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A GENERALIZED HAWKINS SIEVE AND PRIME K-TUPLETS

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ABSTRACT. The Hawkins random sieve, obtained from a simple probabilistic variation of Eratoshenes's sieve, provides a compelling model for the primes. Building on the Hawkins' sieve, we introduce a general random sieve, and prove analogs of both the Prime Number Theorem and Mertens' theorem. Applications include a new probabilistic model for prime k-tuplets.

1. Introduction.

1.1 Purpose. In this paper, we introduce a natural generalization of the Hawkins' random sieve, and prove analogs of both the Prime Number Theorem (PNT) and Mertens' theorem in the more general setting. As an application we present a new probabilistic model for prime k-tuplets.

1.2 Background. When faced with the complexity of prime distribution theory, it is tempting to employ mathematical models. One of the most compelling models for the prime numbers is known as the *Hawkins' primes*. The Hawkins' model, first introduced by David Hawkins [**13**], is based on a simple stochastic variation of the sieve of Eratosthenes. Hawkins' sieve works as follows: Starting with all natural numbers two and larger, we identify $X_1 = 2$ as our first 'sieving number.' In the first step we independently sieve numbers from our list with probability $1/X_1$, and identify X_2 as the smallest surviving number which is larger than X_1. In the second step, we sieve numbers from our remaining list with probability $1/X_2$ and identify X_3 as the smallest surviving number which is larger than X_2. If we carry on with the process, we produce a list $\{X_1, X_2, \dots,\}$ of sieving numbers which are called Hawkins' primes.

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A good model should be accurate enough to yield useful information, yet simple enough to be approachable. The Hawkins' model appears to be accurate, in that it shares two important characteristics with the real primes, see [**13**]:

• The Hawkins' primes have the same asymptotic density as the real primes.

• Like the real primes, the events of Hawkins primality are interdependent.

Further, the model has proved to be approachable. Results established for the Hawkins' primes parallel many of the famous existing results and conjectures concerning the distribution of primes, including PNT, Mertens' theorem, the twin primes conjecture, and the Riemann hypothesis. (A sampling is given in [**3, 14, 19**–**21, 24**]. See [**18**] for a survey of such results.) For example, letting X_n denote the nth member of a Hawkins' prime sequence, the analogs of PNT and Mertens' theorem, respectively, are (see [**21, 24**]):

$$X_n \sim n \log n \quad \text{a.s.} \quad \text{and} \quad \prod_{k \le n} (1 - 1/X_k)^{-1} \sim \log n \quad \text{a.s.}$$

(Unfortunately, but perhaps not surprisingly, the analog of Mertens' theorem predicted by the Hawkins' model fails to detect the tantalizing factor of e^γ which is present in the 'real' Mertens' theorem, and which reflects the special nature of the primes, see [**12**]. We are reminded that the Hawkins' model is not entirely ideal.)

There have been, more or less, two types of techniques used to prove probabilistic results about the Hawkins' primes. The first features clever tinkering with limit theory of various types of random variables, including orthogonal random variables and martingales, e.g., see [**14, 20, 21**]. The second involves a frontal approach in which one directly computes expectations of random variables by constructing an appropriate probability measure on a sample space of sequences, e.g., see [**24**]. The first technique is elegant, the second is flexible, and both are effective. In the sequel, as we consider a generalization of the Hawkins' sieve, we find that a marriage of these techniques is useful.

1.3 Main results. In Hawkins' original model, a sieving number n sieves subsequent numbers with probability $1/n$. We generalize the model by simply allowing sieving numbers n to sieve subsequent numbers with (fixed) probability $\mathfrak{p}(n)$. A sequence generated by such a sieve will be called a set of *Hawkins' $\mathfrak{p}$-primes*. The main questions we must address are:

(1) What results can be established for the new model, and do they bear resemblance to those for Hawkins' original model?

(2) What kinds of interesting integer sequences can be modeled by choosing $\mathfrak{p}(n)$ appropriately?

Regarding the first question, given certain conditions on the decay of $\mathfrak{p}(n)$ we produce an asymptotic formula for the density of Hawkins' $\mathfrak{p}$-primes, see Theorem 2.1. Further, with additional conditions on $\mathfrak{p}(n)$, we obtain asymptotic formulas for both X_n (the nth term of a sequence of Hawkins' $\mathfrak{p}$-primes) and $\prod_{k<n}(1-\mathfrak{p}(X_k))^{-1}$, thereby obtaining probabilistic generalizations of PNT and Mertens' theorem, see Theorem 5.6 and Corollary 4.3, respectively. To obtain these latter results, we employ a combination of techniques previously used separately on the Hawkins' sieve.

For the second question, we show that in the case $\mathfrak{p}(n) = n^{-1}\log^k n$, the sieve provides a new probabilistic model for prime $(k+1)$-tuplets. These results and a precise definition of 'prime k-tuplet' may be found in Section 6.

1.4 Additional comments and references. The present paper is concerned with a collection of random sieves which generalize a random model of the classical sieve. However, we note that there are well-known *nonrandom* sieves which generalize the sieve of Eratosthenes. A key example is Beurling's generalized primes [**2**], which have been studied extensively, e.g., see [**1, 4**–**6**]. Also, those seeking general background in probabilistic number theory might wish to consult, for example, [**7, 8, 15, 16, 22**].

2. Generalized primes and their density. In this section we introduce a random sieving procedure on the natural numbers reminiscent of the Hawkins' sieve, and investigate the density of sequences produced by the sieve.

To begin, we set $\mathfrak{p} : \mathbf{N}_{\geq 2} \to [0,1]$. The sieve works in the following way. Starting with all natural numbers two and larger, we identify $X_1 = 2$ as our first 'sieving number.' In the first step we independently sieve numbers from our list with probability $\mathfrak{p}(X_1)$, and identify X_2 as the smallest surviving number which is larger than X_1. In the second step, we sieve numbers from our remaining list with probability $\mathfrak{p}(X_2)$ and identify X_3 as the smallest surviving number which is larger than X_2. If we carry on with the process, we produce a list $\{X_1, X_2, \ldots, \}$ of sieving numbers which we call *Hawkins' $\mathfrak{p}$-primes*. In the case $\mathfrak{p}(n) = 1/n$, this specializes to the Hawkins' random sieve.

In the following theorem, we determine the local behavior of the Hawkins' $\mathfrak{p}$-primes, subject to some conditions on the decay of $\mathfrak{p}(n)$.

Theorem 2.1. *Let S_n denote the event that a natural number n is a sieving number, and suppose that $\sum \mathfrak{p}^2(k)$ converges while $\sum \mathfrak{p}(k)$ diverges. Then*

$$P(S_n) \sim \left(\sum_{k \leq n} \mathfrak{p}(k) \right)^{-1}.$$

Proof. We first show that

$$P(S_{n+1}) = P(S_n) - \mathfrak{p}(n) P(S_n)^2, \tag{1}$$

which, with T_n denoting the event complementary to S_n, is equivalent to

$$P(T_{n+1}) = P(T_n)^2 + P(T_n)P(S_n) + \mathfrak{p}(n)P(S_n)^2. \tag{2}$$

To verify (2), we begin with

$$P(T_{n+1}) = P(T_{n+1}|T_n)P(T_n) + P(T_{n+1}|S_n)P(S_n), \tag{3}$$

and use elementary facts about conditional probability together with the definition of the sieve to show

$$P(T_{n+1}|T_n) = P(T_n) \quad \text{and} \quad P(T_{n+1}|S_n) = P(T_n) + \mathfrak{p}(n)P(S_n). \tag{4}$$

The results of (4) and (3) together imply (2).

With (1) in hand, we proceed with the proof of the theorem. If we put $g_n = 1/P(S_n)$, then (1) together with basic facts about geometric series gives

$$g_{n+1} = \frac{g_n}{1 - (\mathfrak{p}(n)/g_n)} = \sum_{k=0}^{\infty} g_n \left(\frac{\mathfrak{p}(n)}{g_n} \right)^k = g_n + \mathfrak{p}(n) + \frac{\mathfrak{p}^2(n)}{g_n - \mathfrak{p}(n)}.$$

Therefore,

$$g_n = g_2 + \sum_{k=2}^{n-1} (g_{k+1} - g_k) = g_2 + \sum_{k=2}^{n-1} \mathfrak{p}(k) + \sum_{k=2}^{n-1} \frac{\mathfrak{p}^2(k)}{g_k - \mathfrak{p}(k)}. \tag{5}$$

Our hypotheses on $\mathfrak{p}(n)$ imply that the righthand-most sum in (5) converges as $n \to \infty$. Therefore, from (5) we have

$$\lim_{n \to \infty} g_n \left[\sum_{k \le n} \mathfrak{p}(k) \right]^{-1} = 1,$$

concluding the proof. $\square$

3. Some random variables and their expectations. In order to establish analogs of Mertens' theorem and PNT for the Hawkins' $\mathfrak{p}$-primes, we need the expected values of certain random variables defined on a sample space of sequences. In this section, we specify the sample space and associated probability measure, and we compute the desired expected values. Techniques in this section are drawn from Wunderlich [**24**].

3.1 Sample space and probability measure. Let $\mathcal{X}$ denote the set of all strictly increasing sequences of integers larger than 1. A set $A \subset X$ is said to be *elementary* if there exists a finite sequence $\{a_1, a_2, \dots, a_k\} \in \mathcal{X}$ and an integer $n > a_k$ (called the *order* of A) for which a sequence α lies in A if and only if the terms of α strictly less than n are precisely $a_1, a_2, \dots, a_k$. An elementary set A of order n will be denoted by $(a_1, \dots, a_k; n)$. Finally, let $\mathcal{A}$ denote the σ-field on $\mathcal{X}$ generated by the collection of elementary sets.

Theorem 3.1 [**24**]. *Let* $\mathfrak{p} : \mathbf{N} \to [0,1]$ *be given. There is a probability measure* P *on* $\mathcal{A}$ *defined recursively on elementary sequences as follows*:

(a) $P(\ ;2) = 1$.

(b) $P(a_1, a_2, \dots, a_k, n; n+1) = P(a_1, a_2, \dots, a_k; n) \prod_{i=1}^{k} (1 - \mathfrak{p}(a_i))$.

(c) $P(a_1, a_2, \dots, a_k; n+1) = P(a_1, a_2, \dots, a_k; n) \Big(1 - \prod_{i=1}^{k} (1 - \mathfrak{p}(a_i))\Big)$.

Proof. The proof in case $\mathfrak{p}(n) = 1/n$ is given by Wunderlich [**24**]. The same proof works in the more general setting. □

3.2 Expectations. For the immediate future, our main concern will be the following sequence of random variables.

Definition 3.2. For $n \geq 2$ and $\alpha \in \mathcal{X}$, put

$$y_n(\alpha) = \prod_{\substack{a \in \alpha \\ a < n}} (1 - \mathfrak{p}(a))$$

for $n > 2$, and put $y_2(\alpha) = 1$.

Observe that when $\mathfrak{p}(n) = 1/n$ and α is the sequence of 'real' primes, we obtain

$$y_n(\alpha) = \prod_{\substack{p \in \alpha \\ p < n}} (1 - 1/p),$$

an asymptotic formula for which constitutes the 'real' Mertens' theorem. Also, note that y_n is constant on any elementary set A_n of order n, and we will often write $y(A_n)$ in place of $y_n(\alpha)$ for $\alpha \in A_n$.

Lemma 3.3. *Let* $n, k \in \mathbf{Z}$ *with* $n \geq 2$. *We have*

$$E[y_{n+1}^k] - E[y_n^k] = \left((1 - \mathfrak{p}(n))^k - 1\right) E[y_n^{k+1}].$$

Proof. For $\alpha \in \mathcal{X}$, put

$$z_n(\alpha) = \begin{cases} y_n(\alpha) & \text{if } n \in \alpha, \\ 1 - y_n(\alpha) & \text{if } n \notin \alpha, \end{cases}$$

and observe from Theorem 3.1 that if α is any sequence in an elementary set A_n, then

$$P(A_n) = \prod_{j=3}^{n-1} z_j(\alpha). \tag{6}$$

Now, appealing to (6) and Definition 3.2, we have

$$\begin{aligned} E[y_{n+1}^k] &= \sum_{A_{n+1}} y^k(A_{n+1})P(A_{n+1}) \\ &= \sum_{A_n} P(A_n)(1-y(A_n))y^k(A_n)+P(A_n)y(A_n)y^k(A_n)(1-\mathfrak{p}(n))^k \\ &= \sum_{A_n} P(A_n)((1-\mathfrak{p}(n))^k-1)y^{k+1}(A_n)+P(A_n)y^k(A_n) \\ &= ((1-\mathfrak{p}(n))^k-1)E[y_n^{k+1}]+E[y_n^k], \end{aligned}$$

where the sums extend over all elementary sets of the indicated order, and the two types of terms in the second sum correspond to whether or not n lies in a given A_{n+1}. □

We now set about using the recursive formula in Lemma 3.3 to compute $E[y_n^k]$, but before doing so, we place some conditions on the sieving probabilities $\mathfrak{p}(n)$.

Condition 3.4. *Extend* $\mathfrak{p}$ *to a function* $\mathfrak{p} : [2,\infty) \to [0,1]$ *such that*

(i) $\mathfrak{p}$ *is positive, continuous, and decreasing to zero, with* $\mathfrak{p}(2) < 1$.

(ii) $\sum \mathfrak{p}(n)$ *diverges.*

(iii) $\sum \mathfrak{p}^2(n)$ *converges.*

Lemma 3.5. *Suppose that* $\mathfrak{p}$ *satisfies Condition* 3.4, *and put* $I(n) = \int_2^n \mathfrak{p}(t)\,dt$. *Then*

$$E[y_n^{-1}] = I(n) + O(1) \quad \text{and} \quad E[y_n] = I^{-1}(n) + O(I^{-2}(n)).$$

Proof. By putting $k = -1$ in Lemma 3.3 and adding successive differences, one obtains

$$E[y_n^{-1}] = 1 + \sum_{k=2}^{n-1} \frac{\mathfrak{p}(k)}{1 - \mathfrak{p}(k)}. \tag{7}$$

Meanwhile, by putting $k = 1$ in Lemma 3.3 and recalling that (by the Schwarz inequality) $E^2[y_n] \leq E[y_n^2]$, we have

$$\begin{aligned} \frac{1}{E[y_{n+1}]} - \frac{1}{E[y_n]} &= \left(\frac{\mathfrak{p}^{-1}(n) E^2[y_n]}{E[y_n^2]} - E[y_n] \right)^{-1} \\ &\geq \frac{1}{\mathfrak{p}^{-1}(n) - E[y_n]} \geq \mathfrak{p}(n). \end{aligned}$$

Therefore, by adding successive differences we obtain

$$E^{-1}[y_n] \geq 1 + \sum_{k=2}^{n-1} \mathfrak{p}(k). \tag{8}$$

Another application of the Schwarz inequality together with (7) and (8) gives

$$1 + \sum_{k=2}^{n-1} \mathfrak{p}(k) \leq E^{-1}[y_n] \leq E[y_n^{-1}] = 1 + \sum_{k=2}^{n-1} \frac{\mathfrak{p}(k)}{1 - \mathfrak{p}(k)}. \tag{9}$$

From (9), parts (i) and (iii) of Condition 3.4 imply that $E[y_n^{-1}] = I(n) + O(1)$ and $E^{-1}[y_n] = I(n) + O(1)$. The latter estimate implies that $E[y_n] = I^{-1}(n) + O(I^{-2}(n))$. In all three estimates, part (ii) of Condition 3.4 ensures that the error terms are strictly smaller than the leading terms. $\square$

Proposition 3.6. *If $\mathfrak{p}$ satisfies Condition 3.4 and k is a positive integer, then*

(a) $E[y_n^{-k}] = I^k(n) + O(I^{k-1}(n))$, *and*

(b) $E[y_n^k] = I^{-k}(n) + O(I^{-(k+1)}(n))$.

Proof. First, we note that $I(n)\mathfrak{p}(n)$ is bounded as a consequence of parts (i) and (iii) of Condition 3.4:

$$I(n)\mathfrak{p}(n) = \int_2^n \mathfrak{p}(n)\mathfrak{p}(t)\,dt \le \int_2^\infty \mathfrak{p}^2(t)\,dt < \infty.$$

In particular, this implies that, for $k \ge 1$,

$$\mathfrak{p}^2(n)I^k(n) = O(\mathfrak{p}(n)I^{k-1}(n)). \tag{10}$$

We apply induction to both statements in the theorem. Part (a) holds for $k = 1$ by Lemma 3.5. Suppose now that $k > 1$ and the statement holds for $k - 1$. Using the induction hypothesis, Lemma 3.3, and (10), we have

$$\begin{aligned} E[y_{n+1}^{-k}] - E[y_n^{-k}] &= \left[\left(1 + \frac{\mathfrak{p}(n)}{1-\mathfrak{p}(n)}\right)^k - 1\right] E[y_n^{-(k-1)}] \\ &= \left[\frac{k\mathfrak{p}(n)}{1-\mathfrak{p}(n)} + O\left(\left(\frac{\mathfrak{p}(n)}{1-\mathfrak{p}(n)}\right)^2\right)\right] \\ &\quad \times \left[I^{k-1}(n) + O(I^{k-2}(n))\right] \\ &= \frac{k\mathfrak{p}(n)I^{k-1}(n)}{1-\mathfrak{p}(n)} + O(\mathfrak{p}(n)I^{k-2}(n)) \\ &= k\mathfrak{p}(n)I^{k-1}(n) + O(\mathfrak{p}(n)I^{k-2}(n)), \end{aligned}$$

where the last step is accomplished by expanding $(1 - \mathfrak{p}(n))^{-1}$ as a series and then applying (10) again for the error term. Part (a) then follows by adding successive differences.

By Lemma 3.5, part (b) holds for $k = 1$. Now assume $k > 1$ and the statement holds for $k-1$. Part (a) together with the Schwarz inequality give

$$\begin{aligned} E[y_n^k] &\ge E^{-1}[y_n^{-k}] = (I^k(n) + O(I^{k-1}(n))^{-1} \\ &= I^{-k}(n) + O(I^{-(k+1)}(n)). \end{aligned} \tag{11}$$

Using Lemma 3.3 we find that

$$\begin{aligned} E^{-1}[y_{n+1}^k] - E^{-1}[y_n^k] &= \left(\frac{E[y_{n+1}^k]E[y_n^k]}{E[y_n^k] - E[y_{n+1}^k]}\right)^{-1} \\ &= \left[\frac{E^2[y_n^k] - (k\mathfrak{p}(n) + O(\mathfrak{p}^2(n)))E[y_n^{k+1}]E[y_n]}{(k\mathfrak{p}(n) + O(\mathfrak{p}^2(n)))E[y_n^{k+1}]}\right]^{-1}. \end{aligned}$$

This together with the Schwarz inequality and the induction hypothesis gives

$$
\begin{aligned}
E^{-1}[y_{n+1}^k] - E^{-1}[y_n^k] &\geq \left[\frac{E^2[y_n^k]}{(k\mathfrak{p}(n) + O(\mathfrak{p}^2(n)))E[y_n^{k+1}]}\right]^{-1} \\
&\geq \frac{k\mathfrak{p}(n) + O(\mathfrak{p}^2(n))}{E[y_n^{k-1}]} \\
&= \frac{k\mathfrak{p}(n) + O(\mathfrak{p}^2(n))}{I^{-(k-1)}(n) + O(I^{-k}(n))} \\
&= k\mathfrak{p}(n)I^{k-1}(n) + O(\mathfrak{p}(n)I^{k-2}(n)).
\end{aligned}
$$

Adding successive differences then gives

$$E^{-1}[y_n^k] \geq I^k(n) + O(I^{k-1}(n)),$$

which implies

$$E[y_n^k] \leq I^{-k}(n) + O(I^{-(k+1)}(n)). \tag{12}$$

Equations (11) and (12) together imply part (b) of the theorem. □

4. Mertens' theorem. In this section we prove a strong probabilistic analog of Mertens' theorem for the Hawkins $\mathfrak{p}$-primes, subject to certain conditions on the sieving function $\mathfrak{p}$.

Theorem 4.1 [**24**]. *Suppose that x_n is a sequence of random variables on $\mathcal{X}$ with mean $E[x_n]$ and variance $V[x_n]$. Suppose further that we can choose functions $E(n)$, $V(n)$, and $R(n)$ on $\mathbf{N}$ satisfying*

$$E(n) \sim E[x_n], \quad V(n) = O(V[x_n]), \quad \textit{and} \quad R(n) = V(n)/E^2(n)$$

and

(a) $\lim_{a\to 1}\left(\lim_{n\to\infty} \frac{E(a^n)}{E(a^{n+1})}\right) = 1$ *where* $E(a) = E([a])$,

(b) $R(n)$ *is monotonically decreasing and* $\sum R(k)/k$ *converges*,

(c) x_n *is monotonic in* n *when applied to any* $\alpha \in \mathcal{X}$.

Then $x_n \sim E(n)$ *a.s.*

Condition 4.2. *Extend* $\mathfrak{p}$ *to a continuous function* $\mathfrak{p} : [2, \infty) \to [0, 1]$ *such that*

(i) $\lim_{a\to 1} \big(\lim_{n\to\infty} I(a^{n+1})/I(a^n)\big) = 1$, *and*

(ii) $\sum (n\, I(n))^{-1}$ *converges,*

where $I(n) = \int_2^n \mathfrak{p}(t)\, dt$.

Now we have the following analog of Mertens' theorem.

Corollary 4.3. *Suppose the sieving probability function* $\mathfrak{p}$ *satisfies Conditions* 3.4 *and* 4.2. *Then* $y_n \sim I^{-1}(n)$ *a.s.*

Proof. By applying part (b) of Proposition 3.6 (valid here since we are assuming Condition 3.4), we obtain

$$V[y_n] = I^{-2}(n) + O(I^{-3}(n)) - (I^{-1}(n) + o(I^{-2}(n)))^2 = O(I^{-3}(n)).$$

So, in applying Theorem 4.1 to y_n, we should choose $E(n) = I^{-1}(n)$, $V(n) = I^{-3}(n)$ and $R(n) = I^{-1}(n)$. Condition 4.2 then insures that the three conditions of Theorem 4.1 will be satisfied, and the result follows. □

Remark 4.4. Observe that the special case $\mathfrak{p}(n) = 1/n$ (the original Hawkins' sieve) does not satisfy part (ii) of Condition 4.2. Using these techniques, somewhat different estimates for $V[y_n]$ are needed to obtain an analog of Mertens' theorem (and PNT) in this case. Interested readers should refer to [**24**].

5. Limit theory and the prime number theorem. The techniques used thus far, which have gained us an analog of Mertens' theorem (Corollary 4.3), do not lend themselves well for an analog of PNT for the Hawkins $\mathfrak{p}$-primes. In this section, we use the analog of Mertens' theorem together with limit theory for orthogonal random variables to prove a version of PNT for Hawkins $\mathfrak{p}$-primes. Techniques in this section are reminiscent of those in [**14, 20, 21**]. Those seeking a refresher in conditional expectations and orthogonality should consult either [**9**] or [**17**].

5.1 Limit theory. Before approaching PNT, we establish the limit theory of certain random variables. We begin with notation.

Notation 5.1. Let $n \in \mathbf{Z}^+$ and a sieving probability function $\mathfrak{p}$ be given.

(i) X_n denotes the random variable on $\mathcal{X}$ giving the nth generalized prime.

(ii) $Y_n := \prod_{k \le n}(1 - \mathfrak{p}(X_k))^{-1}$. (One may think of Y_n as the nth partial Euler product of Hawkins $\mathfrak{p}$-primes. Also, observe $Y_n = y_{X_n}^{-1}$, where y_n is as in Definition 3.2.)

(iii) Put $U_{n+1} = Y_n^{-1}(X_{n+1} - X_n)$.

(iv) $\mathcal{B}_n$ represents the sub σ-field of $\mathcal{A}$ generated by $\{X_j, Y_j \mid j \le n\}$.

The following limit theorem for orthogonal random variables is particularly useful, see [**17**, Section 33].

Theorem 5.2. *Let $\{Z_n\}$ be a sequence of orthogonal r.v.'s on a probability space $(\Omega, \mathcal{A}, \mu)$.*

(i) *If $\sum \log^2 n \cdot E|Z_n|^2 < \infty$, then the series $\sum Z_n$ converges in $L^2(\Omega)$ and a.s.*

(ii) *If $\sum (\log n/b_n)^2 E|Z_n|^2 < \infty$ with $\{b_n\}$ increasing to infinity, then*

$$\frac{1}{b_n}\sum_{k=1}^{n} Z_k \longrightarrow 0 \quad a.s.$$

Proposition 5.3. *Given a sieving probability function $\mathfrak{p}$, we have*

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k\le n} U_{k+1}\right) = 1 \quad a.s.$$

Proof. For $j \in \mathbf{Z}^+$, one may use the definition of the sieve (Section 2) to see that $X_{n+1} - X_n$ is geometric, with

$$P(X_{n+1} - X_n = j \mid \mathcal{B}_n) = Y_n^{-1}(1 - Y_n^{-1})^{j-1}. \tag{13}$$

Then, via summation using (13), one obtains

$$E[U_{n+1} - 1 \mid \mathcal{B}_n] = 0, \tag{14}$$

and

$$E[(U_{n+1} - 1)^2 \mid \mathcal{B}_n] = 1 - Y_n^{-1} \leq 1. \tag{15}$$

Since U_k is $\mathcal{B}_n$-measurable for $k \leq n$, it follows from (14) that $U_{n+1} - 1$ is orthogonal to $U_k - 1$ for $k \leq n$, and so $\{U_n - 1 \mid n \in \mathbf{N}\}$ forms a collection of orthogonal random variables.

To finish, from (15) we see that part (ii) of Theorem 5.2 applies with $b_n = n^{1/2+\varepsilon}$ to give

$$\frac{1}{n^{1/2+\varepsilon}} \sum_{k \leq n} U_{k+1} - 1 \longrightarrow 0 \quad \text{a.s.}$$

In case $\varepsilon = 1/2$, this specializes to

$$\frac{1}{n} \sum_{k \leq n} U_{k+1} \longrightarrow 1 \quad \text{a.s.} \qquad \square$$

5.2 The prime number theorem. We now put our generalization of Mertens' theorem (Corollary 4.3) together with the limit theory given above to obtain a generalization of the Prime Number theorem for Hawkins' $\mathfrak{p}$-primes.

Lemma 5.4. *Suppose the sieving probability function* $\mathfrak{p}$ *satisfies parts* (i) *and* (ii) *of Conditions* 3.4. *Then*

(a) Y_n *increases to infinity.*

(b) *If* $(X_{n+1}\, \mathfrak{p}(X_{n+1}))/Y_n \to 0$ *a.s., then* $X_n \sim n\, Y_n$ *a.s.*

Proof. We begin with part (a). For a contradiction, suppose Y_n increases to a finite limit Y. Note that

$$Y_n = \prod_{k \leq n} (1 - \mathfrak{p}(X_k))^{-1} = \prod_{k \leq n} \left(1 + \frac{\mathfrak{p}(X_k)}{1 - \mathfrak{p}(X_k)}\right),$$

yielding

$$Y_n \text{ increases to } Y \longleftrightarrow \sum_{k=1}^{\infty} \frac{\mathfrak{p}(X_k)}{1-\mathfrak{p}(X_k)} < \infty \longleftrightarrow \sum_{k=1}^{\infty} \mathfrak{p}(X_k) < \infty.$$

Meanwhile, appealing to Proposition 5.3, we have

$$\begin{aligned} 0 < \frac{X_{n+1}-X_n}{nY} &= \frac{1}{nY}\left[(X_2-X_1)+\cdots+(X_{n+1}-X_n)\right] \\ &\le \frac{1}{n}\left[\frac{X_2-X_1}{Y_1}+\cdots+\frac{X_{n+1}-X_n}{Y_n}\right] \\ &= \frac{1}{n}\left[U_2+\cdots+U_{n+1}\right] \longrightarrow 1 \quad \text{a.s.} \end{aligned}$$

Since Y is finite, it follows that X_n/n is bounded. This, taken together with part (i) of Condition 3.4s and the fact that $\sum \mathfrak{p}(X_k)$ converges, implies that $\sum \mathfrak{p}(n)$ converges, contradicting part (ii) of Condition 3.4.

For part (b), observe that

$$\begin{aligned} X_{n+1}Y_{n+1}^{-1} - X_nY_n^{-1} &= X_{n+1}(1-\mathfrak{p}\,(X_{n+1}))Y_n^{-1} - X_nY_n^{-1} \\ &= Y_n^{-1}[(X_{n+1}-X_n) - X_{n+1}\,\mathfrak{p}(X_{n+1})] \\ &= U_{n+1} - \frac{X_{n+1}\,\mathfrak{p}(X_{n+1})}{Y_n}. \end{aligned}$$

By adding successive differences and applying Proposition 5.3 and the part (b) hypotheses, we obtain

$$\frac{X_{n+1}}{nY_{n+1}} = \frac{1}{n}\sum_{k\le n} U_{k+1} - \frac{1}{n}\sum_{k\le n}\frac{X_{n+1}\,\mathfrak{p}(X_{n+1})}{Y_n} \longrightarrow 1-0=1 \quad \text{a.s.}$$

Therefore, $X_n \sim nY_n$. □

The previous lemma tells us that Y_n tends to infinity under rather mild conditions on $\mathfrak{p}$, but that something stronger must be imposed in order to force $X_n \sim nY_n$. This leads to our final condition on $\mathfrak{p}$:

Condition 5.5. *Extend the sieving probability function* $\mathfrak{p}$ *to a continuous function* $\mathfrak{p} : [2,\infty) \to [0,1]$ *such that* $n\,\mathfrak{p}(n)/I(n) \to 0$ *as* $n \to \infty$.

Theorem 5.6. *Suppose the sieving probability function* $\mathfrak{p}$ *satisfies Conditions* 3.4*,* 4.2 *and* 5.5*. Then*

$$\frac{1}{n}\,X_n \sim Y_n \sim I(X_n).$$

(Here, the asymptotic equivalence $n^{-1}X_n \sim I(X_n)$ is an analog of PNT, while $Y_n \sim I(X_n)$ is another analog of Mertens' theorem.)

Proof. From Corollary 4.3 and Notation 5.1 we deduce that $I(X_n) \sim Y_n$. Further, note that

$$Y_{n+1}{Y_n}^{-1} = (1-\mathfrak{p}(X_{n+1}))^{-1},$$

so $Y_{n+1}Y_n^{-1} \to 1$ a.s. Putting these facts together with Condition 5.5 gives

$$\begin{aligned} \frac{X_{n+1}\,\mathfrak{p}(X_{n+1})}{Y_n} &= \frac{X_{n+1}\,\mathfrak{p}(X_{n+1})}{I(X_{n+1})}\cdot\frac{I(X_{n+1})}{Y_{n+1}}\cdot\frac{Y_{n+1}}{Y_n} \\ &\to 0\cdot 1\cdot 1 = 0 \quad \text{a.s.} \end{aligned} \tag{16}$$

Due to (16), the hypotheses of part (b) of Lemma 5.4 are satisfied, and we conclude that $X_n \sim nY_n$ a.s. Therefore,

$$I(X_n) \sim Y_n \sim \frac{1}{n}\,X_n \quad \text{a.s.} \qquad \square$$

6. An application: Prime k-tuplets. In this section we present evidence that certain choices for $\mathfrak{p}(n)$ will yield probabilistic sieving models for prime k-tuplets.

To begin, we recall that, given natural numbers $0 < a_1 < a_2 \cdots < a_{k-1}$, the k-tuplet conjecture asserts that the number of primes $p \le x$ such that each of $p, p+2a_1, \ldots, p+2a_{k-1}$ is prime approaches

$$\frac{C_{a_1,\ldots,a_{k-1}}x}{\log^k x}$$

asymptotically, where $C_{a_1,\ldots,a_{k-1}}$ is a constant, and where we assume that there are no divisibility conditions preventing all of $p, p+$

$2a_1, \dots, p + 2a_{k-1}$ from being prime infinitely often. This conjecture, including specific values for $C_{a_1,\dots,a_{k-1}}$, was given first by Hardy and Littlewood [**11**] in the case of 2-tuples, and later generalized, see, for example, [**10**].

Now, we wish to examine our results in the special case $\mathfrak{p}(n) = \log^k n/n$, where k is a fixed nonnegative integer. In the case $k = 0$, this is simply the original Hawkins' sieve modeling the prime numbers, and it is known [**13, 21, 24**] that

$$P(S_n) \sim \frac{1}{\log n},$$

and

$$X_n \sim n \log n \quad \text{a.s.} \quad \text{and} \quad Y_n \sim \log n \quad \text{a.s.}$$

In the case $k \geq 1$, a moment's work shows that the sieving probability function $\mathfrak{p}(n)$ satisfies Conditions 3.4, 4.2 and 5.5, so that Theorem 2.1, Corollary 4.3 and Theorem 5.6 all apply.

Since $I(n) = (k+1)^{-1} \log^{k+1}(n)$, from Theorem 2.1 we conclude that

$$P(S_n) \sim \frac{(k+1)}{\log^{k+1} n},$$

which is (more or less) the conjectured asymptotic density of prime $(k+1)$-tuplets. So, for $k \geq 0$, the sieving probabilities $\mathfrak{p}(n) = \log^k n/n$ are likely candidates for probabilistic sieving models for prime $(k+1)$-tuplets.

We now produce specific probabilistic analogs of Mertens' theorem and PNT for the prime $(k+1)$-tuplets. From Theorem 5.6, in case $\mathfrak{p}(n) = \log^k n/n$ we have

$$Y_n \sim n \, \frac{\log^{k+1} X_n}{k+1} \sim X_n, \tag{17}$$

so that taking logarithms on both sides of the latter equivalence yields

$$\log n + (k+1) \log(\log X_n) - \log(k+1) \sim \log X_n.$$

This implies that $\log n \sim \log X_n$, and substituting back into (17) gives

Theorem 6.1. *In case $\mathfrak{p}(n) = \log^k n/n$, we have*

$$X_n \sim \frac{n}{k+1} \log^{k+1} n \quad a.s. \quad and \quad Y_n \sim \frac{1}{k+1} \log^{k+1} n \quad a.s.$$

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