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ON THE MULTILINEAR GENERALIZATIONS OF THE CONCEPT OF ABSOLUTELY SUMMING OPERATORS

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ABSTRACT. In this paper we investigate the several multilinear generalizations of the concept of absolutely summing operators and their connections. We also introduce the concept of p semi-integral mappings and establish the position of p semi-integral mappings with respect to the other classes.

1. Introduction and notation.

The core of the theory of absolutely summing operators lies in the ideas of A. Grothendieck in the 1950s. Further work of Pietsch [**23**] and Lindenstrauss and Pełczyński [**9**] clarified Grothendieck's insights, and nowadays the ideal of absolutely summing operators is a central topic of investigation. For details on absolutely summing operators we refer to the book by Diestel-Jarchow-Tonge [**7**].

A natural question is how to extend the concept of absolutely summing operators to multilinear mappings and polynomials. A first light in this direction is the work by Alencar-Matos [**1**], where several classes of multilinear mappings between Banach spaces were investigated. Since then, just concerning the idea of lifting the ideal of absolutely summing operators to polynomials and multilinear mappings, several concepts have appeared and so far none of the definitions proposed appears as clearly better or more useful than the rest. However, there seems to be no effort in the direction of comparing all these different classes. The aim of this paper is to investigate the connections between these classes, to introduce the class of p semi-integral mappings and to establish the position of p semi-integral mappings with respect to the other concepts.

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Throughout this paper the letters $E, E_1, \dots, E_n, G_1, \dots, G_n, F, F_0$ represent Banach spaces, the symbol $\mathbf{K}$ represents the field of all scalars (complex or real), and $\mathbf{N}$ denotes the set of all positive integers. Given a natural number $n \geq 2$, the Banach space of all continuous n-linear mappings from $E_1 \times \cdots \times E_n$ into F endowed with the sup norm will be denoted by $\mathcal{L}(E_1, \dots, E_n; F)$. For $i = 1, \dots, n$, we denote by $\Psi_i^{(n)} : \mathcal{L}(E_1, \dots, E_n; F) \to \mathcal{L}(E_i; \mathcal{L}(E_1, \overset{[i]}{\dots}, E_n; F))$ the canonical isometric isomorphism

$$\Psi_i^{(n)}(T)(x_i)(x_1 \overset{[i]}{\cdots} x_n) = T(x_1, \dots, x_n),$$

where $\overset{[i]}{\dots}$ means that the ith coordinate is not involved.

For $p > 0$, the linear space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that $\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty} \|x_j\|^p)^{1/p} < \infty$ is denoted by $l_p(E)$. We represent by $l_p^w(E)$ the linear space of the sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in l_p$ for every continuous linear functional $\varphi : E \to \mathbf{K}$ and define $\|.\|_{w,p}$ in $l_p^w(E)$ by $\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{E'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_p$. If $p = \infty$ we are restricted to the case of bounded sequences and in $l_\infty(E)$ we use the sup norm. One can verify that $\|.\|_p$, respectively $\|.\|_{w,p}$, is a p-norm in $l_p(E)$, respectively $l_p^w(E)$, for $p < 1$ and a norm in $l_p(E)$, respectively $l_p^w(E)$, for $p \geq 1$.

If K is a Hausdorff compact topological space, $C(K)$ denotes the Banach space, under the supremum norm, of all continuous functions on K.

We begin by presenting the several classes of multilinear mappings related to the concept of absolutely summing operators:

• If $p \geq 1$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *p-dominated* ($T \in \mathcal{L}_{d,p}(E_1, \dots, E_n; F)$) if there exist $C \geq 0$ and regular probability measures μ_j on the Borel σ-algebras $\mathcal{B}(B_{E_j'})$ of $B_{E_j'}$ endowed with the weak star topologies $\sigma(E_j', E_j)$, $j = 1, \dots, n$, such that

$$\|T(x_1, \dots, x_n)\| \leq C \prod_{j=1}^{n} \left[\int_{B_{E_j'}} |\varphi(x_j)|^p \, d\mu_j(\varphi) \right]^{1/p}$$

for every $x_j \in E_j$ and $j = 1, \dots, n$. It is well known that $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is p-dominated if and only if there exist Banach spaces $G_1, \dots, G_n$, absolutely p-summing linear operators $u_j \in$

$\mathcal{L}(E_j;G_j)$ and a continuous n-linear mapping $R \in \mathcal{L}(G_1,\dots,G_n;F)$ so that $T = R \circ (u_1,\dots,u_n)$.

• If $p \geq 1$, $T \in \mathcal{L}(E_1,\dots,E_n;F)$ is of *absolutely p-summing type* ($T \in [\Pi_{as(p)}](E_1,\dots,E_n;F)$) if $\Psi_j^{(n)}(T)$ is absolutely p-summing for every $j \in \{1,\dots,n\}$.

• If $p \geq 1$, $T \in \mathcal{L}(E_1,\dots,E_n;F)$ is *p semi-integral* ($T \in \mathcal{L}_{si,p}(E_1,\dots,E_n;F)$) if there exist $C \geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra $\mathcal{B}(B_{E_1'} \times \cdots \times B_{E_n'})$ of $B_{E_1'} \times \cdots \times B_{E_n'}$ endowed with the product of the weak star topologies $\sigma(E_l',E_l)$, $l = 1,\dots,n$, such that

$$\|T(x_1,\dots,x_n)\| \leq C\left(\int_{B_{E_1'}\times\cdots\times B_{E_n'}} |\varphi_1(x_1)\cdots\varphi_n(x_n)|^p \, d\mu(\varphi_1,\dots,\varphi_n)\right)^{1/p}$$

for every $x_j \in E_j$ and $j = 1,\dots,n$. The infimum of the C defines a norm $\|.\|_{si,p}$ for the space of p semi-integral mappings.

• If $p \geq 1$, $T : E_1 \times \cdots \times E_n \to F$ is *fully, or multiple, p-summing* if there exists $C > 0$ such that

$$\left(\sum_{j_1,\dots,j_n=1}^{\infty} \|T(x_{j_1}^{(1)},\dots,x_{j_n}^{(n)})\|^p\right)^{1/p} \leq C \prod_{k=1}^{n} \|(x_j^{(k)})_{j=1}^{\infty}\|_{w,p}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in l_p^w(E_k)$, $k = 1,\dots,n$. The space of all fully p-summing n-linear mappings from $E_1 \times \cdots \times E_n$ into F will be denoted by $\mathcal{L}_{fas,p}(E_1,\dots,E_n;F)$, and the infimum of the C for which the inequality always holds defines a norm $\|.\|_{fas,p}$ for $\mathcal{L}_{fas,p}(E_1,\dots,E_n;F)$.

• If $p \geq 1$, $T \in \mathcal{L}(E_1,\dots,E_n;F)$ is strongly p-summing ($T \in \mathcal{L}_{sas,p}(E_1,\dots,E_n;F)$) if there exists $C \geq 0$ and a regular probability measure μ on the Borel σ-algebra $\mathcal{B}(B_{\mathcal{L}(E_1,\dots,E_n;\mathbf{K})})$ of $B_{\mathcal{L}(E_1,\dots,E_n;\mathbf{K})}$ with the weak star topology such that

$$\|T(x_1,\dots,x_n)\| \leq C\left(\int_{B_{\mathcal{L}(E_1,\dots,E_n;\mathbf{K})}} |\phi(x_1,\dots,x_n)|^p \, d\mu(\phi)\right)^{1/p}$$

for every $x_j \in E_j$ and $j = 1,\dots,n$.

• If $p, q_1, \ldots, q_n > 0$, $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is *absolutely* $(p; q_1, \ldots, q_n)$-*summing, or* $(p; q_1, \ldots, q_n)$-*summing, at the point* $(a_1, \ldots, a_n)$ in $E_1 \times \cdots \times E_n$ when

$$(T(a_1 + x_j^{(1)}, \ldots, a_n + x_j^{(n)}) - T(a_1, \ldots, a_n))_{j=1}^{\infty} \in l_p(F)$$

for every $(x_j^{(s)})_{j=1}^{\infty} \in l_{q_s}^w(E)$, $s = 1, \ldots, n$. In the case that T is $(p; q_1, \ldots, q_n)$-summing at every $(a_1, \ldots, a_n) \in E_1 \times \cdots \times E_n$ we say that T is $(p; q_1, \ldots, q_n)$-summing everywhere. Notation: $\mathcal{L}_{as(p,q_1,\ldots,q_n)}^{ev}(E_1, \ldots, E_n; F)$ or $\mathcal{L}_{as,p}^{ev}(E_1, \ldots, E_n; F)$ if $p = q_1 = \cdots = q_n$.

• If $1/p \leq 1/q_1 + \cdots + 1/q_n$ and T is $(p; q_1, \ldots, q_n)$-summing at $(0, \ldots, 0) \in E_1 \times \cdots \times E_n$, we say that T is $(p; q_1, \ldots, q_n)$-*summing*, and we write $T \in \mathcal{L}_{as(p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)$. When $p = q_1 = \cdots = q_n$ we write $\mathcal{L}_{as,p}(E_1, \ldots, E_n; F)$. It is well known that $\mathcal{L}_{d,p}(E_1, \ldots, E_n; F) = \mathcal{L}_{as((p/n),p,\ldots,p)}(E_1, \ldots, E_n; F)$.

Except perhaps for the concept of p semi-integral mappings, all of the above concepts are well known and individually investigated. The p semi-integral mappings were introduced in [**18**] motivated by the work of Alencar-Matos [**1**]. Dominated mappings were first explored by Schneider [**24**] and Matos [**11**] and more recently in [**3, 5, 14, 16**]. Multilinear mappings of absolutely summing type are motivated by abstract methods of creating ideals and are explored in [**18**]. The ideal of fully summing multilinear mappings was introduced by Matos [**12**] and investigated by Souza [**25**] in her doctoral thesis under his supervision. It was also independently introduced by Bombal et al. (with a different name "multiple summing") and developed in [**2, 22**]. The ideal of strongly summing multilinear mappings was introduced by Dimant [**8**] and the concept of absolutely summing multilinear mappings appears firstly in Alencar-Matos [**1**], Matos [**11**] and have been vastly studied (we mention [**13, 16–19, 22**], for example). In the next section we investigate the class of p semi-integral mappings and in the last section we study the connections between the classes previously introduced.

2. p **semi-integral mappings.** We begin with a characterization of p semi-integral mappings that will be useful in the subsequent section.

Theorem 1. *$T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$ if and only if there exists $C \geq 0$ such that*

$$(2.1)\quad \left(\sum_{j=1}^{m} \|T(x_{1,j}, \dots, x_{n,j})\|^p\right)^{1/p} \leq C\left(\sup_{\varphi_l \in B_{E'_l}, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \dots \varphi_n(x_{n,j})|^p\right)^{1/p}$$

for every $m \in \mathbf{N}$, $x_{l,j} \in E_l$ with $l = 1, \dots, n$ and $j = 1, \dots, m$. Moreover, the infimum of the C in (2.1) *is $\|T\|_{si,p}$.*

Proof. If T is p semi-integral, it is not hard to obtain (2.1). Conversely, suppose that (2.1) holds. The proof follows the idea of the case $p = 1$ in [**1**]. Define

- $\Gamma_1 = \{f \in C(B_{E'_1} \times \cdots \times B_{E'_n}); f < C^{-p}\}$.
- $\Gamma_2 = \mathrm{co}\,\{f \in C(B_{E'_1} \times \cdots \times B_{E'_n})$; there are $x_l \in E_l$, $l = 1, \dots, n$, so that $\|T(x_1, \dots, x_n)\| = 1$ and $f(\varphi_1, \dots, \varphi_n) = |\varphi_1(x_1) \cdots \varphi(x_n)|^p\}$.

where co$\,\{.\}$ denotes the convex hull. Let us show that $\Gamma_1 \cap \Gamma_2 = \phi$.

If $h \in \Gamma_2$, then $h = \sum_{j=1}^{m} \alpha_j f_j$, $\alpha_j > 0$, $\sum_{j=1}^{m} \alpha_j = 1$ and

$$f_j(\varphi_1, \dots, \varphi_n) = |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p$$

for every $\varphi_l \in B_{E'_l}$. By hypothesis we have

$$\begin{aligned}\|h\| &= \left(\sup_{\varphi_l \in B_{E'_l}, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(\alpha_j^{1/p} x_{1,j}) \cdots \varphi_n(x_{n,j})|^p\right)\\ &\geq C^{-p} \sum_{j=1}^{m} (\alpha_j^{1/p})^p \|T(x_{1,j}, \dots, x_{n,j})\|^p = C^{-p}.\end{aligned}$$

Hence $h \notin \Gamma_1$. By the Hahn-Banach separation theorem, there exist $\lambda > 0$ and

$$\psi \in C(B_{E'_1} \times \cdots \times B_{E'_n})'$$

so that $\|\psi\| = 1$ and

$$\psi(f) < \lambda \leq \psi(g)$$

for every $f \in \Gamma_1$, $g \in \Gamma_2$. Since each $f < 0$ belongs to Γ_1, we have $\psi(mf) < \lambda$ for every $m \in \mathbf{N}$. Thus $\psi(f) \leq 0$ and ψ is a positive functional and thus, by the Riesz representation theorem, there exists a regular probability measure μ, defined on the Borel sets of $B_{E_1'} \times \cdots \times B_{E_n'}$ so that

$$\psi(f) = \int_{B_{E_1'} \times \cdots \times B_{E_n'}} f \, d\mu.$$

Defining f_m by $f_m = C^{-p} - (1/m)$, we have $f_m \in \Gamma_1$ for every $m \in \mathbf{N}$. Thus

$$\int_{B_{E_1'} \times \cdots \times B_{E_n'}} f_m \, d\mu = C^{-p} - \frac{1}{m} \leq \lambda \quad \text{for every} \quad m,$$

and hence $\lambda \geq C^{-p}$. Therefore, if $\|T(x_1, \dots, x_n)\| = 1$, defining $f(\varphi_1, \dots, \varphi_n) := |\varphi_1(x_1) \cdots \varphi(x_n)|^p$, we have $f \in \Gamma_2$ and

$$\int_{B_{E_1'} \times \cdots \times B_{E_n'}} f \, d\mu = \psi(f) \geq C^{-p} = C^{-p} \|T(x_1, \dots, x_n)\| \tag{2.2}$$

i.e.,

$$C^p \int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \cdots \varphi(x_n)|^p \, d\mu \geq \|T(x_1, \dots, x_n)\|,$$

and, since $\|T(x_1, \dots, x_n)\| = 1$, we obtain

$$\|T(x_1, \dots, x_n)\| \leq C \bigg(\int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \cdots \varphi(x_n)|^p \, d\mu \bigg)^{1/p}.$$

If $\|T(x_1, \dots, x_n)\| \neq 0$, it suffices to replace x_1 by $x_1 \|T(x_1, \dots, x_n)\|^{-1}$ in (2.2), and we deduce

$$\|T(x_1, \dots, x_n)\| \leq C \bigg(\int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \cdots \varphi(x_n)|^p \, d\mu \bigg)^{1/p}. \qquad \square$$

The preceding theorem has various straightforward consequences whose proof we omit:

Proposition 1. (i) (Ideal property). *If $T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$, $A_k \in \mathcal{L}(D_k; E_k)$, $k = 1, \dots, n$, and $S \in \mathcal{L}(F; G)$, then $S \circ T \circ (A_1, \dots, A_n)$ is p semi-integral and*

$$\|S \circ T \circ (A_1, \dots, A_n)\|_{si,p} \leq \|S\| \|T\|_{si,p} \prod_{k=1}^{n} \|A_k\|.$$

(ii) *If $T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$ and $i : F \to F_0$ is an isometric embedding, then $\|i \circ T\|_{si,p} = \|T\|_{si,p}$.*

(iii) *If $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$, then*

$$\mathcal{L}(E_{j_1}, \dots E_{j_n}; F) = \mathcal{L}_{si,p}(E_{j_1}, \dots E_{j_n}; F),$$

for every $j_1, \dots, j_k$ in $\{1, \dots, n\}$ with $j_r \neq j_s$ for $r \neq s$.

For $r \geq 1$ we have the following characterization of r semi-integral mappings defined in $C(K_1) \times \cdots \times C(K_n)$ whose proof is based on an argument used in [**1**].

Theorem 2. *Let $K_1, \dots, K_n$ be Hausdorff compact topological spaces. If $T \in \mathcal{L}(C(K_1), \dots, C(K_n); F)$ and $r \geq 1$, then the following conditions are equivalent*:

(i) *T is r semi-integral.*

(ii) *There exist $C \geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra $\mathcal{B}(K_1 \times \cdots \times K_n)$ of $K_1 \times \cdots \times K_n$ such that*

$$\|T(f_1, \dots, f_n)\| \leq C \left(\int_{K_1 \times \cdots \times K_n} |f_1(x_1) \cdots f_n(x_n)|^r \, d\mu(x_1, \dots, x_n) \right)^{1/r}$$

for every $f_l \in C(K_l)$ with $l = 1, \dots, n$.

(iii) *There exist* $D \geq 0$ *such that*

$$\left(\sum_{j=1}^{m} \|T(f_{1,j}, \dots, f_{n,j})\|^r\right)^{1/r} \leq D\left(\sup_{x_l \in K_l, l=1,\dots,n} \sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r\right)^{1/r}$$

for every $m \in \mathbf{N}$, $f_{l,j} \in C(K_l)$ *with* $l = 1, \dots, n$ *and* $j = 1, \dots, m$.

Proof. A modification on the proof of Theorem 1 shows the equivalence between (ii) and (iii). Hence we only need to prove the equivalence between (i) and (iii), which follows from Theorem 1 and from the identity

$$\sup_{\varphi_l \in B_{C(K_l)'}, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^r = \sup_{x_l \in K_l, l=1,\dots,n} \sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r$$

for every $f_{l,j} \in C(K_l)$ with $l = 1, \dots, n$, $j = 1, \dots, m$, and $m \in \mathbf{N}$.

Note that, for $x_l \in K_l$, $l = 1, \dots, n$, we have

$$\sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r = \sum_{j=1}^{m} |\delta_{x_1}(f_{1,j})\cdots\delta_{x_n}(f_{n,j})|^r$$

where $\delta_{x_l} \in B_{C(K_l)'}$ is given by $\delta_{x_l}(f) = f(x_l)$, for every $f \in C(K_l)$, $l = 1, \dots, n$. Hence

$$\sup_{x_l \in K_l, l=1,\dots,n} \sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r \leq \sup_{\varphi_l \in B_{C(K_l)'}, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^r.$$

Now we show the converse. We have

$$\begin{aligned}
&\sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^r \\
&= \sup_{|\lambda_j|=1} \left| \sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} \lambda_j \varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j}) \right| \\
&= \sup_{|\lambda_j|=1} \left| \varphi_1\otimes\cdots\otimes\varphi_n \left(\sum_{j=1}^{m} \lambda_j |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} f_{1,j}\otimes\cdots\otimes f_{n,j} \right) \right| \\
&\leq \|\varphi_1\otimes\cdots\otimes\varphi_n\| \sup_{|\lambda_j|=1} \left\| \sum_{j=1}^{m} \lambda_j |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} f_{1,j}\otimes\cdots\otimes f_{n,j} \right\| \\
&\leq \sup_{|\lambda_j|=1, x_l\in K_l, l=1,\dots,n} \left| \sum_{j=1}^{m} \lambda_j |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} f_{1,j}(x_1)\cdots f_{n,j}(x_n) \right| \\
&\leq \sup_{x_l\in K_l, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)| \\
&\leq \sup_{x_l\in K_l, l=1,\dots,n} \left[\left(\sum_{j=1}^{m} \left(|\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^{r-1} \right)^{r/r-1} \right)^{(r-1)/r} \right. \\
&\qquad\qquad \left. \times \left(\sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r \right)^{1/r} \right].
\end{aligned}$$

Hence,

$$\left(\sum_{j=1}^{m} |\varphi_1(f_{1,j})\cdots\varphi_n(f_{n,j})|^r \right)^{1-(r-1/r)} \leq \sup_{x_l\in K_l, l=1,\dots,n} \left(\sum_{j=1}^{m} |f_{1,j}(x_1)\cdots f_{n,j}(x_n)|^r \right)^{1/r},$$

and the proof is done. □

3. Connections between the different classes. Recall that an n-linear mapping T is said to be completely continuous if it takes sequences weakly converging to zero into sequences norm converging to zero. It is well known that every absolutely p-summing operator is weakly compact and completely continuous. So, a natural question is to ask whether their multilinear generalizations still preserve these properties.

Concerning completely continuous mappings, it is not hard to prove that every p semi-integral mapping is completely continuous. On the other hand, contrary to the linear case, the absolutely summing (and strongly summing) multilinear mappings are not completely continuous, in general. For example, $T : l_2 \times l_2 \to \mathbf{K}$ given by $T((x_j)_{j=1}^{\infty}, (y_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} x_j y_j$ is absolutely 1-summing and strongly 2-summing but fails to be completely continuous.

In this section we will obtain certain connections between the different classes investigated in this paper and apply our results to give an alternative direct answer for a question (concerning weak compactness for strongly summing mappings) posed by Dimant [**8**] and recently answered in Carando-Dimant [**6**].

Theorem 3. *Let E and F be Banach spaces. Then*

(i) $[\Pi_{as(p)}](^n E; F) = \mathcal{L}_{as(p;p,\infty,\dots,\infty)}(^n E; F) \cap \dots \cap \mathcal{L}_{as(p;\infty,\dots,\infty,p)} \times (^n E; F)$.

(ii) *If $T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$, then $\Psi_i^{(n)}(T) \in \mathcal{L}_{as,p}(E_i; \mathcal{L}(E_1, \overset{[i]}{\dots}, E_n; F))$ and $\Psi_i^{(n)}(T)(x)$ is p semi-integral for every x in E_i.*

(iii) $\mathcal{L}_{d,p}(^n E; F) \subset \mathcal{L}_{si,p}(^n E; F) \subset [\Pi_{as(p)}](^n E; F)$.

(iv) $\mathcal{L}_{si,p}(^n E; F) \subset \mathcal{L}_{d,np}(^n E; F)$.

(v) $\mathcal{L}_{si,p}(^n E; F) \subset \mathcal{L}_{fas,p}(^n E; F) \subset \mathcal{L}_{as,p}^{ev}(^n E; F) \subset \mathcal{L}_{as,p}(^n E; F)$.

(vi) $\mathcal{L}_{si,p}(^n E; F) \subset \mathcal{L}_{sas,p}(^n E; F)$.

Proof. (i) The case $n = 3$ is illustrative. If $T \in [\Pi_{as(p)}](^3 E; F)$ and $(x_j)_{j=1}^{\infty} \in l_p^w(E), (y_j)_{j=1}^{\infty} \in l_{\infty}(E), (z_j)_{j=1}^{\infty} \in l_{\infty}(E)$ are nonidentically null, we have

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\|T(x_j,y_j,z_j)\|^p\right)^{1/p}\\
&=\left\|(y_j)_{j=1}^{\infty}\right\|_{\infty}\left\|(z_j)_{j=1}^{\infty}\right\|_{\infty}\left(\sum_{j=1}^{\infty}\left\|T\Big(x_j,\frac{y_j}{\left\|(y_j)_{j=1}^{\infty}\right\|_{\infty}},\frac{z_j}{\left\|(z_j)_{j=1}^{\infty}\right\|_{\infty}}\Big)\right\|^p\right)^{1/p}\\
&\le\left\|(y_j)_{j=1}^{\infty}\right\|_{\infty}\left\|(z_j)_{j=1}^{\infty}\right\|_{\infty}\left(\sum_{j=1}^{\infty}\left\|\Psi_1^{(3)}(T)(x_j)\right\|^p\right)^{1/p}\\
&\le\left\|\Psi_1^{(3)}(T)\right\|_{as,p}\left\|(y_j)_{j=1}^{\infty}\right\|_{\infty}\left\|(z_j)_{j=1}^{\infty}\right\|_{\infty}\left\|(x_j)_{j=1}^{\infty}\right\|_{w,p},
\end{aligned}$$

and thus $T\in\mathcal{L}_{as(p;p,\infty,\infty)}({}^3E;F)$. The other cases are similar. The converse is not difficult.

(ii) Let $T\in\mathcal{L}_{si,p}(E_1,\dots,E_n;F)$. Fix a natural number $i\in\{1,\dots,n\}$ and let $x_j\in E_i$, $j=1,\dots,m$. For $\varepsilon>0$, there exist $x_{k,j}\in B_{E_k}$, $k=1,\overset{[i]}{\dots},n$ such that

$$\|\Psi_i^n(T)(x_j)\|^p\le\|\Psi_i^n(T)(x_j)(x_{1,j},\overset{[i]}{\dots},x_{n,j})\|^p+\frac{\varepsilon}{m}.$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{m}\|\Psi_i^n(T)(x_j)\|^p\\
&\qquad\le\varepsilon+\sum_{j=1}^{m}\|\Psi_i^n(T)(x_j)(x_{1,j},\overset{[i]}{\dots},x_{n,j})\|^p\\
&\qquad\le\varepsilon+\|T\|_{si,p}^p\sup_{\varphi_l\in B_{E_l'},l=1,\dots,n}\sum_{j=1}^{m}|\varphi_i(x_j)\varphi_1(x_{1,j})\overset{[i]}{\cdots}\varphi_n(x_{n,j})|^p\\
&\qquad\le\varepsilon+\|T\|_{si,p}^p\sup_{\varphi_i\in B_{E_i'}}\sum_{j=1}^{m}|\varphi_i(x_j)|^p.
\end{aligned}$$

Since ε is arbitrary, we have

$$\sum_{j=1}^{m}\|\Psi_i^n(T)(x_j)\|^p\le\|T\|_{si,p}^p\left\|(x_j)_{j=1}^{m}\right\|_{w,p}^p.$$

Hence $\Psi_i^{(n)}(T) \in \mathcal{L}_{as,p}(E_i;\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F))$ for every $i = 1, \ldots, n$.

Now we show that $\Psi_i^{(n)}(T)(x)$ is p semi-integral for every $x \in E_i$, $i = 1, \ldots, n$. Since

$$\begin{aligned}
&\bigg(\sum_{j=1}^{m} \|\Psi_i^{(n)}(T)(x)(x_{1,j}, \overset{[i]}{\ldots}, x_{n,j})\|^p\bigg)^{1/p} \\
&= \bigg(\sum_{j=1}^{m} \|T(x_{1,j}, \ldots, x, \ldots, x_{n,j})\|^p\bigg)^{1/p} \\
&\leq \|T\|_{si,p}\bigg(\sup_{\varphi_l \in B_{E_l'}, l=1,\ldots,n} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_i(x) \cdots \varphi_n(x_{n,j})|^p\bigg)^{1/p} \\
&\leq \|T\|_{si,p}\|x\|\bigg(\sup_{\varphi_l \in B_{E_l'}, l=1,\overset{[i]}{\ldots},n} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \overset{[i]}{\cdots} \varphi_n(x_{n,j})|^p\bigg)^{1/p},
\end{aligned}$$

we get $\|\Psi_i^{(n)}(T)(x)\|_{si,p} \leq \|T\|_{si,p}\|x\|$, for every $x \in E_i$, $i = 1, \ldots, n$.

(iii) The proof that $\mathcal{L}_{si,p}(^nE;F) \subset [\Pi_{as(p)}](^nE;F)$ is a consequence of (ii). If $T \in \mathcal{L}_{d,p}(^nE;F)$, then

$$\begin{aligned}
\|T(x_1, \ldots, x_n)\| &\leq C\bigg(\int_{B_{E_1'}} |\varphi_1(x_1)|^p \, d\mu_1(\varphi_1)\bigg)^{1/p} \\
&\qquad\qquad \cdots \bigg(\int_{B_{E_n'}} |\varphi_n(x_n)|^p \, d\mu_n(\varphi_n)\bigg)^{1/p} \\
&= C\bigg(\int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \\
&\qquad \cdots \varphi_n(x_n)|^p \, d(\mu_1 \otimes \cdots \otimes \mu_n)(\varphi_1, \ldots, \varphi_n)\bigg)^{1/p},
\end{aligned}$$

and hence $T \in \mathcal{L}_{si,p}(^nE;F)$.

(iv) Suppose that T is p semi-integral. Then, from Theorem 1 we deduce

$$\begin{aligned}
&\Big(\sum_{j=1}^{m} \|T(x_{1,j}, \dots, x_{n,j})\|^p\Big)^{1/p} \\
&\quad \le C\Big(\sup_{\varphi_l \in B_{E_l'}, l=1,\dots,n} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p\Big)^{1/p} \\
&\quad \le C \sup_{\varphi_l \in B_{E_l'}, l=1,\dots,n} \Bigg[\Big(\sum_{j=1}^{m} |\varphi_1(x_{1,j})|^{np}\Big)^{1/np} \\
&\qquad\qquad \cdots \Big(\sum_{j=1}^{m} |\varphi_n(x_{n,j})|^{np}\Big)^{1/np}\Bigg] \\
&\quad = C \left\|(x_{1,j})_{j=1}^m\right\|_{w,np} \cdots \left\|(x_{n,j})_{j=1}^m\right\|_{w,np}.
\end{aligned}$$

(v) If $T \in \mathcal{L}_{si,p}({}^nE;F)$, then

$$\begin{aligned}
&\sum_{j_1,\dots j_n=1}^{m} \|T(x_{1,j_1}, \dots, x_{n,j_n})\|^p \\
&\le C^p \int_{B_{E_1'} \times \cdots \times B_{E_n'}} \sum_{j_1,\dots j_n=1}^{m} |\varphi_1(x_{1,j_1}) \cdots \varphi_n(x_{n,j_n})|^p \, d\mu(\varphi_1, \dots, \varphi_n).
\end{aligned}$$

So

$$\begin{aligned}
&\Big(\sum_{j_1,\dots j_n=1}^{m} \|T(x_{1,j_1}, \dots, x_{n,j_n})\|^p\Big)^{1/p} \\
&\quad \le C\Big(\int_{B_{E_1'} \times \cdots \times B_{E_n'}} \sum_{j_1,\dots j_n=1}^{m} |\varphi_1(x_{1,j_1}) \cdots \varphi_n(x_{n,j_n})|^p \, d\mu \\
&\qquad\qquad \times (\varphi_1, \dots, \varphi_n)\Big)^{1/p} \\
&\quad \le C \sup_{\varphi_l \in B_{E_l'}, l=1,\dots,n} \Big(\sum_{j_1,\dots j_n=1}^{m} |\varphi_1(x_{1,j_1}) \cdots \varphi_n(x_{n,j_n})|^p\Big)^{1/p}
\end{aligned}$$

$$= C \sup_{\varphi_l \in B_{E_l'}, l=1,\dots,n} \left[\left(\sum_{j_1=1}^{m} |\varphi_1(x_{1,j_1})|^p \right)^{1/p} \cdots \left(\sum_{j_n=1}^{m} |\varphi_n(x_{n,j_n})|^p \right)^{1/p} \right],$$

and thus $T \in \mathcal{L}_{fas,p}(^n E; F)$. Now let us consider $T \in \mathcal{L}_{fas,p}(^n E; F)$. The case $n = 2$ is illustrative and indicates the proof. If $(x_j)_{j=1}^\infty, (y_j)_{j=1}^\infty \in l_p^w(E)$, we have

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} \|T(a + x_j, b + y_j) - T(a, b)\|^p \right)^{1/p}
&\leq \left(\sum_{j=1}^{\infty} \|T(a, y_j)\|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} \|T(x_j, b)\|^p \right)^{1/p} \\
&\quad + \left(\sum_{j=1}^{\infty} \|T(x_j, y_j)\|^p \right)^{1/p} \\
&\leq \left(\sum_{j,k=1}^{\infty} \|T(z_k, y_j)\|^p \right)^{1/p} + \left(\sum_{j,k=1}^{\infty} \|T(x_j, w_k)\|^p \right)^{1/p} \\
&\quad + \left(\sum_{j,k=1}^{\infty} \|T(x_j, y_k)\|^p \right)^{1/p} < \infty,
\end{aligned}$$

where $(z_k)_{k=1}^\infty = (a, 0, 0, \dots)$ and $(w_k)_{k=1}^\infty = (b, 0, 0, \dots)$.

The proof of (vi) is easy and we omit it. □

Remark 1. Obviously, each one of the assertions in Theorem 3 holds for spaces $E_1, \dots, E_n$ instead of $E, \dots, E$. The inclusion $\mathcal{L}_{si,1}(E_1, \dots, E_n; F) \subset \mathcal{L}_{sas,1}(E_1, \dots, E_n; F)$ is sometimes strict. In fact, if $T : l_1 \times l_1 \to \mathbf{K}$ is given by

$$T((x_i)_{j=1}^\infty, (y_j)_{j=1}^\infty) = \sum_{j=1}^{\infty} y_j \sum_{k=1}^{j} x_k,$$

then T fails to be semi-integral, see [**1**], but T is obviously strongly 1-summing.

The inclusion $\mathcal{L}_{si,1}(E_1,\dots,E_n;F) \subset \mathcal{L}_{fas,1}(E_1,\dots,E_n;F)$ is also sometimes strict, since

$$\mathcal{L}(l_2,l_1;\mathbf{K}) = \mathcal{L}_{fas,1}(l_2,l_1;\mathbf{K}) \quad [\mathbf{20}]$$

and

$$\mathcal{L}(l_2,l_1;\mathbf{K}) \neq \mathcal{L}_{si,1}(l_2,l_1;\mathbf{K}).$$

In fact, if we had $\mathcal{L}(l_2,l_1;\mathbf{K}) = \mathcal{L}_{si,1}(l_2,l_1;\mathbf{K})$, by Theorem 3 (ii) we would have

$$\mathcal{L}(l_2,l_\infty) = \mathcal{L}_{as,1}(l_2,l_\infty),$$

and it is a contradiction. The inclusion

$$\mathcal{L}_{fas,1}(E_1,\dots,E_n;F) \subset \mathcal{L}_{as,1}(E_1,\dots,E_n;F)$$

is also strict, see [**12**].

In general $\mathcal{L}_{sas,p}(E_1,\dots,E_n;F)$ is not contained in $\mathcal{L}_{as,p}(E_1,\dots,E_n;F)$ and $\mathcal{L}_{as,p}(E_1,\dots,E_n;F)$ is not contained in $\mathcal{L}_{sas,p}(E_1,\dots,E_n;F)$. In fact, $\mathcal{L}_{as,1}(^2l_1;l_1) = \mathcal{L}(^2l_1;l_1)$ and $\mathcal{L}_{sas,1}(^2l_1;l_1) \neq \mathcal{L}(^2l_1;l_1)$. On the other hand, $\mathcal{L}_{sas,2}(^2l_2;\mathbf{K}) = \mathcal{L}(^2l_2;\mathbf{K})$ and $\mathcal{L}_{as,2}(^2l_2;\mathbf{K}) \neq \mathcal{L}(^2l_2;\mathbf{K})$.

In [**4**], Botelho proves that $P_n : l_1 \to l_1 : P_n((\alpha_i)_{i=1}^\infty) = ((\alpha_i)^n)_{i=1}^\infty$ is n-dominated and is not weakly compact. The same occurs with the symmetric n-linear mapping associated to P.

The question, "Is every strongly p-summing n-linear mapping weakly compact?" appears in [**8**] and was recently answered by Carando-Dimant in [**6**]. However, by Theorem 3, since $\mathcal{L}_{d,p}(E_1,\dots,E_n;F) \subset \mathcal{L}_{sas,p}(E_1,\dots,E_n;F)$, one can see that Botelho's counterexample is a direct answer to this question.

The next result shows that the spaces of semi-integral and dominated mappings coincide in some situations. Firstly, let us recall the concept of cotype: Let $2 \leq q \leq \infty$ and $(r_j)_{j=1}^\infty$ be Rademacher functions. We say that E has cotype q if there exists $C \geq 0$ such that for any choice of $k \in \mathbf{N}$ and $x_1,\dots,x_k \in E$ we have

$$\bigg(\sum_{j=1}^k \|x_j\|^q\bigg)^{1/q} \leq C\bigg(\int_0^1 \|\sum_{j=1}^k r_j(t)x_j\|^2\,dt\bigg)^{1/2}.$$

In the case $q=\infty$, we replace $(\sum_{j=1}^k \|x_j\|^q)^{1/q}$ by $\max\{\|x_j\|;\ 1\leq j\leq k\}$.

Theorem 4. *If E has cotype 2, then $\mathcal{L}_{si,1}(^2E;F) = \mathcal{L}_{d,1}(^2E;F)$ for every F.*

Proof. If E has cotype 2, we know that $\mathcal{L}_{as,1}(E;F) = \mathcal{L}_{as,2}(E;F)$ for every Banach space F. Thus, if $T \in \mathcal{L}_{d,2}(^2E;F)$, then $T = R\circ(u_1,u_2)$, with $R \in \mathcal{L}(^2G;F)$ and $u_1,u_2 \in \mathcal{L}_{as,2}(E;G) = \mathcal{L}_{as,1}(E;G)$. Hence, $T \in \mathcal{L}_{d,1}(^2E;F)$ and thus

$$\mathcal{L}_{d,1}(^2E;F) = \mathcal{L}_{d,2}(^2E;F)$$

for every F. Since $\mathcal{L}_{d,1}(^2E;F) \subset \mathcal{L}_{si,1}(^2E;F) \subset \mathcal{L}_{d,2}(^2E;F)$, we thus have $\mathcal{L}_{si,1}(^2E;F) = \mathcal{L}_{d,1}(^2E;F)$. $\square$

Remark 2. We can find Banach spaces E and F so that $\mathcal{L}_{d,1}(^2E;F) \subsetneq [\Pi_{as(1)}](^2E;F)$. The following example is suggested by M.C. Matos.

Consider $T : l_2 \times l_2 \to \mathbf{K}$ given by $T(x,y) = \sum_{j=1}^{\infty}(1/j^{\alpha})x_jy_j$ with $\alpha = 1/2+\varepsilon$ and $\varepsilon \in \,]0,1/2[$. We will show that $T \in [\Pi_{as(1)}](^2l_2;\mathbf{K}) \setminus \mathcal{L}_{d,1}(^2E;F)$. Since

$$\left(\sum_{j=1}^{m}\|T(e_j,e_j)\|^{1/2}\right)^2 = \left[\sum_{j=1}^{m}\left(\frac{1}{j^{\alpha}}\right)^{1/2}\right]^2 \geq \left[\sum_{j=1}^{m}\left(\frac{1}{m^{\alpha/2}}\right)\right]^2 = m^{2-\alpha},$$

if we had

$$\left(\sum_{j=1}^{m}\|T(e_j,e_j)\|^{1/2}\right)^2 \leq C\left\|(e_j)_{j=1}^{m}\right\|_{w,1}^2,$$

for every m, we would obtain $m^{2-\alpha} \leq C(m^{1/2})^2 = Cm$, a contradiction.

In order to prove that $\Psi_1^{(2)}(T) \in \mathcal{L}_{as,1}(l_2;l_2)$, observe that

$$\Psi_1^{(2)}(T)((x_j)_{j=1}^{\infty}) = \left(\frac{1}{j^{\alpha}}\,x_j\right)_{j=1}^{\infty}.$$

Now it suffices to show that $\Psi_1^{(2)}(T)$ is a Hilbert-Schmidt operator, see [**15**]. But it is easy to check, since

$$\sum_{k=1}^{\infty}\left\|\Psi_1^{(2)}(T)(e_k)\right\|_{l_2}^2 = \sum_{k=1}^{\infty}\left[\frac{1}{k^{\alpha}}\right]^2 < \infty.$$

Hence the inclusion is strict. Since l_2 has cotype 2, Theorem 4 yields that

$$\mathcal{L}_{si,1}(^2l_2;\mathbf{K}) = \mathcal{L}_{d,1}(^2l_2;\mathbf{K}) \subsetneq [\Pi_{as(1)}](^2l_2;\mathbf{K}).$$

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Note added in proof. A recent paper [**21**] due to David Pérez-García presents nice information comparing the classes of multiple (fully) summing, dominated and absolutely summing multilinear mappings.

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