, if $\lambda_1, \lambda_2 \in \mathbf{R}$ are such that $\lambda_1 \lambda_2 \geq 0$ and $|\lambda_1| \leq |\lambda_2|$, then $\Delta(P - \lambda_1 P') \leq \Delta(P - \lambda_2 P')$.*

Proof. It is clearly enough to prove only the last assertion of the corollary. Let P be a strictly hyperbolic polynomial in \mathcal{H}_n with zeros $x_1 < \dots < x_n$. Denote by $x_i(\lambda)$, $1 \leq i \leq n$, the zeros of the strictly hyperbolic polynomial $P - \lambda P'$, $\lambda \in \mathbf{R}$, and assume that these are labeled so that $x_i(0) = x_i$, $1 \leq i \leq n$. The arguments in the proof of Theorem 2 show that $x_i(\lambda) < x_{i+1}(\lambda)$ for all $\lambda \in \mathbf{R}$ and

$1 \leq i \leq n-1$. Moreover, by using (6) and (7) with $Q = P'$ we see that $\mathbf{R} \ni \lambda \mapsto x_n(\lambda) - \lambda$ is a strictly convex function with a global minimum at $\lambda = 0$ while $\mathbf{R} \ni \lambda \mapsto x_1(\lambda) - \lambda$ is a (strictly) concave function with a global maximum at $\lambda = 0$, which proves the corollary in the generic case when P has simple zeros. If P has multiple zeros, then we consider the strictly hyperbolic polynomial P_ε with zeros $x_k + k\varepsilon$, $1 \leq k \leq n$, where $\varepsilon > 0$. Let $\lambda_1, \lambda_2 \in \mathbf{R}$ be as in the corollary. The desired conclusion follows by letting $\varepsilon \rightarrow 0$ in the inequality $\Delta(P_\varepsilon - \lambda_1 P'_\varepsilon) \leq \Delta(P_\varepsilon - \lambda_2 P'_\varepsilon)$, which holds thanks to the first part of the proof since P_ε has only simple zeros. \square

The study of the geometrical structure of \mathcal{H}_n was initiated by Arnold in [1]. Subsequently, the convex subsets of \mathcal{H}_n were characterized in [10]. In view of the above results one may ask whether twisted root maps are Gårding convex when restricted to arbitrary convex subsets of \mathcal{H}_n . This is definitely not true, as one can see by considering for instance the subset $\{(1 - \alpha)P_1 + \alpha P_2 \mid \alpha \in [0, 1]\}$ of \mathcal{H}_2 , where $P_1(x) = x^2 - 1$ and $P_2(x) = x^2 - 2$. However, we can show that the following analog of Theorem 2 holds for arbitrary segments of hyperbolic polynomials:

Corollary 3. *Let P_1 and P_2 be distinct polynomials in \mathcal{H}_n such that the segment $[P_1, P_2] := \{(1 - \alpha)P_1 + \alpha P_2 \mid \alpha \in [0, 1]\}$ is contained in \mathcal{H}_n . There exists a (not necessarily unique) polynomial $P_3 \in \mathcal{H}_n$ such that for any $T \in \mathcal{A}_n(\mathbf{R})$ the T -twisted root maps $\mathcal{Z} \circ T|_{[P_1, P_3]}$ and $\mathcal{Z} \circ T|_{[P_2, P_3]}$ are Gårding convex.*

Proof. Let $x_1 \leq \dots \leq x_n$ and $y_1 \leq \dots \leq y_n$ be the zeros of P_1 and P_2 , respectively. According to [10, Theorem 2.1], the segment $[P_1, P_2]$ is contained in \mathcal{H}_n if and only if $\max(x_i, y_i) \leq \min(x_{i+1}, y_{i+1})$ for $1 \leq i \leq n-1$. Let z_i , $1 \leq i \leq n$, be such that $z_n \geq \max(x_n, y_n)$ and $\max(x_i, y_i) \leq z_i \leq \min(x_{i+1}, y_{i+1})$, $1 \leq i \leq n-1$. Denote by \mathcal{L}_k , $k = 1, 2$, the real line through P_k and P_3 , $k = 1, 2$, where $P_3 \in \mathcal{H}_n$ is such that $\mathcal{Z}(P_3) = (z_1, \dots, z_n)$. Since both pairs of polynomials $\{P_1, P_3\}$ and $\{P_2, P_3\}$ have weakly interlacing zeros Theorem 3 implies that both \mathcal{L}_1 and \mathcal{L}_2 are polynomial pencils contained in \mathcal{H}_n . The result follows readily from Theorem 2. \square

3. A characterization of hyperbolic LD -pencils. Hyperbolic pencils of logarithmic derivative type are particularly interesting for at least two reasons. On the one hand, they are obviously related to the action of linear differential operators on the manifold \mathcal{H}_n . On the other hand, these pencils have interesting connections with classical majorization via the partial ordering on \mathcal{H}_n defined in the introduction. Indeed, the main result below states that the class of hyperbolic LD -pencils is actually characterized by a local minimum property with respect to the spectral order. This hints at possibly even deeper connections between hyperbolic polynomials, classical majorization, and differential operators, which we shall further investigate in the next sections.

Definition 3. Let \mathcal{L} be a polynomial pencil in \mathcal{R}_n with a basis $\{P, Q\}$ as in (3). The set $\mathcal{L}_s(P, Q) := \{P(x + \lambda) - \lambda Q(x + \lambda) \mid \lambda \in \mathbf{R}\}$ is called the $\{P, Q\}$ -shift of \mathcal{L} .

Note that all polynomials in the $\{P, Q\}$ -shift of \mathcal{L} have the same zero sum as P .

Theorem 4. A polynomial pencil $\mathcal{L} \subset \mathcal{H}_n$ is an LD -pencil if and only if there is a shift $\mathcal{L}_s(P, Q)$ of \mathcal{L} such that the root map $\mathcal{Z}|_{\mathcal{L}_s(P, Q)}$ has a local minimum with respect to the spectral order, i.e., there exist $\lambda_0 \in \mathbf{R}$ and $\varepsilon > 0$ such that

$$P(x + \lambda_0) - \lambda_0 Q(x + \lambda_0) \preceq P(x + \lambda) - \lambda Q(x + \lambda) \text{ for } \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon),$$

where \preceq denotes the partial ordering on \mathcal{H}_n .

The proof of the sufficiency part of Theorem 4 relies on the following lemma.

Lemma 1. Let \mathcal{L} be a polynomial pencil in \mathcal{H}_n with a basis $\{P, Q\}$ as in (3), and assume that there exists $\varepsilon > 0$ such that

$$(8) \quad P(x) \preceq P(x + \mu) - \mu Q(x + \mu) \text{ for } \mu \in (-\varepsilon, \varepsilon).$$

Then $Q = P'$, so that \mathcal{L} is an LD -pencil with canonical basis $\{P, P'\}$.

Proof. Let S denote the greatest common monic divisor of P and Q . Then either $S \equiv 1$ or $\deg S \geq 1$, in which case we denote by w_1, \dots, w_k the distinct zeros of S with multiplicities s_1, \dots, s_k , respectively. Hence, we may write

$$\begin{aligned} S(x) &= \prod_{i=1}^k (x - w_i)^{s_i}, \\ \tilde{P}(x) &:= \frac{P(x)}{S(x)} = \prod_{i=1}^d (x - x_i), \\ \tilde{Q}(x) &:= \frac{Q(x)}{S(x)} = n \prod_{i=1}^{d-1} (x - y_i), \end{aligned}$$

where $d \geq 1$, $\sum_{i=1}^k s_i = n - d$, $x_1 < y_1 < x_2 < \dots < x_{d-1} < y_{d-1} < x_d$ if $d \geq 2$, with the usual understanding that empty products are equal to one while empty sums equal zero. For $\mu \in \mathbf{R}$ let $\tilde{R}_\mu = \tilde{P} - \lambda \tilde{Q}$ and $R_\mu = P - \mu Q = S \tilde{R}_\mu$. Denote by $x_i(\mu)$, $1 \leq i \leq d$, the zeros of \tilde{R}_μ , which we label so that $x_i(0) = x_i$ for $1 \leq i \leq d$. Since \tilde{R}_μ and \tilde{Q} have strictly interlacing zeros if $d \geq 2$, all of these zeros are simple. We may therefore use a computation similar to (6) to get

$$(9) \quad x'_i(\mu) = \frac{\tilde{Q}(x_i(\mu))}{\tilde{R}'_\mu(x_i(\mu))} \quad \text{for } \mu \in \mathbf{R}, \text{ and } 1 \leq i \leq d.$$

Choose $c \in \mathbf{R}$ such that $\zeta + c \neq 0$ whenever $P(\zeta) = 0$, and consider the sequence of convex functions $\{f_m\}_{m=1}^\infty$ given by $f_m(x) = (x + c)^m$. Note that the complete list of zeros of R_μ consists of $w_j - \mu$, $1 \leq j \leq k$, with multiplicities s_1, \dots, s_k , respectively, and $x_i(\mu) - \mu$, $1 \leq i \leq d$, and that condition (8) reads $R_0 \preccurlyeq R_\mu$ for $|\mu| < \varepsilon$. By Theorem 1 this implies that the differentiable function

$$\mathbf{R} \ni \mu \mapsto \sum_{i=1}^d f_m(x_i(\mu) - \mu) + \sum_{j=1}^k s_j f_m(w_j - \mu)$$

has a local minimum at $\mu = 0$ for any fixed $m \in \mathbf{N}$. Differentiation with respect to μ and formula (9) then yield the identities

$$(10) \quad \sum_{i=1}^d \left(\frac{\tilde{Q}(x_i)}{\tilde{P}'(x_i)} - 1 \right) (x_i + c)^{m-1} - \sum_{j=1}^k s_j (w_j + c)^{m-1} = 0 \quad \text{for } m \in \mathbf{N}.$$

From (10) and the choice of c , we deduce that the following relations must hold:

$$\begin{aligned}
 &k \leq d; \quad \tilde{P}(w_j) = 0 \text{ for } 1 \leq j \leq k; \\
 &P(x) = \prod_{i=1}^d (x - x_i)^{n_i}, \text{ where } n_i = s_i + 1 \\
 &\text{if } S(x_i) = 0 \text{ and } n_i = 1 \text{ otherwise;} \\
 &\text{and } \frac{\tilde{Q}(x_i)}{\tilde{P}'(x_i)} = n_i \text{ for } 1 \leq i \leq d.
 \end{aligned}$$

Using these relations and a partial fractional decomposition, we obtain

$$\frac{Q(x)}{P(x)} = \frac{S(x)\tilde{Q}(x)}{S(x)\tilde{P}(x)} = \frac{\tilde{Q}(x)}{\tilde{P}(x)} = \sum_{i=1}^d \frac{\tilde{Q}(x_i)}{\tilde{P}'(x_i)} \frac{1}{x - x_i} = \sum_{i=1}^d \frac{n_i}{x - x_i} = \frac{P'(x)}{P(x)}$$

for all $x \neq x_i, 1 \leq i \leq d$. It follows that $Q = P'$, as required. □

The proof of the necessity part of Theorem 4 is based on a criterion for classical majorization due to Hardy, Littlewood, and Pólya [12]. It should be mentioned that there are no known analogs of this criterion for multivariate majorization.

Theorem 5. *Let $X = (x_1 \leq x_2 \leq \dots \leq x_n)$ and $Y = (y_1 \leq y_2 \leq \dots \leq y_n)$ be two n -tuples of real numbers. Then $X \prec Y$ if and only if x_i and $y_i, 1 \leq i \leq n$, satisfy the following conditions:*

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \text{ and } \sum_{i=0}^k x_{n-i} \leq \sum_{i=0}^k y_{n-i} \text{ for } 0 \leq k \leq n - 2.$$

Lemma 2. *If $P \in \mathcal{H}_n$, then there exists $\varepsilon > 0$ such that for all real λ with $|\lambda| < \varepsilon$ one has $P(x) \preceq P(x + \lambda) - \lambda P'(x + \lambda)$.*

Proof. We show first that for any $P \in \mathcal{H}_n$ there exists $\varepsilon_1 = \varepsilon_1(P) > 0$ such that if $\lambda \in [0, \varepsilon_1)$ then $P(x) \preceq P(x + \lambda) - \lambda P'(x + \lambda)$. Let

$x_1 < \dots < x_d$ denote the distinct zeros of P with multiplicities n_1, \dots, n_d , respectively. Set $S(x) = \prod_{i=1}^d (x - x_i)^{n_i - 1}$, $\tilde{P} = P/S$, $Q = P'/S$, and $\tilde{R}_\lambda = \tilde{P} - \lambda Q$, $\lambda \in \mathbf{R}$. Clearly, \tilde{R}_λ is a strictly hyperbolic polynomial for all $\lambda \in \mathbf{R}$. Denote its zeros by $x_1(\lambda), \dots, x_d(\lambda)$, and label these so that $x_i(0) = x_i$, $1 \leq i \leq d$. An argument similar to the one used in the proof of Theorem 2 shows that all these zeros are increasing functions of λ . Moreover, by analogy with (6) and (7) and some straightforward computations, we obtain

$$x'_i(0) = \frac{Q(x_i(0))}{\tilde{P}'(x_i(0))} = \frac{Q(x_i)}{\tilde{P}'(x_i)} = n_i$$

and

$$\begin{aligned} (11) \quad x''_i(0) &= (x'_i(0))^2 \left[\frac{2Q'}{Q} - \frac{\tilde{R}''_0}{\tilde{R}'_0} \right] (x_i(0)) \\ &= n_i^2 \left[\frac{2Q'(x_i)}{Q(x_i)} - \frac{\tilde{P}''(x_i)}{\tilde{P}'(x_i)} \right] = 2 \sum_{\substack{j=1 \\ j \neq i}}^d \frac{n_i n_j}{x_i - x_j} \\ &\text{for } 1 \leq i \leq d; \quad \sum_{i=1}^d x''_i(0) = 0; \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^k x''_i(0) &= 2 \sum_{i=1}^k \sum_{j=k+1}^d \frac{n_i n_j}{x_i - x_j} < 0 \\ &\text{if } k \leq d - 1 \text{ since } x_1 < \dots < x_d. \end{aligned}$$

Assume for now that $\lambda \geq 0$, and set $R_\lambda = P - \lambda P' = S\tilde{R}_\lambda$. Let $z_m(\lambda)$, $1 \leq m \leq n$, be the zeros of R_λ , which we label as follows. Given $m \in \{1, 2, \dots, n\}$, there is a unique $i = i(m) \in \{1, \dots, d\}$ such that $\sum_{j=0}^{i-1} n_j < m \leq \sum_{j=0}^i n_j$, where $n_0 := 0$. Then we set $z_m(\lambda) = x_i$, $\lambda \geq 0$, if $n_i \geq 2$ and $m < \sum_{j=0}^i n_j$, and we let $z_m(\lambda) = x_i(\lambda)$, $\lambda \geq 0$, otherwise. Note that with this labeling we have $z_1(\lambda) \leq z_2(\lambda) \leq \dots \leq z_n(\lambda)$ for any $\lambda \geq 0$. Furthermore, if

$1 \leq m \leq n - 1$, then for all small $\lambda \geq 0$, we get

$$(12) \quad \sum_{j=1}^m (z_j(\lambda) - \lambda) - \sum_{j=1}^m z_j(0) = - \left(m - \sum_{j=0}^{i(m)-1} n_j \right) \lambda + \mathcal{O}(\lambda^2)$$

if $m < \sum_{j=0}^{i(m)} n_j$,

$$\sum_{j=1}^m (z_j(\lambda) - \lambda) - \sum_{j=1}^m z_j(0) = \frac{1}{2} \left(\sum_{j=1}^{i(m)} x_j''(0) \right) \lambda^2 + \mathcal{O}(\lambda^3)$$

if $m = \sum_{j=0}^{i(m)} n_j$.

From (11) and (12) we see that there exists $\varepsilon_1 = \varepsilon_1(P) > 0$ such that if $\lambda \in [0, \varepsilon_1]$ and $1 \leq m \leq n - 1$, then $\sum_{j=1}^m (z_j(\lambda) - \lambda) \leq \sum_{j=1}^m z_j(0)$, which is the same as

$$\sum_{i=1}^k z_{n-i}(0) \leq \sum_{i=1}^k (z_{n-i}(\lambda) - \lambda) \text{ for } \lambda \in [0, \varepsilon_1] \text{ and } 0 \leq k \leq n - 2$$

since $\sum_{j=1}^n (z_j(\lambda) - \lambda) = \sum_{j=1}^n z_j(0)$ whenever $\lambda \geq 0$. By Theorem 5 this means that $P(x) \preceq P(x + \lambda) - \lambda P'(x + \lambda)$ for $\lambda \in [0, \varepsilon_1]$, as required. In order to complete the proof let $P_1(x) = (-1)^n P(-x) \in \mathcal{H}_n$. The above arguments applied to P_1 show that there exists some $\varepsilon_2 = \varepsilon_2(P) > 0$ such that $P_1(x) \preceq P_1(x + \mu) - \mu P_1'(x + \mu)$ for all $\mu \in [0, \varepsilon_2]$. Since $\mathcal{Z}(P_1(x + \mu) - \mu P_1'(x + \mu)) = -\mathcal{Z}(P(x + \lambda) - \lambda P'(x + \lambda))$, where $\lambda = -\mu$, it follows that $P(x) \preceq P(x + \lambda) - \lambda P'(x + \lambda)$ for any real λ with $|\lambda| < \varepsilon := \min(\varepsilon_1, \varepsilon_2)$. This finishes the proof of the lemma. \square

Proof of Theorem 4. Let \mathcal{L} be a polynomial pencil in \mathcal{H}_n such that there exists a shift $\mathcal{L}_s(P, Q)$ of \mathcal{L} that satisfies the local minimum property stated in the theorem with $\lambda_0 \in \mathbf{R}$ and $\varepsilon > 0$. Set $\widehat{P}(x) = P(x + \lambda_0) - \lambda_0 Q(x + \lambda_0)$, $\widehat{Q}(x) = Q(x + \lambda_0)$, and $\mu = \lambda - \lambda_0$. Clearly, the local minimum condition translates into

$$\widehat{P}(x) \preceq \widehat{P}(x + \mu) - \mu \widehat{Q}(x + \mu) \text{ for } \mu \in (-\varepsilon, \varepsilon).$$

Applying Lemma 1 to the polynomial pencil $\widehat{\mathcal{L}} := \{\widehat{P} - \mu\widehat{Q} \mid \mu \in \mathbf{R}\}$, we get $\widehat{Q} = \widehat{P}'$. Hence, $Q = P' - \lambda_0 Q'$, so that $(P - \lambda_0 Q)' = (P - \lambda_0 Q) - (P - (\lambda_0 + 1)Q)$ and thus \mathcal{L} is an LD -pencil by Definition 2. Conversely, if \mathcal{L} is an LD -pencil in \mathcal{H}_n with canonical basis $\{P, P'\}$ then Lemma 2 shows that the shift $\mathcal{L}_s(P, P')$ satisfies the local minimum property stated in the theorem. \square

Remark 3. The necessity part of Theorem 4 may also be seen as a corollary of Theorem 6 below, where it is shown that LD -pencils in \mathcal{H}_n satisfy in fact a global minimum property with respect to the spectral order. Note also that, unlike the property stated in Theorem 4, the minimum property for (twisted) span functions obtained in Corollary 1 is not specific for the class of hyperbolic LD -pencils.

4. Spectral order and differential operators of Laguerre-Pólya type. The monoid $\mathcal{A}_n(\mathbf{R}) = \{T \in \text{End } \Pi \mid T(\mathcal{H}_n) \subseteq \mathcal{H}_n\}$ was previously defined only for $n \geq 1$. Let us extend this notation to $n = 0$ by putting $\mathcal{H}_0 = \{1\} \subset \Pi$ and $\mathcal{A}_0(\mathbf{R}) = \{T \in \text{End } \Pi \mid T(\mathcal{H}_0) = \mathcal{H}_0\}$. Given a nonconstant polynomial $P \in \Pi$ we denote by $\sigma(P)$ the sum of the zeros of P . Set

$$\mathcal{A} = \bigcap_{n=0}^{\infty} \mathcal{A}_n(\mathbf{R})$$

and

$$\mathcal{A}' = \{T \in \mathcal{A} \mid \sigma(T(P)) = \sigma(P) \text{ if } P \in \Pi, \deg P \geq 1\}.$$

Thus \mathcal{A} is the largest monoid of linear operators that act on each \mathcal{H}_n for $n \geq 0$ while \mathcal{A}' is the largest submonoid of \mathcal{A} consisting of operators whose action on Π preserves the average of the zeros of any nonconstant polynomial. As we already saw in Section 2, hyperbolic LD -pencils may be described by means of a local minimum property that involves the spectral order on \mathbf{R} . More generally, the study of the relative location of the zero sets $\mathcal{Z}(T(P))$ and $\mathcal{Z}(P)$ for $T \in \mathcal{A}'$ and $P \in \mathcal{H}_n$ reveals some interesting connections between the action of hyperbolicity-preserving linear operators on the manifold \mathcal{H}_n and classical majorization. Indeed, the main result of this section shows that if $n \geq 1$ then any polynomial in \mathcal{H}_n is the global minimum of its \mathcal{A}' -orbit with respect to the partial ordering on \mathcal{H}_n :

Theorem 6. *If $n \geq 1$ and $P \in \mathcal{H}_n$, then $P \preceq T(P)$ for any $T \in \mathcal{A}'$.*

Before embarking on the proof let us point out that [7, Theorem 1] and the Hermite-Poulain theorem yield actually a complete description of the structure of the monoids \mathcal{A} and \mathcal{A}' :

Theorem 7. *An operator $T \in \text{End} \Pi$ belongs to \mathcal{A} if and only if $T = \varphi(D)$, where $D = d/dx$ and φ is a real entire function in the Laguerre-Pólya class of the form*

$$\varphi(x) = e^{-a^2x^2+bx} \prod_{k=1}^{\infty} (1 - \alpha_k x) e^{\alpha_k x}$$

with $a, b, \alpha_k \in \mathbf{R}$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. In particular, \mathcal{A} is a commutative monoid.

Corollary 4. *The monoid \mathcal{A}' consists of linear operators of the form $\varphi(D)$, where $D = d/dx$ and φ is a real entire function in the Laguerre-Pólya class given by*

$$\varphi(x) = e^{-a^2x^2} \prod_{k=1}^{\infty} (1 - \alpha_k x) e^{\alpha_k x}$$

with $a, \alpha_k \in \mathbf{R}$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Thus, $\mathcal{A} = \mathcal{A}' \times \langle e^{bD} \mid b \in \mathbf{R} \rangle$.

In view of Theorem 7 it seems reasonable to adopt the following terminology: an operator $T \in \text{End} \Pi$ is said to be a *differential operator of Laguerre-Pólya type* if $T = \varphi(D)$, where $D = d/dx$ and φ is a real entire function in the Laguerre-Pólya class. Such operators were studied in, e.g., [9] in connection with various generalizations of the Pólya-Wiman conjecture. Since \mathcal{A}' contains only differential operators of Laguerre-Pólya type, it is enough to check that Theorem 6 is true for the “building blocks” of these operators, that is, differential operators of the form $(1 - \lambda D)e^{\lambda D}$ or $e^{-\lambda^2 D}$ with $\lambda \in \mathbf{R}$. To do this we need the following lemma.

Lemma 3. *Let $n \geq 2$, $\mathbf{x} = (x_1 < \dots < x_n) \in \mathbf{R}^n$, and $P(x) = \prod_{i=1}^n (x - x_i) \in \mathcal{H}_n$. For $\lambda \in \mathbf{R}$, denote by $\zeta_i = \zeta_i(\lambda; \mathbf{x})$, $1 \leq i \leq n$, the zeros of the strictly hyperbolic polynomial $P - \lambda P'$. If*

these are labeled so that $\zeta_i(0; \mathbf{x}) = x_i$, $1 \leq i \leq n$, then

$$\begin{aligned} (\zeta_i(\lambda; \mathbf{x}) - x_j)^2 \frac{\partial \zeta_i}{\partial x_j}(\lambda; \mathbf{x}) &= \lambda^2 \frac{\partial \zeta_i}{\partial \lambda}(\lambda; \mathbf{x}) > 0 \\ \text{for } \lambda \neq 0 \text{ and } 1 \leq i, j \leq n. \end{aligned}$$

In particular, for any fixed values $x_1 < \cdots < x_{n-1}$ and $\lambda \neq 0$, each of the functions $(x_{n-1}, \infty) \ni x_n \mapsto \zeta_i(\lambda; \mathbf{x})$, $1 \leq i \leq n$, is increasing.

Proof. Let $1 \leq i, j \leq n$, $\mathbf{x} \in \mathbf{R}^n$, and set $P(x) = (x - x_j)Q(x)$. For $\lambda \in \mathbf{R}$ we get

$$\begin{aligned} P(x) - \lambda P'(x) &= (x - x_j) [Q(x) - \lambda Q'(x)] - \lambda Q(x), \\ P'(x) - \lambda P''(x) &= Q(x) - 2\lambda Q'(x) + (x - x_j) [Q'(x) - \lambda Q''(x)], \\ P(\zeta_i(\lambda; \mathbf{x})) &= \lambda P'(\zeta_i(\lambda; \mathbf{x})) = (\zeta_i(\lambda; \mathbf{x}) - x_j) Q(\zeta_i(\lambda; \mathbf{x})), \\ \text{and } [P'(\zeta_i(\lambda; \mathbf{x})) - \lambda P''(\zeta_i(\lambda; \mathbf{x}))] &\frac{\partial \zeta_i}{\partial \lambda}(\lambda; \mathbf{x}) = P'(\zeta_i(\lambda; \mathbf{x})). \end{aligned}$$

The arguments in the proof of Theorem 2 show that if \mathbf{x} is fixed then for any λ one has $\partial \zeta_i(\lambda; \mathbf{x}) / \partial \lambda > 0$ and $\zeta_i(\lambda; \mathbf{x}) \neq x_k$, $1 \leq k \leq n$, if $\lambda \neq 0$. By differentiating the identity

$$(\zeta_i(\lambda; \mathbf{x}) - x_j) [Q(\zeta_i(\lambda; \mathbf{x})) - \lambda Q'(\zeta_i(\lambda; \mathbf{x}))] = \lambda Q(\zeta_i(\lambda; \mathbf{x}))$$

with respect to x_j and using the relations listed above, we arrive at the desired conclusion. \square

Proof of Theorem 6. The theorem holds trivially for $n = 1$ since $T|_{\mathcal{H}_1} = \text{Id}_{\mathcal{H}_1}$ if $T \in \mathcal{A}'$ by Corollary 4. Hence we may assume that $T \in \mathcal{A}'$ and $P \in \mathcal{H}_n$ with $n \geq 2$.

Step 1. P is strictly hyperbolic and $T = (1 - \lambda D)e^{\lambda D}$ for some $\lambda \in \mathbf{R}$. Using the notations of Lemma 3 we denote the zeros of P by $x_1 < \cdots < x_n$ and those of the (strictly hyperbolic) polynomial $P_\lambda := P - \lambda P'$ by $\zeta_i = \zeta_i(\lambda; \mathbf{x})$, $1 \leq i \leq n$, where $\mathbf{x} = (x_1 < \cdots < x_n) \in \mathbf{R}^n$. We further assume that the latter are labeled so that $\zeta_i(0; \mathbf{x}) = x_i$, $1 \leq i \leq n$. As in the first step of the proof of Theorem 2 we see that this labeling of

the zeros yields $\zeta_1(\lambda; \mathbf{x}) < \dots < \zeta_n(\lambda; \mathbf{x})$ for all $\lambda \in \mathbf{R}$. By Theorem 5 the relation $P \preccurlyeq T(P)$ is equivalent to the following inequalities:

$$(13) \quad \sum_{i=1}^j (\zeta_i(\lambda; \mathbf{x}) - \lambda) \leq \sum_{i=1}^j x_i \text{ for } 1 \leq j \leq n - 1.$$

We now prove these inequalities by induction on n . Clearly, if $P(x) = x^2 + ax + b$ with $a, b \in \mathbf{R}$ such that $a^2 > 4b$, then

$$-2(\zeta_1(\lambda; \mathbf{x}) - \lambda) = a + \sqrt{a^2 - 4b + 4\lambda^2} \geq a + \sqrt{a^2 - 4b} = -2x_1$$

for $\lambda \in \mathbf{R}$.

Thus, (13) is true for $n = 2$. Let $n \geq 3$ and assume that (13) holds for all monic strictly hyperbolic polynomials of degree at most $n - 1$. Then we may write

$$P(x) = (x - x_n)Q(x) \quad \text{and} \quad P_\lambda(x) = (x - x_n)Q_\lambda(x) - \lambda Q(x),$$

where

$$Q(x) = \prod_{i=1}^{n-1} (x - x_i), \quad Q_\lambda(x) := Q(x) - \lambda Q'(x) = \prod_{i=1}^{n-1} (x - \omega_i(\lambda; \mathbf{x}')),$$

$$\mathbf{x}' = (x_1 < \dots < x_{n-1}) \in \mathbf{R}^{n-1}, \quad \omega_i(0; \mathbf{x}') = x_i, \quad 1 \leq i \leq n - 1.$$

Note that with this labeling we get $\omega_1(\lambda; \mathbf{x}') < \dots < \omega_{n-1}(\lambda; \mathbf{x}')$ for all $\lambda \in \mathbf{R}$ and that if we fix λ then $\zeta_i(\lambda; \mathbf{x}) \rightarrow \omega_i(\lambda; \mathbf{x}')$ for $1 \leq i \leq n - 1$ while $\zeta_n(\lambda; \mathbf{x}) \rightarrow \infty$ as $x_n \rightarrow \infty$. By Lemma 3 and the induction assumption applied to Q , we obtain

$$\sum_{i=1}^j (\zeta_i(\lambda; \mathbf{x}) - \lambda) < \sum_{i=1}^j (\omega_i(\lambda; \mathbf{x}') - \lambda) \leq \sum_{i=1}^j x_i$$

for $\lambda \neq 0$ and $1 \leq j \leq n - 1$,

which proves (13). We conclude that the theorem is true in this generic case.

Step 2. The general case. Let P be an arbitrary polynomial in \mathcal{H}_n , and consider first an operator T of the form $(1 - \lambda D)e^{\lambda D}$ for some

fixed $\lambda \in \mathbf{R}$. As in the proof of Corollary 2 we denote by P_ε the strictly hyperbolic polynomial with zeros $x_k + k\varepsilon$, $1 \leq k \leq n$, where $\varepsilon > 0$ and $x_1 \leq \dots \leq x_n$ are the (possibly multiple) zeros of P . By Step 1 we have $P_\varepsilon \preceq T(P_\varepsilon)$ for all $\varepsilon > 0$. Using a standard continuity argument we see that the relation $P \preceq T(P)$ is just the limit case $\varepsilon \rightarrow 0$ of the aforementioned relations. Alternatively, we may approximate P with the polynomial $(1 - \varepsilon D)^{n-1}P$, which is strictly hyperbolic for all $\varepsilon \neq 0$ by [9, Lemma 4.2]. Finally, if T is an operator of the form $e^{-\lambda^2 D}$ with $\lambda \in \mathbf{R}$, then

$$P \preceq \lim_{m \rightarrow \infty} \left[\left(1 - \frac{\lambda D}{\sqrt{m}} \right) e^{\lambda D / \sqrt{m}} \right]^m \left[\left(1 + \frac{\lambda D}{\sqrt{m}} \right) e^{-\lambda D / \sqrt{m}} \right]^m P = T(P)$$

since $P \preceq (1 - \mu D)e^{\mu D}P$ for $\mu \in \mathbf{R}$. This completes the proof of the theorem. \square

Note that Theorem 7 and Corollary 2 imply that operators in \mathcal{A} do not decrease the span of hyperbolic polynomials. We can actually deduce an even more general result from Theorem 6 and some well-known properties of classical majorization. Recall that a function $F : \mathbf{R}^n \rightarrow \mathbf{R}$ is called *Schur convex* if $F(X) \leq F(Y)$ for all $X, Y \in \mathbf{R}^n$ with $X \prec Y$, cf., e.g., [14]. Clearly, any such function is symmetric on \mathbf{R}^n and may therefore be viewed as a function on \mathbf{R}^n / Σ_n , where Σ_n denotes as before the symmetric group on n elements. Theorem 6 and (11) yield the following conditions on the relative geometry of $\mathcal{Z}(P)$ and $\mathcal{Z}(T(P))$ for $P \in \mathcal{H}_n$ and $T \in \mathcal{A}'$:

Corollary 5. *Let $n \geq 2$, $P \in \mathcal{H}_n$, and denote the zeros of P by $x_i(P)$, $1 \leq i \leq n$.*

(i) *If $T \in \mathcal{A}'$, then $\min \mathcal{Z}(T(P)) \leq \min \mathcal{Z}(P)$ and $\max \mathcal{Z}(P) \leq \max \mathcal{Z}(T(P))$, so that $\Delta(P) \leq \Delta(S(P))$ for any operator $S \in \mathcal{A}$. All these inequalities are strict unless $T = S = \text{Id}_\Pi$.*

(ii) *The inequality $(F \circ \mathcal{Z})(P) \leq (F \circ \mathcal{Z})(T(P))$ holds for any Schur convex function F on \mathbf{R}^n / Σ_n and any operator $T \in \mathcal{A}'$. In particular, if $f : \mathbf{R} \rightarrow \mathbf{R}$ is a convex function then $\sum_{i=1}^n f(x_i(P)) \leq \sum_{i=1}^n f(x_i(T(P)))$.*

5. Related topics and open problems.

5.1. Toward an analytic theory of classical majorization.

Although we did not explicitly address the question of describing all operators in the monoid $\mathcal{A}_n(\mathbf{R})$, Problem 1 and the results of the previous sections are certainly a good motivation for studying this question. Indeed, these results show that even a partial knowledge of operators in $\mathcal{A}_n(\mathbf{R})$ can provide some interesting information on the relative geometry of the zeros of a hyperbolic polynomial and the zeros of its images under such operators. We therefore propose the following general problem.

Problem 2. Let $n \geq 2$, and set $P_{\preceq} = \{Q \in \mathcal{H}_n \mid P \preceq Q\}$ for $P \in \mathcal{H}_n$. Define the monoid $\mathcal{B}_n = \{T \in \mathcal{A}_n(\mathbf{R}) \mid P \preceq T(P) \text{ if } P \in \mathcal{H}_n\}$ and note that $\mathcal{A}' \subseteq \mathcal{B}_n$ by Theorem 6 and that $\mathcal{B}_n P \subseteq P_{\preceq}$ for all $P \in \mathcal{H}_n$, where $\mathcal{B}_n P = \{T(P) \mid T \in \mathcal{B}_n\}$.

(i) Is the inclusion $\mathcal{A}' \subseteq \mathcal{B}_n$ strict for all $n \geq 2$? Describe all operators in \mathcal{B}_n .

(ii) Is it possible to describe classical majorization by means of the action of linear (differential) operators on hyperbolic polynomials? In other words, is it true that $\mathcal{B}_n P = P_{\preceq}$ for all $P \in \mathcal{H}_n$ if $n \geq 2$?

(iii) Characterize all operators in the monoid $\mathcal{A}_n(\mathbf{R})$.

Using Corollary 5 it is not difficult to show that $\mathcal{A}' P \subsetneq P_{\preceq}$ whenever $n \geq 3$ and $P \in \mathcal{H}_n$ is such that $\Delta(P) > 0$, where $\mathcal{A}' P = \{T(P) \mid T \in \mathcal{A}'\}$. In particular, if P is strictly hyperbolic, then $\mathcal{A}' P \subsetneq P_{\preceq}$. Thus, if the answer to the first question in Problem 2 (i) were negative, then it would not be possible to get a description of the spectral order on \mathbf{R} as suggested in part (ii) of Problem 2. Nevertheless, it seems likely that $\mathcal{A}' \subsetneq \mathcal{B}_n$ for $n \geq 2$.

Note also that an affirmative answer to Problem 2 (ii) would actually provide a description of classical majorization which in a way would be dual to the usual characterization by means of doubly stochastic matrices (cf. Theorem 1): the former deals with the set $P_{\preceq} = \{Q \in \mathcal{H}_n \mid P \preceq Q\}$, while the latter deals with the “polytope” $P_{\succ} := \{Q \in \mathcal{H}_n \mid Q \succ P\}$, where P is an arbitrary polynomial in \mathcal{H}_n . We refer to [4] for a further study of these questions and related topics.

5.2. Pencils of complex polynomials. The manifold \mathcal{C}_n is a natural context for discussing possible extensions of the results in Sections 1–3 to the complex case. Indeed, by analogy with the hyperbolic case, we may view (\mathcal{C}_n, \preceq) as a partially ordered set, where the ordering relation \preceq is now induced by the spectral order on n -tuples of vectors in \mathbf{R}^2 , cf. Theorem 1 and Birkhoff's theorem. This means that zero sets of polynomials in \mathcal{C}_n are viewed as subsets of \mathbf{R}^2 and that if $P, Q \in \mathcal{C}_n$ then $P \preceq Q$ if and only if $Z(P) \prec Z(Q)$. The following example shows that if the partial ordering \preceq on \mathcal{C}_n is defined in this way then one cannot expect a complex analog of Theorem 6.

Proposition 1. *Let $P(z) = z^n - 1$ and $\lambda \in \mathbf{C}^*$. If $n \geq 5$ and $|\lambda|$ is small enough, then $(1 - \lambda D)e^{\lambda D}P$ and P are incomparable as elements of the poset (\mathcal{C}_n, \preceq) .*

Proof. Let $z_k(\lambda)$, $1 \leq k \leq n$, denote the zeros of $(1 - \lambda D)e^{\lambda D}P$, which we label so that $z_k(0) = z_k := e^{(2\pi ki)/n}$ for $1 \leq k \leq n$. Since $z_k(\lambda)$, $1 \leq k \leq n$, are analytic functions of λ in a neighborhood of 0, we can use formulas (6)–(7) to show that $2(z_k(\lambda) - \lambda) = 2z_k + (n-1)\bar{z}_k\lambda^2 + \mathcal{O}(\lambda^3)$, $1 \leq k \leq n$. Hence,

$$|z_k(\lambda) - \lambda| = 1 + \frac{n-1}{2}\Re(\bar{z}_k\lambda^2) + \mathcal{O}(\lambda^3) \text{ for } 1 \leq k \leq n.$$

It is geometrically clear that if $n \geq 5$ and $\lambda \neq 0$ then there exist distinct indices k_1 and k_2 such that $\Re(\bar{z}_{k_1}\lambda^2) > 0$ and $\Re(\bar{z}_{k_2}\lambda^2) < 0$. This implies that if $n \geq 5$ and $|\lambda|$ is a small enough positive number then $|z_{k_1}(\lambda) - \lambda| > 1$ and $|z_{k_2}(\lambda) - \lambda| < 1$ for some $1 \leq k_1 \neq k_2 \leq n$. It follows that for these values of λ there can be no inclusion relation between the convex hulls of the zeros of $(1 - \lambda D)e^{\lambda D}P$ and P , so that these polynomials are incomparable as elements of the poset (\mathcal{C}_n, \preceq) . \square

We also note that the results of the previous sections concerning hyperbolic polynomials are valid only for real values of the parameter λ :

Proposition 2. *For any $n \geq 3$ and $\theta \in (0, \pi) \cup (\pi, 2\pi)$, there exists $\varepsilon = \varepsilon(n, \theta) > 0$ with the following property: if $\lambda = re^{i\theta}$ and $r \in (0, \varepsilon)$, then one can find $P \in \mathcal{H}_n$ such that P and $(1 - \lambda D)e^{\lambda D}P$ are incomparable as elements of the poset (\mathcal{C}_n, \preceq) .*

Proof. Given a function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ and $z \in \mathbf{C}$, we shall write $f(z)$ instead of $f(x, y)$, where $z = x + iy$. Set $I = (0, \pi) \cup (\pi, 2\pi)$ and assume first that $\theta \in I \setminus \{\pi/2, 3\pi/2\}$. Let P be a strictly hyperbolic polynomial in \mathcal{H}_n with zeros z_j , $1 \leq j \leq n$, and denote the zeros of $P - \lambda P'$ by $z_j(\lambda)$, $1 \leq j \leq n$, where $\lambda = re^{i\theta}$ and $r \geq 0$. If r is small enough, we may label these zeros so that $z_j(0) = z_j$, $1 \leq j \leq n$. Note that

$$\sum_{j=1}^n \frac{z_j P''(z_j)}{P'(z_j)} = n(n-1)$$

and

$$z_j(\lambda) - \lambda = z_j + \frac{z_j P''(z_j)}{2P'(z_j)} \lambda^2 + \mathcal{O}(\lambda^3), \quad 1 \leq j \leq n,$$

by (6) and (7). Fix $a \in \mathbf{R}$ such that $|a| > |\cot 2\theta|$, and consider the convex functions $f_{\pm a} : \mathbf{R}^2 \rightarrow \mathbf{R}$ defined by $f_{\pm a}(x, y) = (x \pm ay)^2$. Using the above relations, we get

$$\begin{aligned} F_{\pm a}(\lambda) &:= \sum_{j=1}^n f_{\pm a}(z_j(\lambda) - \lambda) - \sum_{j=1}^n f_{\pm a}(z_j) \\ &= r^2(\cos 2\theta \pm a \sin 2\theta) \sum_{j=1}^n \frac{z_j P''(z_j)}{P'(z_j)} + \mathcal{O}(r^3) \\ &= n(n-1)(\cos 2\theta \pm a \sin 2\theta)r^2 + \mathcal{O}(r^3). \end{aligned}$$

Clearly, these formulas and the choice of a show that there exists $\varepsilon_1 = \varepsilon_1(n, \theta) > 0$ such that $F_a(\lambda)F_{-a}(\lambda) < 0$ whenever $|\lambda| \in (0, \varepsilon_1)$. By Theorem 1 we see that P and $(1 - \lambda D)e^{\lambda D}P$ are incomparable with respect to the partial ordering \preceq on \mathcal{C}_n .

Let us now consider the case when $\theta \in \{\pi/2, 3\pi/2\}$, that is, $\lambda = bi$ with $b \in \mathbf{R}$. Set $P(x) = x^n - x^{n-1} \in \mathcal{H}_n$, and note that if $|\lambda|$ is small enough then there is a unique zero of $P - \lambda P'$ with largest real part. We denote this zero of $P - \lambda P'$ by $z(\lambda)$. A computation shows that $z(\lambda) - \lambda = 1 + (n-1)\lambda^2 + \mathcal{O}(\lambda^3)$ and so there exists $\varepsilon_2 = \varepsilon_2(n) > 0$ such that $\Re(z(\lambda)) = 1 - (n-1)b^2 + \mathcal{O}(b^3) < 1$ if $|b| \in (0, \varepsilon_2)$. Since $n \geq 3$ the polynomial $(1 - \lambda D)e^{\lambda D}P$ has at least one zero at $-\lambda \notin \mathbf{R}$. It follows that in this case there can be no inclusion relation between the convex hulls of the zeros of $(1 - \lambda D)e^{\lambda D}P$ and P . Thus, as in the proof of Proposition 1 we conclude that these polynomials are incomparable as

elements of the poset (\mathcal{C}_n, \preceq) . To complete the proof of the proposition, we simply let $\varepsilon(n, \theta) = \min(\varepsilon_1, \varepsilon_2)$. \square

Propositions 1 and 2 suggest that complex generalizations of Theorem 6, if any, should involve only classical majorization and real values of the parameter λ . Using the computations in (11) and (12), it is not difficult to show that if P is a polynomial in \mathcal{C}_n whose zeros have distinct real parts then there exists $\varepsilon = \varepsilon(P) > 0$ such that $\Re Z(P) \prec \Re Z((1 - \lambda D)e^{\lambda D} P)$ if λ is real and $|\lambda| \leq \varepsilon$. Based on extensive numerical experiments, we make the following conjecture.

Conjecture 1. *If $T \in \mathcal{A}'$ and $n \geq 1$, then for any $P \in \mathcal{C}_n$ at least one of the relations $\Re Z(P) \prec \Re Z(T(P))$, $\Im Z(T(P)) \prec \Im Z(P)$ is valid.*

5.3. The Laguerre-Pólya class of functions. There are several known extensions of majorization to infinite sequences of real numbers [14, page 16]. A natural question is whether these extensions could yield infinite-dimensional analogs of Theorem 6. For this one would need to find both a suitable set of functions in the Laguerre-Pólya class and an appropriate submonoid of \mathcal{A}' that acts on this set. One could for instance consider the set of functions of genus 0 or 1 in the Laguerre-Pólya class. Indeed, using [9, Lemmas 3.1 and 3.2] one can show that this set is closed under the action of operators in \mathcal{A}' . Finally, it would be interesting to know whether there are any analogs of Conjecture 1 for transcendental entire functions with finitely or infinitely many complex zeros.

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ENDNOTES

1. Note added in the proof. The question of describing all linear operators that preserve the set of univariate hyperbolic (real-rooted) polynomials has been settled quite recently by the author jointly with P. Brändén, *Pólya-Schur master theorems for circular domains and their boundaries*, to appear in *Annals of Math.*, see www.arxiv.org/abs/math.CV/0607416.

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