ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 6, 2006

NONEXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

H. DANG, S. ORUGANTI AND R. SHIVAJI

ABSTRACT. We consider the system

$$\begin{aligned} -\Delta u &= \lambda\, f(v); \quad x \in \Omega \\ -\Delta v &= \mu\, g(u); \quad x \in \Omega \\ u = 0 &= v; \quad x \in \partial\Omega, \end{aligned}$$

where Ω is a ball in R^N, $N \geq 1$ and $\partial\Omega$ is its boundary, λ, μ are positive parameters bounded away from zero, and f, g are smooth functions that are negative at the origin and grow at least linearly at infinity. We establish the nonexistence of positive solutions when $\lambda\,\mu$ is large. Our proofs depend on energy analysis and comparison methods.

1. Introduction. Consider the system

$$\begin{aligned} -\Delta u &= \lambda\, f(v); \quad x \in \Omega \\ -\Delta v &= \mu\, g(u); \quad x \in \Omega \\ u = 0 &= v; \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded region in R^N, $\partial\Omega$ is its boundary, λ, $\mu \geq \varepsilon_0$ where $\varepsilon_0 > 0$, and f and g are smooth functions that grow at least linearly at infinity. Such systems arise naturally as steady states in reaction diffusion processes with unequal diffusion coefficients. It is of great interest to find regions of the parameters involved (diffusion coefficients) for which positive steady states cease to exist. If f(0) and g(0) are positive, then the nonexistence of positive solutions to (1.1) follows rather easily, see Appendix A. However the case when $f(0) < 0$ and $g(0) < 0$ is nontrivial.

The main purpose of this paper is to study this strictly semi-positone case. While the case when Ω is any bounded region remains open,

Received by the editors on March 31, 2004.

we establish such a nonexistence result when Ω is a ball. Namely, we assume

(C1) $f, g : [0, \infty) \to R$ are continuous, nondecreasing, $f(0) < 0$ and $g(0) < 0$,

(C2) there exist positive numbers K_i and M_i, $i = 1, 2$ such that $f(z) \geq K_1 z - M_1$ and $g(z) \geq K_2 z - M_2$ for all $z \geq 0$,

and establish:

Theorem 1.1. *Let* (C1)–(C2) *hold, and let Ω be a ball in R^N, $N \geq 1$. Then there exists a positive number σ such that the system* (1.1) *has no positive solutions for $\lambda \mu > \sigma$.*

When Ω is a ball and $N > 1$ by [**8**] all nonnegative solutions are positive componentwise. Hence by [**16**] solutions are radially symmetric and decreasing. The proofs of our main results rely heavily on this property. We will prove Theorem 1.1 in Section 2.

For an existence result for positive solutions for classes of superlinearities satisfying (C1), $\lambda = \mu$ and λ small, see [**10, 13**]. Also see [**11**] for a similar existence result for a class of p-Laplacian systems in an annulus and [**7**] for a recent survey on semipositone systems. In the single equation case, see [**1, 4, 5**] for nonexistence results and [**1**–**4, 6, 9, 12, 14, 15**] for existence results.

2. Proofs of main results. Without loss of generality we assume that Ω is the unit ball in R^N. Let (u, v) be a positive solution of (1.1). Then u and v are radial, decreasing and satisfy

$$
\begin{aligned}
-(r^{(N-1)}u')' &= \lambda\, r^{N-1} f(v); \quad 0 < r < 1 \\
-(r^{(N-1)}v')' &= \mu\, r^{N-1} g(u); \quad 0 < r < 1 \\
u'(0) &= 0 = v'(0) \\
u(1) &= 0 = v(1).
\end{aligned} \tag{2.1}
$$

We first establish some preliminary results.

Lemma 2.1. *There exists a positive constant C such that for $\lambda \mu$ large, $u(1/4) + v(1/4) \leq C$.*

Proof. Let λ_1 be the principal eigenvalue of the $-\Delta$ with Dirichlet boundary conditions and ϕ a corresponding positive eigenfunction. First note that (1.1) and (C2) imply, see (A.6),

$$\int_0^1 \left[\lambda_1 - \lambda\,\mu\,\frac{K_1K_2}{\lambda_1}\right] v\phi\, r^{N-1}\,dr \geq \int_0^1 \mu\left[-\,\lambda\,\frac{K_2M_1}{\lambda_1} - M_2\right]\phi\, r^{N-1}\,dr$$

and hence if $\lambda\,\mu$ is large enough, we have

$$\int_0^1 \frac{\lambda\,\mu}{2}\,K_1K_2\,v\phi\, r^{N-1}\,dr \leq \int_0^1 \lambda\,\mu\left[K_2M_1 + \frac{M_2\lambda_1}{\lambda}\right]\phi\, r^{N-1}\,dr.$$

This implies

$$\int_0^1 \frac{K_1K_2}{2}\,v\,\phi\, r^{N-1}\,dr \leq \int_0^1 \left[K_2M_1 + \frac{M_2\lambda_1}{\varepsilon_0}\right]\phi\, r^{N-1}\,dr. \tag{2.2}$$

Similarly,

$$\int_0^1 \frac{K_1K_2}{2}\,u\,\phi\, r^{N-1}\,dr \leq \int_0^1 \left[K_1M_2 + \frac{M_1\lambda_1}{\varepsilon_0}\right]\phi\, r^{N-1}\,dr. \tag{2.3}$$

Adding (2.2) and (2.3),

$$\begin{aligned}\int_0^1 (u+v)\,\phi\, r^{N-1}\,dr &\leq \frac{2}{K_1K_2}\int_0^1 \left[K_1M_2 + \frac{M_1\lambda_1}{\varepsilon_0} + K_2M_1 + \frac{M_2\lambda_1}{\varepsilon_0}\right]\phi\, r^{N-1}\,dr \\ &= C_0 \quad \text{(say)},\end{aligned} \tag{2.4}$$

and, since u and v are decreasing, we obtain

$$u\left(\frac{1}{4}\right) + v\left(\frac{1}{4}\right) \leq \frac{C_0}{\int_0^{1/4}\phi\, r^{N-1}\,dr} = C.$$

Hence the result.

Lemma 2.2. *For $\lambda\,\mu$ sufficiently large, $u(3/4) \le \beta_2$ or $v(3/4) \le \beta_1$, where β_1 and β_2 are the unique positive zeros of f and g, respectively.*

Proof. Suppose $u(3/4) > \beta_2$ and $v(3/4) > \beta_1$.

Case 1. $u(1/2) > \rho_2$ or $v(1/2) > \rho_1$ where $\rho_1 = (\beta_1 + \theta_1)/2$ and $\rho_2 = (\beta_2 + \theta_2)/2$ with θ_1 and θ_2 being the unique positive zeros of $F(z) = \int_0^z f(t)\,dt$ and $G(z) = \int_0^z g(t)\,dt$, respectively.

Now, if $u(1/2) > \rho_2$, then

$$-(r^{N-1}\,v')' = \mu\,r^{N-1}\,g(u) \ge \varepsilon_0\,r^{N-1}\,g(\rho_2) \quad \text{in } J := \left(\frac{1}{4}, \frac{1}{2}\right)$$

and $v(r) \ge \beta_1$ on $\bar{J}$. Let w be the unique solution of

$$\begin{aligned} -(r^{N-1}\,w')' &= \varepsilon_0\,r^{N-1}\,g(\rho_2) \quad \text{on } J \\ w &= \beta_1 \quad \text{on } \partial J. \end{aligned}$$

Then, by comparison arguments, $v(r) \ge w(r) = \varepsilon_0\,g(\rho_2) w_0(r) + \beta_1$ in $\bar{J}$, where w_0 is the unique (positive) solution of

$$\begin{aligned} -(r^{N-1}\,w_0')' &= 1 \quad \text{in } J \\ w_0 &= 0 \quad \text{on } \partial J. \end{aligned}$$

In particular, there exists $\bar{\beta}_1 > \beta_1$ such that $v(5/12) \ge w(5/12) \ge \bar{\beta}_1$ and hence $v(r) \ge \bar{\beta}_1$ on $J^* = (1/3, 5/12)$. Then

$$\begin{aligned} -(r^{N-1}(u-\beta_2)')' &= \lambda\,r^{N-1}\,f(v) \ge \lambda\,r^{N-1}\,f(\bar{\beta}_1) \\ &\ge \left(\frac{\lambda\,f(\bar{\beta}_1)}{C}\right) r^{N-1}\,(u-\beta_2) \quad \text{on } J^*, \end{aligned}$$

where C is as in Lemma 2.1. Since $u - \beta_2 > 0$ on $\bar{J}^*$, it follows that

$$\frac{\lambda\,f(\bar{\beta}_1)}{C} \le \lambda_1(J^*), \tag{2.5}$$

where $\lambda_1(J^*)$ is the principal eigenvalue of (B.2) with $(a, b) = J^*$, see Appendix B.

Next consider

$$-(r^{N-1}(v-\beta_1)')' = \mu\, r^{N-1}\, g(u) \geq \mu\, r^{N-1}\, g(\rho_2)$$
$$\geq \left(\frac{\mu\, g(\rho_2)}{C}\right) r^{N-1}(v-\beta_1) \quad \text{in } J.$$

Since $v-\beta_1>0$ on $\bar{J}$, it follows that (see Appendix B)

$$\frac{\mu\, g(\rho_2)}{C} \leq \lambda_1(J), \tag{2.6}$$

where $\lambda_1(J)$ is the principal eigenvalue of (B.2) with $(a,b)=J$.

Combining (2.5) and (2.6), we get

$$\frac{\lambda\,\mu f(\bar{\beta}_1)g(\rho_2)}{C^2} \leq \lambda_1(J^*)\lambda_1(J). \tag{2.7}$$

But, in the above inequality, $f(\bar{\beta}_1)$, $g(\rho_2)$ and C are fixed positive constants. Hence, (2.7) cannot hold for large $\lambda\,\mu$, a contradiction. A similar contradiction can be reached for the case when $v(1/2)>\rho_1$.

Case 2. $u(1/2)\leq\rho_2$ and $v(1/2)\leq\rho_1$. Then $\beta_2<u\leq\rho_2$ and $\beta_1<v\leq\rho_1$ on $J_1=[1/2,3/4]$. Then, by the mean value theorem, there exist $C_1,C_2\in J_1$ such that $|u'(C_2)|\leq 4\rho_2$ and $|v'(C_1)|\leq 4\rho_1$. Since $-(r^{N-1}\,u')'\geq 0$ on $[1/2,3/4)$, it follows that

$$-\,r^{N-1}\,u'(r) \leq -\,C_2^{N-1}\,u'(C_2) \quad \text{on} \quad J_2=[1/2,C_2),$$

and so

$$|u'(r)| \leq \frac{C_2^{N-1}}{r^{N-1}}\,|u'(C_2)| \leq \frac{1}{(1/2)^{N-1}}\left(\frac{3}{4}\right)^{N-1} 4\,\rho_2 \quad \text{on} \quad J_2.$$

Similarly, $|v'(r)|\leq 4(3/2)^{N-1}\,\rho_1$ on $J_3=[1/2,C_1)$. Hence, there exists $r_0\in[1/2,3/4)$ such that

$$|u'(r_0)|\leq\tilde{C},\ \ |v'(r_0)|\leq\tilde{C},$$

where $\tilde{C}=4\,(3/2)^{N-1}\,\max(\rho_2,\rho_1)$. Now define

$$E(r)=u'(r)v'(r)+\lambda\,F(v(r))+\mu\,G(u(r)).$$

Then

$$E'(r) = -\,\frac{2(N-1)}{r}\,u'v' \le 0$$

and hence $E \ge 0$ on $[0,1]$ since $E(1) = u'(1)v'(1) \ge 0$. But

$$E(r_0) \le \widetilde{C}^2 + \lambda\, k_1 + \mu\, k_2, \tag{2.8}$$

where $k_1 = F(\rho_1)$ and $k_2 = G(\rho_2)$. But $F(\rho_1) < 0$ and $G(\rho_2) < 0$. Hence (2.8) implies that $E(r_0) < 0$ for $\lambda\,\mu$ large, which is a contradiction. Thus Lemma 2.2 is proven. □

Proof of Theorem 1.1. Assume that $\lambda\,\mu$ is large enough so that both Lemmas 2.1 and 2.2 hold. First we take the case when $u(3/4) \le \beta_2$. Then

$$-(r^{N-1}v')' = \mu\, r^{N-1}\, g(u) \le 0 \quad \text{on} \quad J_3 = \left(\frac{3}{4},\, 1\right),$$
$$v\left(\frac{3}{4}\right) \le C, \quad v(1) = 0.$$

Thus by comparison arguments $v(r) \le \tilde{w}(r)$, where $\tilde{w}$ is the solution of

$$-(r^{N-1}\,\tilde{w}')' = 0 \quad \text{on} \quad J_3 = \left(\frac{3}{4},\, 1\right)$$
$$\tilde{w}\left(\frac{3}{4}\right) = C, \quad \tilde{w}(1) = 0.$$

But $\tilde{w}(r) = C/(\int_{3/4}^1 s^{1-N})\int_r^1 s^{N-1}\,ds$ decreases from C to 0 on $[3/4,1]$ and hence there exists $r_1 \in (3/4,1)$ (which is independent of $\lambda\,\mu$) such that $\tilde{w}(r_1) = \beta_1/2$. (Here we assume without loss of generality that $\beta_1/2 < C$. If not, we can choose r_1 such that $\tilde{w}(r_1) = \beta_1/N_0$ where N_0 is large enough so that $\beta_1/N_0 < C$.) Hence, $v(r_1) \le \beta_1/2$ and

$$\begin{aligned} -(r^{N-1}(\beta_2 - u)')' &= -\,\lambda\, r^{N-1} f(v) \ge -\,\lambda\, r^{N-1} f\left(\frac{\beta_1}{2}\right) \\ &\ge \lambda\left[-\,f\left(\frac{\beta_1}{2}\right)\right] r^{N-1}\,\frac{(\beta_2 - u)}{\beta_2} \quad \text{on} \quad J_4 = (r_1, 1). \end{aligned}$$

Since $\beta_2 - u > 0$ on $\bar{J}_4$, we have (see Appendix B)

$$\frac{\lambda\widetilde{K_1}}{\beta_2} \leq \lambda_1(J_4), \tag{2.9}$$

where $\widetilde{K_1} = -f(\beta_1/2)$ and $\lambda_1(J_4)$ is the principal eigenvalue of (B.2) with $(a,b) = J_4$.

By comparison arguments $v(r) \leq \tilde{w}(r)$, where $\tilde{w}(r)$ is as before, and in particular, there exists r_2 (independent of $\lambda\,\mu$) $\in (3/4, 1)$ such that $v(r_2) < \beta_1/2$. (Again, without loss of generality, we assume $\beta_1/2 < C$.) Then

$$\begin{gathered} -(r^{N-1}\,u')' = \lambda\, r^{N-1}\, f(v) \leq 0 \quad \text{on} \quad J_5 = (r_2, 1) \\ u(r_2) \leq C, \quad u(1) = 0. \end{gathered}$$

Hence, by comparison arguments, we obtain

$$u(r) \leq w_1(r) = \frac{C}{\int_{r_2}^1 s^{1-N}\,ds} \int_r^1 s^{1-N}\,ds$$

which satisfies

$$\begin{gathered} -(r^{N-1}\,w_1')' = 0 \quad \text{on} \quad J_5 \\ w_1(r_2) = C, \quad w_1(1) = 0. \end{gathered}$$

Arguing as before there exists r_3, independent of $\lambda\,\mu$, $\in (r_2, 1)$ such that $u(r_3) \leq w_1(r_3) \leq \beta_2/2 < C$. Hence,

$$\begin{aligned} -(r^{N-1}(\beta_1 - v)')' &= -\,\mu\, r^{N-1}\, g(u) \geq -\,\mu\, r^{N-1}\, g\left(\frac{\beta_2}{2}\right) \\ &\geq \mu\left[-\,g\left(\frac{\beta_2}{2}\right)\right]\frac{(\beta_1 - v)}{\beta_1} \quad \text{on} \quad J_6 = (r_3, 1). \end{aligned}$$

Since $\beta_1 - v > 0$ on $\bar{J}_6$, it follows that (see Appendix B)

$$\frac{\mu\widetilde{K_2}}{\beta_1} \leq \lambda_1(J_6), \tag{2.10}$$

where $\widetilde{K_2} = -g(\beta_2/2)$ and $\lambda_1(J_6)$ is the principal eigenvalue of (B.2) with $(a,b) = J_6$. Hence, combining (2.9) and (2.10), we have

$$\frac{\lambda\,\mu\widetilde{K_1}\widetilde{K_2}}{\beta_1\beta_2} \leq \lambda_1(J_4)\lambda_1(J_6), \tag{2.11}$$

a contradiction for $\lambda\,\mu$ large.

A similar contradiction can be reached for the case $v(3/4) \leq \beta_1$. Hence, Theorem 1.1 is proven.

Appendix

A. Consider the system

$$\begin{aligned} -\Delta u &= \lambda\, f(v); \quad x \in \Omega \\ -\Delta v &= \mu\, g(u); \quad x \in \Omega \\ u = 0 &= v; \quad x \in \partial\Omega, \end{aligned} \tag{A.1}$$

where Ω is a smooth bounded region in R^N, $\partial\Omega$ is its boundary and λ, μ are nonnegative parameters. Let $f, g : [0,\infty) \to R$ be continuous, and assume that there exist $\sigma_1 > 0$, $\sigma_2 > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$f(v) \geq \sigma_1 v + \varepsilon_1, \quad \text{for all} \quad v \in [0,\infty) \tag{A.2}$$

and

$$g(u) \geq \sigma_2 u + \varepsilon_2, \quad \text{for all} \quad u \in [0,\infty). \tag{A.3}$$

Then we prove:

Theorem A. *Let* (A.2)–(A.3) *hold. Then the system* (A.1) *has no positive solutions if* $\lambda\,\mu > \lambda_1^2/(\sigma_1\sigma_2)$ *where* λ_1 *is the first eigenvalue of the* $-\Delta$ *with Dirichlet boundary conditions.*

Proof. Multiplying the first equation in (A.1) by a positive eigenfunction, say ϕ, corresponding to λ_1 and using (A.2), we obtain

$$-\int_\Omega \Delta u\,\phi\,dx \geq \int_\Omega \lambda\,(\sigma_1 v + \varepsilon_1)\,\phi\,dx.$$

That is,

$$\int_\Omega u\,\lambda_1\,\phi\,dx \geq \int_\Omega \lambda\,(\sigma_1 v + \varepsilon_1)\,\phi\,dx. \tag{A.4}$$

Similarly using the second equation in (A.1) and (A.3), we obtain

$$\int_\Omega v\,\lambda_1\,\phi\,dx \geq \int_\Omega \mu(\sigma_2 u + \varepsilon_2)\,\phi\,dx. \tag{A.5}$$

Combining (A.4) and (A.5), we obtain

$$\int_\Omega \left[\lambda_1 - (\lambda\,\mu)\,\frac{\sigma_1\sigma_2}{\lambda_1}\right] v\,\phi\,dx \geq \int_\Omega \mu\left[\lambda\,\frac{\sigma_2\varepsilon_1}{\lambda_1} + \epsilon_2\right]\phi\,dx. \tag{A.6}$$

Inequality (A.6) clearly leads to a contradiction if $\lambda\,\mu > \lambda_1^2/(\sigma_1\sigma_2)$. Hence the result.

Remark. Note that in the case when (A.2)–(A.3) is satisfied with $\varepsilon_1 = \varepsilon_2 = 0$, (A.6) gives a contradiction if $\lambda\,\mu > \lambda_1^2/(\sigma_1\sigma_2)$. Hence this nonexistence result holds in this case as well.

B. Assume that there exists $z \geq 0$, $z \not\equiv 0$, on $\bar{I}$ where $I = (a,b)$ and a constant σ such that

$$-(r^{N-1}\,z')' \geq \sigma\, r^{N-1}\,z; \quad r \in I. \tag{B.1}$$

Let $\lambda_1 = \lambda_1(I) > 0$ denote the principal eigenvalue of

$$\begin{gathered} -(r^{N-1}\,\phi')' = \lambda\, r^{N-1}\,\phi; \quad r \in (a,b) \\ \phi(a) = 0 = \phi(b), \end{gathered} \tag{B.2}$$

where $0 < a < b \leq 1$.

Then we prove:

Theorem B. *Let* (B.1) *hold. Then* $\sigma \leq \lambda_1(I)$.

Proof. Multiplying (B.1) by $\phi(> 0)$, an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$, and integrating by parts (twice) we obtain

$$\int_a^b [\sigma - \lambda_1(I)]\, r^{N-1} z\,\phi\,dr \leq b^{N-1}\,\phi'(b)z(b) - a^{N-1}\,\phi'(a)z(a). \tag{B.3}$$

But $\phi'(b) < 0$ and $\phi'(a) > 0$. Hence the right-hand side of (B.3) ≤ 0, and thus $\sigma \leq \lambda_1(I)$.

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Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762
E-mail address: dang@ra.msstate.edu

Department of Mathematics, School of Science, Penn State Erie, The Behrend, Erie, PA 16563
E-mail address: sxo12@psu.edu

Department of Mathematics, Georgia Southern University, Statesboro, GA 30460-8093
E-mail address: shivaji@ra.msstate.edu