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POSITIVE SOLUTION OF MULTI-POINT BOUNDARY VALUE PROBLEM FOR THE ONE-DIMENSIONAL P-LAPLACIAN WITH SINGULARITIES

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ABSTRACT. In the paper, we get positive solutions of the following multi-point singular boundary value problem with p-Laplacian operator

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u,u') = 0 & 0 < t < 1, \\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), & u'(1) = \sum_{i=1}^{n} \beta_i u'(\xi_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s, p > 1$; $\xi_i \in (0,1)$, $i = 1,2,\dots,n$, $0 \le \alpha_i$, $\beta_i < 1$, $i = 1,2,\dots,n$, $0 \le \sum_{i=1}^n \alpha_i$, $\sum_{i=1}^n \beta_i < 1$ and $f(t,u,u')$ may be singular at $u = 0$, $u' = 0$.

1. Introduction. In this paper we study the singular boundary value problem (BVP for short)

$$(1.1)\qquad \begin{cases} (\phi_p(u'))' + q(t)f(t,u,u') = 0 & 0 < t < 1, \\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), & u'(1) = \sum_{i=1}^{n} \beta_i u'(\xi_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$; $\xi_i \in (0,1)$, $i = 1,2,\dots,n$, $0 \le \alpha_i$, $\beta_i < 1$, $i = 1,2,\dots,n$, $0 \le \sum_{i=1}^n \alpha_i$, $\sum_{i=1}^n \beta_i < 1$ and $f(t,u,u')$ may be singular at $u = 0$, $u' = 0$, $q(t) \in C[0,1]$. The singular differential boundary value problem arises in many branches of both applied and basic mathematics and it has been extensively studied in the literature, for details, we refer the reader to [**2**].

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When $f(t,u,u') = f(t,u)$ has no singularity at $u = 0$, Bai [**4**] and Ma [**8**] studied two problems similar to (1.1) respectively, i.e.,

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u) = 0 & 0 < t < 1 \\ u'(0) = \displaystyle\sum_{i=1}^{n} \alpha_i u'(\xi_i), & u(1) = \displaystyle\sum_{i=1}^{n} \beta_i u(\xi_i), \end{cases}$$

and

$$\begin{cases} u'' + q(t)f(t,u) = 0 & 0 < t < 1 \\ u'(0) = \displaystyle\sum_{i=1}^{n} \alpha_i u'(\xi_i), & u(1) = \displaystyle\sum_{i=1}^{n} \beta_i u(\xi_i), \end{cases}$$

The tools used in [**4, 8**] are fixed point index theory and fixed point theorem in cones due to Krasnoselskii, respectively. When $p = 2$, and $f(t,u,u')$ has no singularity at $u = 0$, $u' = 0$, (1.1) has been also studied in [**5**] and its references. But we may see easily the method used in [**4, 5, 8**] is of no effect to (1.1) since $f(t,u,u')$ may be singular at $u = 0$, $u' = 0$ in our paper. BVP (1.1) contains the following BVP as a special case,

$$\begin{cases} u'' + q(t)f(t,u,u') = 0 & 0 < t < 1 \\ u(0) = 0, & u'(1) = 0, \end{cases} \tag{1.2}$$

when $f(t,u,u')$ may be singular at $u = 0$, $u' = 0$. Equation (1.2) has been studied extensively in [**2**].

In fact, when $f(t,u,u') = f(t,u)$ has singularity at $u = 0$, the first differential equation of (1.1) subjected to some other boundary conditions has been studied, for example,

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u) = 0 & 0 < t < 1 \\ u(0) = 0, & u(1) = 0, \end{cases} \tag{1.3}$$

when $f(t,u) = f(u)$ has singularity at $u = 0$, (1.3) has been studied in [**9**], when $f(t,u)$ has singularity at $u = 0$, (1.3) has also been studied in [**1**]; and

$$\begin{cases} (\phi_p(u'))' + q(t)f(t,u) = 0 & 0 < t < 1 \\ u(0) = 0, & u(1) + B(u'(1)) = 0, \end{cases} \tag{1.4}$$

when $f(t,u)$ has singularity at $u=0$, (1.4) has been studied in [**8**]. In regards to (1.1), to our knowledge there is not a paper in the literature which discusses it. As is known, one difficulty that appears is that, for $p\neq 2$, the differential operator $(\phi_p(u'))'$ is nonlinear, and thus, it is very difficult to change the differential equation in (1.1) to an equivalent integral equation, but in this paper, we use a technique to solve this. The method used in this paper is different from those of [**1**, **4**–**9**].

We shall denote by $C[0,1]$, respectively $C^1[0,1]$, the classical space of continuous, respectively continuously differentiable, real-valued functions on the interval $[0,1]$. The norm in $C[0,1]$ is denoted by $\|w\|_0=\max_{t\in[0,1]}|w(t)|$. The norm in $C^1[0,1]$ is denoted by $\|w\|=\max\{\|w\|_0,\|w'\|_0\}$. Then both $C[0,1]$ and $C^1[0,1]$ are Banach spaces.

In this paper, we say a function $w(t)$ is a positive solution to problem (1.1) if it satisfies the following conditions:

(i) $w\in C[0,1]\cap C^1[0,1]$,

(ii) $w(t)>0$ and $w'(t)>0$ for any $t\in(0,1)$,

(iii) $(\phi_p(w'))'(t)\in L^1[0,1]$ and

$$\begin{cases}(\phi_p(w'))'+q(t)f(t,w,w')=0 & 0<t<1\\ w(0)=\sum_{i=1}^n \alpha_i w(\xi_i), & w'(1)=\sum_{i=1}^n \beta_i w'(\xi_i).\end{cases}$$

We recall that a function w is said to be concave on $[0,1]$, if

$$w(\lambda t_2+(1-\lambda)t_1)\geq \lambda w(t_2)+(1-\lambda)w(t_1),\quad t_1,t_2,\lambda\in[0,1],$$

and a function is said to be monotone on $[0,1]$, if $w(t)$ is nondecreasing or nonincreasing. We denote

$$\begin{aligned}C^1_+[0,1]&=\{w\in C^1[0,1]: w(t)\geq 0,\ w'(t)\geq 0,\ t\in[0,1]\},\\ P&=\{w\in C^1_+[0,1]: w(t) \text{ is concave on } [0,1]\}.\end{aligned}$$

It is easy to see that P is a cone in $C^1[0,1]$.

We know easily that, when $p>1$, $\phi_p(s)$ is strictly increasing on $(-\infty,+\infty)$. So ϕ_p^{-1} exists. Moreover, $\phi_p^{-1}=\phi_q$, where $(1/p)+(1/q)=1$.

The following conditions are needed in this paper:

(H1) $q(t) \in C[0,1]$ with $q(t) > 0$, $t \in (0,1)$.

(H2) $f : [0,1] \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ is continuous;

(H3) there exist $A = 0$ or $A \geq 1$; $B \geq 1$; $C = 0$ or $C \geq 1$ and $0 \leq k \leq \min\{p-1, 1\}$, $l > 0$ such that $0 \leq f(t,u,u') \leq [f_1(u)+f_2(u)][A(u')^k + B(u')^{-l} + C]$ on $[0,1] \times (0,+\infty) \times (0,+\infty)$ with $f_1 > 0$ continuous, nonincreasing on $(0,+\infty)$ and $\int_0^L f_1(u)\,du < +\infty$ for any fixed $L > 0$; $f_2 \geq 0$ is continuous on $[0,+\infty)$;

(H4) for any $K > 0$, $N > 0$, there exists a function $\psi_{K,N}$ continuous on $[0,1]$ and positive on $(0,1)$ with $f(t,u,v) \geq \psi_{K,N}(t)$, $t \in (0,1)$, on $[0,1] \times (0,K] \times (0,N]$;

(H5) $(\phi_p^{-1}(\int_t^1 \psi(s)q(s)\,ds))^{-l} \in L^1[0,1]$ and $f_1(ct) \in L^1[0,1]$, $f_1(ct)$ $(\phi_p^{-1}(\int_t^1 \psi(s)q(s)\,ds))^{-l} \in L^1[0,1]$ for any fixed $c > 0$.

When $c > 0$, let

$$G(c) = \int_0^c (f_1(u) + f_2(u))\,du,$$
$$I(c) = \int_0^c \frac{\phi_p^{-1}(t)}{A(\phi_p^{-1}(t)+1)^k + B(\phi_p^{-1}(t))^{-l} + C}\,dt.$$

Then both $G(c)$ and $I(c)$ are strictly increasing about c. So $(I\phi_p)^{-1}(c) = \phi_p^{-1}(I^{-1}(c))$ exists on $(0,+\infty)$.

We state our main result as follows.

Theorem 3.1. *Assume* (H1)–(H5) *hold and*

$$\sup_{c \in (0,+\infty)} \frac{c}{(I\phi)^{-1}(G(c))\Gamma} > 1,$$

where

$$\Gamma = \frac{(1 - \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \xi_i)}{(1 - \sum_{i=1}^n \alpha_i)(1 - \sum_{i=1}^n \beta_i)}\,(I\phi)^{-1}(\|q\|_0).$$

Then (1.1) *has at least one positive solution.*

The paper is organized as follows. After this section, some lemmas will be established in Section 2. In Section 3, we prove our main results, Theorem 3.1. An example is also given to show our results.

2. Preliminaries. In this section, we suppose $F:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous and $q(t)$ satisfies (H1).

Lemma 2.1. *Suppose $y\in C^1[0,1]$ with $(\phi_p(y'))'\in C[0,1]$ satisfying*

$$\begin{cases} -(\phi_p(y'))'(t)\geq 0 & 0<t<1,\\ y(0)=\sum_{i=1}^n \alpha_i y(\xi_i), & y'(1)=\sum_{i=1}^n \beta_i y'(\xi_i). \end{cases}$$

Then, $y(t)$ is concave and $y(t)\geq 0$, $y'(t)\geq 0$ on $[0,1]$, i.e., $y\in P$.

Proof. The proof is very easy since $0\leq\sum_{i=1}^n\alpha_i<1$, $0\leq\sum_{i=1}^n\beta_i<1$, and we omit it.

For any $x\in C_+^1[0,1]$, suppose u is a solution of the following BVP,

$$\begin{cases} (\phi_p(u'))'+q(t)F(t,x,x')=0 & 0<t<1,\\ u(0)=\sum_{i=1}^n \alpha_i u(\xi_i), & u'(1)=\sum_{i=1}^n \beta_i u'(\xi_i). \end{cases} \tag{2.1}$$

Then

$$\begin{aligned} u'(t)&=\phi_p^{-1}\left[A_x+\int_t^1 q(s)F(s,x(s),x'(s))\,ds\right],\\ u(t)&=B_x+\int_0^t\phi_p^{-1}\left[A_x+\int_s^1 q(r)F(r,x(r),x'(r))\,dr\right]ds, \end{aligned}$$

where A_x, B_x satisfy the boundary conditions, i.e.,

$$\begin{aligned} \phi_p^{-1}A_x&=\sum_{i=1}^n\beta_i\phi_p^{-1}\left(A_x+\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\right)\\ B_x&=\sum_{i=1}^n\alpha_i\left[B_x+\int_0^{\xi_i}\phi_p^{-1}\left(A_x+\int_s^1 q(r)F(r,x(r),x'(r))\,dr\right)ds\right]. \end{aligned} \tag{2.2}$$

So,

$$u(t) = \frac{\sum_{i=1}^{n} \alpha_i \int_0^{\xi_i} \phi_p^{-1}\big(A_x + \int_s^1 q(r)F(r,x,x')\,dr\big)\,ds}{1 - \sum_{i=1}^{n} \alpha_i} + \int_0^t \phi^{-1}\left(A_x + \int_s^1 q(r)F(r,x,x')\,dr\right) ds,$$

where A_x satisfies (2.2).

Lemma 2.2. *For any* $x \in C_+^1[0,1]$*, there exists a unique* $A_x \in (-\infty,+\infty)$ *satisfying* (2.2)*. Therefore, for any* $x \in C_+^1[0,1]$*,* (2.1) *has a solution.*

Proof. For any $x \in C_+^1[0,1]$, define

$$H(c) = \phi^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(c + \int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\right),$$

then $H(c) \in C((-\infty,+\infty),R)$ and

$$H(0) = -\sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\right) \le 0.$$

In what follows, we will divide into two cases to prove that $H(c)=0$ has a unique solution on $(-\infty,+\infty)$, which means that there exists a unique $A_x \in (-\infty,+\infty)$ satisfying (2.2). And, as a result,

$$u(t) = \frac{\sum_{i=1}^{n} \alpha_i \int_0^{\xi_i} \phi_p^{-1}\Big(A_x + \int_s^1 q(r)F(r,x,x')\,dr\Big)\,ds}{1 - \sum_{i=1}^{n} \alpha_i} + \int_0^t \phi^{-1}\left(A_x + \int_s^1 q(r)F(r,x,x')\,dr\right) ds$$

is a solution of (2.1).

Case 1. $H(0)=0$. Then

$$\sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\right) = 0.$$

So,

$$\beta_i \phi_p^{-1}\bigg(\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\bigg) = 0, \quad i = 1,2,\dots,n.$$

Therefore,

$$\phi_p(\beta_i)\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds = 0, \quad i = 1,2,\dots,n.$$

Then,

$$\begin{aligned}
H(c) &= \phi_p^{-1}(c) - \sum_{i=1}^n \beta_i \phi_p^{-1}\bigg(c + \int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\bigg)\\
&= \phi_p^{-1}(c) - \sum_{i=1}^n \phi_p^{-1}\bigg(\phi_p(\beta_i)\bigg(c + \int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\bigg)\bigg)\\
&= \phi_p^{-1}(c)\\
&\quad - \sum_{i=1}^n \phi_p^{-1}\bigg(\phi_p(\beta_i)\bigg(c + \phi_p(\beta_i)\int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\bigg)\bigg)\\
&= \phi_p^{-1}(c) - \sum_{i=1}^n \beta_i\phi_p^{-1}(c) = \bigg(1 - \sum_{i=1}^n \beta_i\bigg)\phi^{-1}(c).
\end{aligned}$$

Obviously, there exists a unique $c = 0$ satisfying $H(c) = 0$.

Case 2. $H(0) \neq 0$. Then $H(0) < 0$. (i) When $c \in (-\infty, 0)$,

$$\begin{aligned}
H(c) &= \phi_p^{-1}(c) - \sum_{i=1}^n \beta_i \phi_p^{-1}\bigg(c + \int_{\xi_i}^1 q(s)F(s,x(s),x'(s))\,ds\bigg)\\
&\le \phi_p^{-1}(c) - \sum_{i=1}^n \beta_i \phi_p^{-1}(c)\\
&= \bigg(1 - \sum_{i=1}^n \beta_i\bigg)\phi^{-1}(c) < 0.
\end{aligned}$$

So when $c \in (-\infty, 0)$, $H(c) \neq 0$.

(ii) When $c \in (0, +\infty)$,

$$\begin{aligned} H(c) &= \phi_p^{-1}(c) - \sum_{i=1}^n \beta_i \phi_p^{-1}\left(c + \int_{\xi_i}^1 q(s)F(s, x(s), x'(s))\, ds\right) \\ &= \phi_p^{-1}(c)\left[1 - \sum_{i=1}^n \beta_i \phi_p^{-1}\left(1 + \frac{\int_{\xi_i}^1 q(s)F(s, x(s), x'(s))\, ds}{c}\right)\right] \\ &= \phi_p^{-1}(c)\overline{H}(c), \end{aligned}$$

where

$$\overline{H}(c) = 1 - \sum_{i=1}^n \beta_i \phi_p^{-1}\left(1 + \frac{\int_{\xi_i}^1 q(s)F(s, x(s), x'(s))\, ds}{c}\right).$$

Since $H(0) \neq 0$, that is,

$$\sum_{i=1}^n \beta_i \phi_p^{-1}\left(\int_{\xi_i}^1 q(s)F(s, x(s), x'(s))\, ds\right) \neq 0.$$

As a result, there must exist $i_0 \in \{1, 2, \ldots, n\}$ such that

$$\beta_{i_0} \phi_p^{-1}\left(\int_{\xi_{i_0}}^1 q(s)F(s, x(s), x'(s))\right) ds \neq 0.$$

Thus, we get $\overline{H}(c)$ is strictly increasing on $(0, +\infty)$;

$$\int_0^1 q(s)F(s, x(s), x'(s))\, ds > 0$$

and $\sum_{i=1}^n \beta_i > 0$. Let

$$\overline{c} = \frac{\phi_p\left(\sum_{i=1}^n \beta_i\right)}{1 - \phi_p\left(\sum_{i=1}^n \beta_i\right)} \int_0^1 q(s)F(s, x(s), x'(s))\, ds,$$

then $\overline{c} > 0$ and we have

$$\begin{aligned} \overline{H}(\overline{c}) &= 1 \\ &\quad - \sum_{i=1}^n \beta_i \phi_p^{-1}\left(1 + \frac{\left(1 - \phi_p\left(\sum_{i=1}^n \beta_i\right)\right) \int_{\xi_i}^1 q(s)F(s, x(s), x'(s))\, ds}{\phi_p\left(\sum_{i=1}^n \beta_i\right) \int_0^1 q(s)F(s, x(s), x'(s))\, ds}\right) \\ &\quad \geq 0. \end{aligned}$$

So, $H(\overline{c}) = \phi_p^{-1}(\overline{c})\overline{H}(\overline{c}) \geq 0$. The mean value theorem guarantees that there exists $c_0 \in (0, \overline{c}] \subset (0, +\infty)$ such that $H(c_0) = 0$. If there exist two constants $c_i \in (0, +\infty)$, $i = 1, 2$, satisfying $H(c_1) = H(c_2) = 0$, then $\overline{H}(c_1) = \overline{H}(c_2) = 0$. So $c_1 = c_2$ since $\overline{H}(c)$ is strictly increasing on $(0, +\infty)$. Therefore, $H(c) = 0$ has a unique solution on $(0, +\infty)$.

Combining (i), (ii) and $H(0) \neq 0$, we obtain that $H(c) = 0$ has a unique solution on $(-\infty, +\infty)$. The proof of Lemma 2.2 is completed. $\square$

Remark 1. From the proof of Lemma 2.2, we know that for any $x \in C_+^1[0,1]$, if we let A_x be the unique constant satisfying equation (2.2) corresponding to x, then

$$A_x \in \left[0, \frac{\phi_p\left(\sum_{i=1}^n \beta_i\right)}{1 - \phi_p\left(\sum_{i=1}^n \beta_i\right)} \int_0^1 q(s) F(s, x(s), x'(s))\, ds\right].$$

For any $x \in C_+^1[0,1]$, let A_x be the unique constant satisfying equation (2.2) corresponding to x. Then the following conclusion holds.

Lemma 2.3. *$A_x : C_+^1[0,1] \to R$ is continuous.*

Proof. Suppose $\{x_n\} \in C_+^1[0,1]$ with $x_n \to x_0 \in C_+^1[0,1]$ in $C_+^1[0,1]$. Then, $\|x_n - x_0\|_0 \to 0$ and $\|x_n' - x_0'\|_0 \to 0$. Let $\{A_n\}$, $n = 0, 1, 2, \ldots$, be constants decided by equation (2.2) corresponding to x_n, $n = 0, 1, 2, \ldots$. Since $\|x_n - x_0\|_0 \to 0$, $\|x_n' - x_0'\|_0 \to 0$ and $F : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous, we get that, for $\varepsilon = 1$, there exists $N > 0$, when $n > N$, for any $r \in [0,1]$,

$$\begin{aligned} 0 &\leq F(r, x_n(r), x_n'(r)) \leq [1 + F(r, x_0(r), x_0'(r))] \\ &\leq \left[1 + \max_{r \in [0,1]} F(r, x_0(r), x_0'(r))\right]. \end{aligned} \tag{2.3}$$

So, by Remark 1,

$$A_n \in \left[0, \frac{\phi_p\left(\sum_{i=1}^n \beta_i\right)}{1 - \phi_p\left(\sum_{i=1}^n \beta_i\right)} \left[1 + \max_{r \in [0,1]} F(r, x_0(r), x_0'(r))\right] \|q\|_0\right],$$

which means that $\{A_n\}$ is bounded.

Suppose A_n does not converge to A_0. Then there exist two subsequences $\{A_{n_k}^{(1)}\}$ and $\{A_{n_k}^{(2)}\}$ of $\{A_n\}$ with $A_{n_k}^{(1)} \to c_1$ and $A_{n_k}^{(2)} \to c_2$ since $\{A_n\}$ is bounded, but $c_1 \neq c_2$.

By construction of $\{A_n\}$, $n = 0, 1, 2, \dots$, we have

$$\phi_p^{-1}(A_{n_k}^{(1)}) = \sum_{i=1}^{n} \beta_i \phi_p^{-1}\biggl((A_{n_k}^{(1)}) + \int_{\xi_i}^{1} q(s)F(s, x_{n_k}^{(1)}(s), (x_{n_k}^{(1)})'(s))\, ds\biggr). \tag{2.4}$$

Using (2.3) and letting $n_k \to +\infty$ in (2.4), we get

$$\begin{aligned}
&\phi_p^{-1}(c_1)\\
&= \lim_{n_k \to \infty} \sum_{i=1}^{n} \beta_i \phi_p^{-1}\biggl((A_{n_k}^{(1)}) + \int_{\xi_i}^{1} q(s)F(s, x_{n_k}^{(1)}(s), (x_{n_k}^{(1)})'(s))\, ds\biggr)\\
&= \sum_{i=1}^{n} \beta_i \phi_p^{-1}\biggl(\lim_{n_k \to \infty}(A_{n_k}^{(1)}) + \lim_{n_k \to \infty}\int_{\xi_i}^{1} q(s)F(s, x_{n_k}^{(1)}(s), (x_{n_k}^{(1)})'(s))\, ds\biggr)\\
&= \sum_{i=1}^{n} \beta_i \phi_p^{-1}\biggl(c_1 + \int_{\xi_i}^{1} q(s)F(s, x_0(s), x_0'(s))\, ds\biggr).
\end{aligned}$$

Since $\{A_n\}$, $n = 0, 1, 2, 3 \dots$, is unique, we get $c_1 = A_0$.

Similarly, $c_2 = A_0$. So $c_1 = c_2$, which is a contradiction. Therefore, for any $x_n \to x_0$, $A_n \to A_0$, which means that $A_x : C^+[0,1] \to R$ is continuous.

The proof of Lemma 2.3 is completed. □

For any $x \in C_+^1[0,1]$, define

$$\begin{aligned}
(Tx)(t) = {}& \frac{\sum_{i=1}^{n} \alpha_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x + \int_s^1 q(r)F(r, x(r), x'(r))\, dr\right) ds}{1 - \sum_{i=1}^{n} \alpha_i}\\
&+ \int_0^t \phi^{-1}\biggl(A_x + \int_s^1 q(r)F(r, x(r), x'(r))\, dr\biggr) ds,
\end{aligned}$$

where A_x is the unique constant in equation (2.2) corresponding to x. By Lemma 2.2, we know Tx is well defined and

$$(Tx)'(t) = \phi_p^{-1}\biggl(A_x + \int_t^1 q(r)F(r, x(r), x'(r))\, dr\biggr).$$

Furthermore, we have the following result.

Lemma 2.4. *$T : P \to P$ is completely continuous, i.e., T is continuous and compact.*

Proof. For any $x \in P$, from the definition of Tx, we know $(Tx) \in C^1[0,1]$, $(\phi_p((Tx)'))' \in C[0,1]$ and

$$\begin{cases} -(\phi_p((Tx)'))'(t) = q(t)F(t,x(t),x'(t)) \geq 0 & 0 < t < 1, \\ (Tx)(0) = \displaystyle\sum_{i=1}^{n} \alpha_i (Tx)(\xi_i), \qquad (Tx)'(1) = \displaystyle\sum_{i=1}^{n} \beta_i (Tx)'(\xi_i). \end{cases}$$

By Lemma 2.1, Tx is concave and $(Tx)(t) \geq 0$, $(Tx)'(t) \geq 0$ on $[0,1]$, i.e., $Tx \in P$. So $TP \subset P$.

The continuity of T is obvious since we have proved A_x is continuous about x in Lemma 2.3. Now, we prove T is compact. Let $\Omega \subset P$ be a bounded set. Then there exists R such that $\Omega \subset \{x \in P \,|\, \|x\|_0 \leq R,\ \|x'\|_0 \leq R\}$. For any $x \in \Omega$, we have $0 \leq \int_0^1 q(s)F(s,x(s),x'(s))\,ds \leq \max_{s\in[0,1],u\in[0,R],v\in[0,R]} F(s,u,v)\|q\|_0 =: M$. From Remark 1, we get

$$|A_x| \leq \frac{\phi_p\left(\sum_{i=1}^n \beta_i\right) \int_0^1 q(s)F(s,x(s),x'(s))\,ds}{1-\phi_p\left(\sum_{i=1}^n \beta_i\right)} \leq \frac{\phi_p\left(\sum_{i=1}^n \beta_i\right) M}{1-\phi_p\left(\sum_{i=1}^n \beta_i\right)}.$$

Therefore,

$$\|(Tx)\|_0 \leq \frac{\left(1-\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \xi_i\right) \phi_p^{-1}(M)}{\left(1-\sum_{i=1}^n \alpha_i\right) \phi_p^{-1}\left(1-\phi_p\left(\sum_{i=1}^n \beta_i\right)\right)},$$

$$\|(Tx)'\|_0 \leq \frac{\phi_p^{-1}(M)}{\phi_p^{-1}\left(1-\phi_p\left(\sum_{i=1}^n \beta_i\right)\right)}, \qquad \|(\phi_p((Tx)'))'\|_0 \leq M.$$

The Arzela-Ascoli theorem guarantees that $T\Omega$ is relatively compact in P, which means T is compact.

The proof of Lemma 2.4 is completed. □

The following lemma is very important in the proof of our main result.

Lemma 2.5 [**3**]. *Assume Ω is a relatively open subset of a convex set K in a normal space E. Let $A : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then either*

(a) *A has a fixed point in $\overline{\Omega}$, or*

(b) *there is an $x \in \partial\Omega$ and a $0 < \lambda < 1$ such that $x = \lambda A(x)$.*

The following properties of $(I\phi_p)^{-1}$ are needed in our paper.

Lemma 2.6. *Assume $A = 0$ or $A \geq 1$, $B \geq 1$, $C = 0$ or $C \geq 1$ and $0 \leq k \leq \min\{p-1, 1\}$, $l > 0$. Then, when $u > 0$, $v > 0$,*

(i) $(I\phi_p)^{-1}(u+v) \leq (I\phi_p)^{-1}(u) + (I\phi_p)^{-1}(v)$;

(ii) $(I\phi_p)^{-1}(uv) \leq (I\phi_p)^{-1}(u)(I\phi_p)^{-1}(v)$.

Proof. (i) For any $c_1 > 0$, $c_2 > 0$,

$$
\begin{aligned}
&(I\phi_p)(c_1+c_2)\\
&\quad = I(\phi_p(c_1+c_2))\\
&\quad = \int_0^{\phi_p(c_1+c_2)} \frac{\phi_p^{-1}(t)}{A(\phi_p^{-1}(t)+1)^k + B(\phi_p^{-1}(t))^{-l} + C}\, dt\\
&\quad = \int_0^{\phi_p(c_1)} \frac{\phi_p^{-1}(t)}{A(\phi_p^{-1}(t)+1)^k + B(\phi_p^{-1}(t))^{-l} + C}\, dt\\
&\qquad + \int_{\phi_p(c_1)}^{\phi_p(c_1+c_2)} \frac{\phi_p^{-1}(t)}{A(\phi_p^{-1}(t)+1)^k + B(\phi_p^{-1}(t))^{-l} + C}\, dt\\
&\quad = (I\phi_p)(c_1) + \int_{c_1}^{c_1+c_2} \frac{(p-1)u^{p-1}}{A(u+1)^k + B(u)^{-l} + C}\, du\\
&\quad = (I\phi_p)(c_1) + \int_0^{c_2} \frac{(p-1)(u+c_1)^{p-1}}{A(u+c_1+1)^k + B(u+c_1)^{-l} + C}\, du\\
&\quad \geq (I\phi_p)(c_1) + \int_0^{c_2} \frac{(p-1)(u)^{p-1}}{A(u+1)^k + B(u)^{-l} + C}\, du\\
&\quad = (I\phi_p)(c_1) + \int_0^{\phi_p(c_2)} \frac{\phi_p^{-1}(t)}{A(\phi_p^{-1}(t)+1)^k + B(\phi_p^{-1}(t))^{-l} + C}\, dt\\
&\quad = (I\phi_p)(c_1) + (I\phi_p)(c_2).
\end{aligned}
$$

Thus, we get

$$(I\phi_p)(c_1 + c_2) \geq (I\phi_p)(c_1)(I\phi_p)(c_2). \tag{2.5}$$

When $u > 0$, $v > 0$, let $c_1 = (I\phi_p)^{-1}(u) > 0$ and $c_2 = (I\phi_p)^{-1}(v) > 0$ in (2.5) to obtain

$$(I\phi_p)^{-1}(u + v) \leq (I\phi_p)^{-1}(u) + (I\phi_p)^{-1}(v).$$

(ii) For any $x > 0$, $y > 0$,

$$\begin{aligned}
I'(xy) &\left[A(\phi_p^{-1}(xy) + 1)^k + \frac{B}{(\phi_p^{-1}(xy))^l} + C\right] \\
&= I'(x)\left[A(\phi_p^{-1}(x) + 1)^k + \frac{B}{(\phi_p^{-1}(x))^l} + C\right] \\
&\quad \times I'(y)\left[A(\phi_p^{-1}(y) + 1)^k + \frac{B}{(\phi_p^{-1}(y))^l} + C\right] \\
&\geq I'(x)I'(y)\left[A^2(\phi_p^{-1}(xy) + 1)^k + \frac{B^2}{(\phi_p^{-1}(xy))^l} + C^2\right] \\
&\geq I'(x)I'(y)\left[A(\phi_p^{-1}(xy) + 1)^k + \frac{B}{(\phi_p^{-1}(xy))^l} + C\right].
\end{aligned}$$

Thus,

$$I'(xy) \geq I'(x)I'(y). \tag{2.6}$$

For any $z > 0$, integrate (2.6) from 0 to z to obtain $I(xz) \geq xI'(x)I(z)$. Remembering, since $0 \leq k \leq 1$,

$$I'(c) = \frac{\phi_p^{-1}(c)}{A(\phi_p^{-1}(c) + 1)^k + B(\phi_p^{-1}(c))^{-l} + C}$$

is increasing about c when $c > 0$ and $I(0) = 0$, we get $xI'(x) \geq I(x)$. So,

$$I(xz) \geq I(x)I(z), \quad x > 0, \quad z > 0. \tag{2.7}$$

When $u > 0$, $v > 0$, let $x = I^{-1}(u) > 0$ and $z = I^{-1}(v) > 0$ in (2.7) to obtain

$$I^{-1}(uv) \leq I^{-1}(u)I^{-1}(v). \tag{2.8}$$

The conclusion (ii) follows easily after (2.8) since $\phi_p^{-1}(uv)=\phi_p^{-1}(u)\phi_p^{-1}(v)$.

3. Proof of Theorem 3.1.

Proof. Since

$$\sup_{c\in(0,+\infty)} \frac{c}{(I\phi)^{-1}(G(c))\Gamma} > 1,$$

there must exist $M_1 > 0$ such that

$$\frac{M_1}{(I\phi)^{-1}(G(M_1))\Gamma} > 1.$$

Choose $1 > \varepsilon > 0$ to satisfy

$$\frac{M_1}{(I\phi)^{-1}(G(M_1+\varepsilon))\Gamma} > 1.$$

Choose $n_0 \in \{1, 2, 3, \dots\}$ with $1/n_0 < \varepsilon$, and let $N_0 = \{n_0, n_0+1, n_0+2, \dots\}$. In the following, we will show for each $m \in N_0$,

$$(3.1)^m \quad \begin{cases} (\phi_p(u'))' + q(t)f\left(t, u + \dfrac{1}{m}, u' + \dfrac{1}{m}\right) = 0 \qquad 0 < t < 1, \\ u(0) = \displaystyle\sum_{i=1}^n \alpha_i u(\xi_i), \qquad u'(1) = \sum_{i=1}^n \beta_i u'(\xi_i), \end{cases}$$

has a solution in P. Obviously, for each $m \in N_0$, $F_m(t,u,u') = f(t, u+(1/m), u'+(1/m)) \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$.

To show that $(3.1)^m$ has a solution in P for each $m \in N_0$, we will apply Lemma 2.5. So now, for any $x \in P$, define

$$(T_m x)(t) = \frac{\sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x + \int_s^1 q(r)F_m(r, x(r), x'(r))\,dr\right)ds}{1 - \sum_{i=1}^n \alpha_i} + \int_0^t \phi^{-1}\left(A_x + \int_s^1 q(r)F_m(r, x(r), x'(r))\,dr\right)ds.$$

Then, by Lemma 2.4, $T_m : P \to P$ is completely continuous. It is well known that a fixed point of operator T_m in P must be a solution of $(3.1)^m$ in P.

Define

$$\Omega = \Big\{x \in P \mid \|x\|_0 < M_1,\ \|x'\|_0 < \frac{(I\phi_p)^{-1}(|q\|_0)(I\phi_p)^{-1}(G(M_1+1))}{1-\sum_{i=1}^n \beta_i} := M_2\Big\}.$$

In what follows, we will prove for each $m \in N_0$ that T_m has a fixed point in $\overline{\Omega}$.

We first show that

$$u \neq \lambda T_m u, \quad \text{for} \quad \lambda \in (0,1), \quad u \in \partial\Omega. \tag{3.2}$$

Otherwise, then there exists a $\lambda \in (0,1)$ and $u \in \partial\Omega$ with $u = \lambda T_m u$. Then by the definition of $T_m u$,

$$\begin{cases} -(\phi_p(u'))' = \phi_p(\lambda)q(t)f\Big(t, u+\dfrac{1}{m},\, u'+\dfrac{1}{m}\Big) \geq 0 \quad 0<t<1, \\ u(0) = \displaystyle\sum_{i=1}^n \alpha_i u(\xi_i), \qquad u'(1) = \sum_{i=1}^n \beta_i u'(\xi_i) \end{cases} \tag{3.3}$$

by Lemma 2.1 we get that $u(t)$ is concave and $u(t) \geq 0$, $u'(t) \geq 0$ on $[0,1]$.

Also, notice by (H3) that

$$\begin{aligned} &-(\phi_p(u'))' \\ &\leq q(t)\Big(f_1\Big(u+\frac{1}{m}\Big)+f_2\Big(u+\frac{1}{m}\Big)\Big)\Big(A\Big(u'+\frac{1}{m}\Big)^k+B\Big(u'+\frac{1}{m}\Big)^{-l}+C\Big) \\ &\leq q(t)\Big(f_1\Big(u+\frac{1}{m}\Big)+f_2\Big(u+\frac{1}{m}\Big)\Big)(A(u'+1)^k+B(u')^{-l}+C). \end{aligned}$$

Multiply the above inequality by u', $u' \geq 0$, to obtain

$$\frac{-(\phi_p(u'))'u'}{A(u'+1)^k+B(u')^{-l}+C} \leq \|q\|_0\Big(f_1\Big(u+\frac{1}{m}\Big)+f_2\Big(u+\frac{1}{m}\Big)\Big)u'. \tag{3.4}$$

Integrating (3.4) from t to 1, we obtain

$$\int_{\phi_p(u'(1))}^{\phi_p(u'(t))} \frac{\phi_p^{-1}(z)}{A(\phi_p^{-1}(z)+1)^k + B(\phi_p^{-1}(z))^{-l} + C}\,dz \leq \|q\|_0 \int_{u(t)+1/m}^{u(1)+1/m} [f_1(z)+f_2(z)]\,dz.$$

Therefore,

$$(I\phi_p)(u'(t)) \leq (I\phi_p)(u'(1)) + \|q\|_0 G(u(1)+\varepsilon)).$$

By Lemma 2.6, we get

$$\begin{aligned} 0 &\leq u'(t) \leq (I\phi_p)^{-1}\big[(I\phi_p)(u'(1)) + \|q\|_0\, G(u(1)+\varepsilon)\big] \\ &\leq u'(1) + (I\phi_p)^{-1}(\|q\|_0\, G(u(1)+\varepsilon)) \\ &\leq u'(1) + (I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)), \quad t \in [0,1]. \end{aligned} \tag{3.5}$$

Thus,

$$0 \leq u'(\xi_i) \leq u'(1) + (I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)), \quad i = 1,2,\dots,n. \tag{3.6}$$

Combining (3.6) and $u'(1) = \sum_{i=1}^n \beta_i u'(\xi_i)$, we get

$$0 \leq u'(1) \leq \frac{\sum_{i=1}^n \beta_i}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)). \tag{3.7}$$

So,

$$0 \leq u'(t) \leq \frac{1}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)). \tag{3.8}$$

Integrate (3.8) from 0 to ξ_i to obtain

$$0 \leq u(\xi_i) \leq u(0) + \frac{\xi_i}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)), \quad i = 1,2,\dots,n. \tag{3.9}$$

Combining (3.9) and $u(0) = \sum_{i=1}^n \alpha_i u(\xi_i)$, we get

$$u(0) \le \frac{\sum_{i=1}^n \alpha_i \xi_i}{(1-\sum_{i=1}^n \alpha_i)\,(1-\sum_{i=1}^n \beta_i)}\,(I\phi_p)^{-1}(\|q\|_0) \\ \times (I\phi_p)^{-1}(G(u(1)+\varepsilon)).$$

Integrate (3.8) from 0 to 1 to obtain

$$\begin{aligned} u(1) &\le u(0) + \frac{1}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(u(1)+\varepsilon)) \\ &\le \frac{1+\sum_{i=1}^n \alpha_i\xi_i - \sum_{i=1}^n \alpha_i}{(1-\sum_{i=1}^n \alpha_i)\,(1-\sum_{i=1}^n \beta_i)} \\ &= (I\phi_p)^{-1}(\|q\|_0) = (I\phi_p)^{-1}(G(u(1)+\varepsilon)) \\ &= (I\phi)^{-1}(G(u(1)+\varepsilon))\Gamma, \end{aligned}$$

where

$$\Gamma = \frac{(1-\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i\xi_i)}{(1-\sum_{i=1}^n \alpha_i)\,(1-\sum_{i=1}^n \beta_i)}\,(I\phi)^{-1}(\|q\|_0).$$

So

$$\frac{u(1)}{(I\phi)^{-1}(G(u(1)+\varepsilon))\Gamma} \le 1,$$

which means that

$$\|u\|_0 = u(1) \ne M_1, \tag{3.10}$$

and, as a result, $\|u\|_0 = u(1) < M_1$ since $u \in \partial\Omega$. At the same time, by (3.8), we have,

(3.11)

$$\begin{aligned} \|u'\|_0 = u'(0) &\le \frac{1}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(M_1+\varepsilon)) \\ &< \frac{1}{1-\sum_{i=1}^n \beta_i}\,(I\phi_p)^{-1}(\|q\|_0)(I\phi_p)^{-1}(G(M_1+1)) = M_2. \end{aligned}$$

Obviously, (3.10) and (3.11) show a contradiction to $u \in \partial\Omega$ and consequently (3.2) is true.

Now, Lemma 2.5 implies T_m has a fixed point u_m in $\overline{\Omega}$, which means that $(3.1)^m$ has a solution u_m in $\overline{\Omega}$ for each $m \in N_0$.

We next show that (1.1) has a solution. To see this, we will conclude

$$\{u_m\}_{n\in N_0},\ \{u'_m\}_{n\in N_0} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{3.12}$$

To do that, since $\{u_m\}_{n\in N_0} \in \overline{\Omega}$, we only need to show $\{u'_m\}_{n\in N_0}$ is an equicontinuous family on $[0,1]$. Now (H4) implies that there is a continuous function $\psi : [0,1] \to (0,+\infty)$, independent of m, with

$$f\bigg(t, u_m(t) + \frac{1}{m},\ u'_m(t) + \frac{1}{m}\bigg) \geq \psi(t), \quad t \in (0,1),$$

i.e.,

$$-(\phi_p(u_m)')' \geq \psi(t)q(t), \quad t \in (0,1). \tag{3.13}$$

Integrate (3.13) from t to 1 to obtain

$$\phi_p(u_m)'(t) \geq \phi_p(u_m)'(1) + \int_t^1 \psi(s)q(s)\,ds \geq \int_t^1 \psi(s)q(s)\,ds, \quad t \in (0,1),$$

i.e.,

$$(u_m)'(t) \geq \phi_p^{-1}\bigg(\int_t^1 \psi(s)q(s)\,ds\bigg) =: \delta_1(t) > 0, \quad t \in (0,1). \tag{3.14}$$

Integrate (3.14) from ξ_n to 1 to obtain

$$\begin{aligned}(u_m)(\xi_n) &\geq (u^m)(0) + \int_0^{\xi_n} \phi_p^{-1}\bigg(\int_s^1 \psi(r)q(r)\,dr\bigg)\,ds \\ &\geq \int_0^{\xi_n} \phi_p^{-1}\bigg(\int_s^1 \psi(r)q(r)\,dr\bigg)\,ds =: \theta > 0.\end{aligned}$$

For any $m \in N_0$, since u_m is nondecreasing and concave, we have when $t \in [0,\xi_n]$, $u_m(t) \geq (\theta/\xi_n t)$; when $t \in [\xi_1, 1]$, $u_m(t) \geq \theta$.

Let

$$\delta_2(t) = \begin{cases} \dfrac{\theta}{\xi_n}\, t & t \in [0,\xi_n] \\ \theta & t \in [\xi_n, 1]. \end{cases}$$

Then for any $m \in N_0$,

$$u_m(t) \geq \delta_2(t), \quad t \in [0,1].$$

Since $f_1(y)$ is nonincreasing about y, we have

(3.15)

$$\begin{aligned}0 \leq -(\phi_p(u_m)')'(t) &= q(t) f\left(t, u_m(t) + \frac{1}{m}, u'_m(t) + \frac{1}{m}\right) \\ &\leq \|q\|_0 \Big[f_1(\delta_2(t)) + \max_{0 \leq r \leq (M_1+1)} f_2(r)\Big]\Big[A(M_2+1)^k + \frac{B}{(\delta_1(t))^l} + C\Big].\end{aligned}$$

By (H5), the right-hand function of above inequality is Lebesgue integrable. Thus, by the absolute continuity of integral interval, we get $\{\phi_p((u_m)')\}_{m=n_0}^{+\infty}$ is equicontinuous and, as a result, $\{(u_m)'\}_{m=n_0}^{+\infty}$ is equicontinuous. So, (3.12) holds.

The Arzela-Ascoli theorem guarantees that both $\{((u_m)')\}_{m=n_0}^{+\infty}$ and $\{u_m\}_{m=n_0}^{+\infty}$ are compact in $C[0,1]$. So there is a subsequence $N^* \subset N_0$ and a function $z^{(j)} \in C[0,1] (j = 0,1)$ with $u_m \to z$ and $u'_m \to z'$ uniformly on $[0,1]$ as $m \to +\infty$ through N^*. By the definition of $u_m(t)$, we have

(3.16)

$$\begin{cases} \phi_p((u_m)'(t)) = \phi_p((u_m)'(0)) \\ \quad - \displaystyle\int_0^t q(s) f\left(s, u_m(s) + \frac{1}{m}, u'_m(s) + \frac{1}{m}\right) ds & 0 < t < 1, \\ u_m(0) = \displaystyle\sum_{i=1}^n \alpha_i u_m(\xi_i), \qquad u'_m(1) = \sum_{i=1}^n \beta_i u'_m(\xi_i). \end{cases}$$

Letting $m \to +\infty$ through N^* and using Lebesgue's dominated convergence in (3.16), we get

$$\begin{cases} \phi_p(z'(t)) = \phi_p(z'(0)) - \displaystyle\int_0^t q(s) f(s, z(s), z'(s))\, ds & 0 < t < 1, \\ z(0) = \displaystyle\sum_{i=1}^n \alpha_i z(\xi_i), \qquad z'(1) = \sum_{i=1}^n \beta_i z'(\xi_i), \end{cases}$$

i.e.,

$$\begin{cases} (\phi_p(z'(t)))' + q(t) f(t, z(t), z'(s)) = 0 & 0 < t < 1, \\ z(0) = \displaystyle\sum_{i=1}^n \alpha_i z(\xi_i), \qquad z'(1) = \sum_{i=1}^n \beta_i z'(\xi_i). \end{cases}$$

From $M_1 \geq u_m(t) \geq \delta_2(t)$, $t \in [0,1]$, we have $M_1 \geq z(t) \geq \delta_2(t)$, $t \in [0,1]$, so $z(t) > 0$, $t \in (0,1)$. From $M_2 \geq u_m'(t) \geq \delta_1(t)$, $t \in (0,1)$, we have $M_2 \geq z'(t) \geq \delta_1(t)$, $t \in (0,1)$, so $z'(t) > 0$, $t \in (0,1)$. Moreover, by

$$\begin{aligned} 0 &\leq -\left(\phi_p((z)'(t))\right)' = q(t)f(t,z,z') \\ &\leq \|q\|_0 \left[f_1(\delta_2(t)) + \max_{0\leq r\leq M_1} f_2(r)\right] \cdot \left[AM_2^k + \frac{B}{(\delta_1(t))^l} + C\right] \in L^1[0,1], \end{aligned}$$

we get $(\phi_p((z)'(t)))' \in L^1[0,1]$. Above all, $z(t)$ is a positive solution to (1.1).

An example. Now, we give an example to show our result. Consider (4.1)

$$\begin{cases} (|u'|^{-2/3}u')' + \mu e^t \left[u^{11/6} + \dfrac{1}{u^{1/2}} + \left|\sin \dfrac{1}{u^{1/2}}\right|\right] \dfrac{1}{(u')^{1/3}} = 0 & 0<t<1, \\ u(0) = \dfrac{1}{2}\, u\, \dfrac{1}{2} + \dfrac{1}{4}\, u\, \dfrac{1}{4}, \qquad u'(1) = \dfrac{1}{2}\, u'\, \dfrac{1}{2} + \dfrac{1}{4}\, u'\, \dfrac{1}{4}. \end{cases}$$

Comparing to Theorem 3.1, conditions (H1)–(H5) are all satisfied. Moreover, if

$$0 < \mu < \frac{17^{1/2}}{25e3^{(23)/6}\,2^{5/2}},$$

then

$$\sup_{c\in(0,+\infty)} \frac{c}{(I\phi_p)^{-1}\,(G(c))\Gamma} > 1.$$

According to Theorem 3.1, (4.1) has a positive solution when

$$0 < \mu < \frac{17^{1/2}}{25e3^{(23)/6}2^{5/2}}.$$

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