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FULL ELASTICITY IN ATOMIC MONOIDS AND INTEGRAL DOMAINS

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ABSTRACT. Let M be a commutative cancellative atomic monoid and M^* its set of nonunits. Let $\rho(x)$ denote the elasticity of factorization of $x \in M^*$, $\mathcal{R}(M) = \{\rho(x) \mid x \in M^*\}$ the set of elasticities of elements of M, and $\rho(M) = \sup \mathcal{R}(M)$ the elasticity of M. We say M is *fully elastic* if $\mathcal{R}(M) = \mathbf{Q} \cap [1, \rho(M)]$. We call an atomic integral domain D fully elastic if its multiplicative monoid, denoted $D^\bullet$, is fully elastic. We examine the full elasticity property in the context of Krull monoids with finite divisor class groups, numerical monoids and certain integral domains. For every real number $\alpha \geq 1$, we construct a fully elastic Dedekind domain D with $\rho(D) = \alpha$. In particular, while we show that noncyclic numerical monoids are never fully elastic, we do verify that several large classes of Krull monoids, and hence certain Krull domains, are fully elastic.

1. Introduction and definitions. Let M be a commutative cancellative monoid with M^* its set of nonunits and $\mathcal{A}(M)$ its set of irreducibles (or atoms). We suppose M is *atomic* (i.e., every element of M^* is a sum of atoms). Much recent literature has been devoted to the study of monoids in which elements fail to factor uniquely. In particular, a central topic of this work has been the *elasticity* of elements of M, which measures their failure to factor uniquely. While much is known about the supremum of the set of elasticities, we study here the complete set of elasticities in several important classes of monoids and integral domains.

We begin with some definitions and notations. For $x \in M^*$, define

$$\mathcal{L}(x) = \{n \mid x = \alpha_1, \dots , \alpha_n \text{ with each } \alpha_i \in \mathcal{A}(M)\}$$

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to be the set of lengths of factorizations of x into irreducibles and

$$\mathcal{L}(M) = \{\, \mathcal{L}(x) \mid x \in M^* \}$$

to be the set of lengths of M. Define

$$L(x) = \sup \mathcal{L}(x), \quad l(x) = \inf \mathcal{L}(x), \quad \text{and} \quad \rho(x) = \frac{L(x)}{l(x)}$$

to be their quotient. $\rho(x)$ is called the *elasticity* of x. Moreover, set

$$\mathcal{R}(M) = \{\rho(x) \mid x \in M^*\}$$

to be the set of elasticities of nonunits in M, and $\rho(M) = \sup \mathcal{R}(M)$ to be the supremum of this set. $\rho(M)$ is called the elasticity of M. The notion of elasticity was introduced by Valenza in [**17**] in the context of rings of integers of algebraic number fields. A good general reference on elasticity of factorizations is [**1**].

By a well-known result of Geroldinger [**13**], if M is a Krull monoid with torsion divisor class group and x in M^*, then $\mathcal{L}(x)$ is an *almost arithmetical multiprogression* (more information on Krull monoids can be found in [**8**, Section 2] and an extended study of almost arithmetic multiprogressions in [**10**]). The following surprising result shows that Geroldinger's structure theorem for sets of lengths cannot be extended to Krull monoids whose divisor class groups are not torsion.

Theorem 1.1 (Kainrath [**16**]). *Let M be a Krull monoid with infinite divisor class group $\mathfrak{C}(M)$ such that every divisor class of $\mathfrak{C}(M)$ contains a prime divisor of M. If S is any finite subset of $\mathbf{N} - \{1\}$, then $S \in \mathcal{L}(M)$.*

For M as described in Theorem 1.1, Kainrath's result implies that $\mathcal{R}(M) = \mathbf{Q} \cap [1, \infty)$ and raises an interesting question. For an arbitrary commutative cancellative atomic monoid M, is it true that $\mathcal{R}(M) = \mathbf{Q} \cap [1, \rho(M)]$ (or $\mathcal{R}(M) = \mathbf{Q} \cap [1, \infty)$ in the case where $\rho(M) = \infty$)? We will show that the answer to this question is "no" and hence make the following definition.

Definition 1.2. Let M be an atomic commutative cancellative monoid. If $\rho(M) < \infty$, then M is *fully elastic* if

$$\mathcal{R}(M) = \mathbf{Q} \cap [1, \rho(M)].$$

If $\rho(M) = \infty$, then M is fully elastic if $\mathcal{R}(M) = \mathbf{Q} \cap [1, \infty)$. An atomic integral domain D is fully elastic if its multiplicative monoid of nonzero elements, denoted $D^\bullet$, is fully elastic.

Following Zaks in [**18**], we say the atomic monoid M is a *half-factorial monoid* (HFM) if, for all nonunits $x \in M$, every irreducible factorization of x has the same length. Note that an HFM is trivially fully elastic. By the definition, if $\rho(M)$ is rational, then a necessary condition for M to be fully elastic is that the elasticity of M is accepted (i.e., there exists a nonunit x in M with $\rho(x) = \rho(M)$).

Example 1.3. The existing literature can be used to provide examples of some integral domains which are fully elastic and some which are not. For instance, consider the domains $D_1 = \mathbf{Z}[\sqrt{8}]$ and $D_2 = \mathbf{Z}[3\imath]$ where $\imath = \sqrt{-1}$. By [**15**, Examples 3, 4], the elasticity of each domain is $3/2$ and $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ both consist of all sets of the form $\{a, a+1, \ldots, b\}$ where a and b are in $\mathbf{N}$ with $a \leq b \leq (3/2)a$. Clearly both D_1 and D_2 are fully elastic.

On the other hand, while a large class of Krull domains have accepted elasticity, many do not. Let D be a Krull domain with divisor class group $\mathbf{Z}$ and set $S = \{-m, -1, s_1, s_2, \ldots\}$ of divisor classes which contain height-one prime ideals where infinitely many of the s_i's are congruent to 1 modulo $m \geq 2$. By [**6**, Example 2.4], $\rho(D) = m$ but $\rho(x) < m$ for every $x \in D$. Clearly D cannot be fully elastic.

We split our work into two sections. In Section 2, we show that any numerical monoid S which requires more than one generator is not fully elastic. We also show that for such an S, the set $\mathcal{R}(S)$ has exactly one limit point. We then use these results to construct a class of integral domains which are not fully elastic. In Section 3, we consider the full elasticity property for certain Krull monoids. We construct an example of an integral domain which is fully elastic but not integrally closed. Moreover, for each real number $\alpha \geq 1$, we use block monoids to mimic a construction of Anderson and Anderson in [**1**] to produce a Dedekind domain D with $\rho(D) = \alpha$ which is fully elastic. We close with a sequence of results which produces a large class of Krull monoids which are fully elastic. This, in turn, allows us to conclude that certain

Krull domains and certain rings of algebraic integers are fully elastic. Finally, we list two open problems which we believe are of great interest.

Let M be an atomic commutative cancellative monoid. We will use the following basic facts about elasticity freely throughout the remainder of this paper.

(i) If $x \in M^*$, then $1 \le \rho(x) \le \rho(M) \le \infty$.

(ii) If M is a bounded factorization monoid (or BFM) (i.e., the set $\mathcal{L}(x)$ is bounded above for all $x \in M^*$), then $\rho(x)$ is rational for all $x \in M^*$.

(iii) If M is finitely generated, then $\rho(M) = (m/n) \in \mathcal{Q}$ and has accepted elasticity [**2**, Theorem 7].

2. Full elasticity in numerical monoids. Let $a_1, \dots, a_t$ be positive integers. We define the *numerical monoid generated by* $a_1, \dots, a_t$ to be

$$S = \langle a_1, \dots, a_t \rangle = \{x_1 a_1 + \dots + x_t a_t \mid x_1, \dots, x_t \in \mathbf{N}_0\},$$

which is a submonoid of $\mathbf{N}_0$. Every numerical monoid S has a minimal set of generators, which are precisely the atoms of S. S is also clearly commutative, cancellative, and atomic. For more information on numerical monoids, see [**11**]. Our first result gives the elasticity of an arbitrary numerical monoid.

Theorem 2.1. *Let $S = \langle a_1, \dots, a_t \rangle$ be a numerical monoid, where $a_1 < a_2 < \dots < a_t$ is a minimal set of generators for S. Then $\rho(S) = a_t/a_1$.*

Proof. Let $n \in S$ and suppose $n = x_1 a_1 + \dots + x_t a_t$. Then

$$\frac{n}{a_t} = \frac{a_1}{a_t} x_1 + \dots + \frac{a_t}{a_t} x_t \le x_1 + \dots + x_t \le \frac{a_1}{a_1} x_1 + \dots + \frac{a_t}{a_1} x_t = \frac{n}{a_1}.$$

Thus $L(n) \le n/a_1$ and $l(n) \ge n/a_t$ for all $n \in S$, from which $\rho(S) \le a_t/a_1$. Also, $\rho(S) \ge \rho(a_1 a_t) = a_t/a_1$, so we have equality. $\square$

Note that if $S = \langle a \rangle$ is generated by a single element, then S is a factorial monoid so S is trivially fully elastic. In the results that follow, we will assume the minimal number of generators of S is $t \geq 2$.

Theorem 2.2. *Let $S = \langle a_1, \dots, a_t \rangle$ be a numerical monoid, where $a_1, \dots, a_t \in \mathbf{N}$ minimally generate S and $t \geq 2$. Then S is not fully elastic.*

Proof. Suppose without loss of generality that $1 < a_1 < \cdots < a_t$. Let $n \in S$ with maximal length factorization $n = x_1a_1 + \cdots + x_ta_t$. If $x_i \geq a_{i-1}$ for any $i \in \{2, \dots, t\}$, then

$$n = x_1a_1 + \cdots + (x_{i-1} + a_i)a_{i-1} + (x_i - a_{i-1})a_i + \cdots + x_ta_t$$

is a factorization with longer length. Thus $x_2 < a_1, \dots, x_t < a_{t-1}$ and $x_2a_2 + \cdots + x_ta_t < a_1a_2 + \cdots + a_{t-1}a_t$. Let $s = a_1a_2 + \cdots + a_{t-1}a_t$. Then

$$L(n) = x_1 + \cdots + x_t \geq x_1 = \frac{n - (x_2a_2 + \cdots + x_ta_t)}{a_1} > \frac{n - s}{a_1}.$$

Now suppose $n = y_1a_1 + \cdots + y_ta_t$ is a factorization of minimal length. Then, by a parallel argument, we have $y_1 < a_2, \dots, y_{t-1} < a_t$, and so $y_1a_1 + \cdots + y_{t-1}a_{t-1} < s$. Thus,

$$y_1 + \cdots + y_{t-1} \leq \frac{a_1}{a_1}\, y_1 + \cdots + \frac{a_{t-1}}{a_1}\, y_{t-1} = \frac{y_1a_1 + \cdots + y_{t-1}a_{t-1}}{a_1} < \frac{s}{a_1}.$$

Also,

$$y_t = \frac{n - (y_1a_1 + \cdots + y_{t-1}a_{t-1})}{a_t} \leq \frac{n}{a_t},$$

and, combining this with the previous result, we have

$$l(n) = y_1 + \cdots + y_t < \frac{s}{a_1} + \frac{n}{a_t} = \frac{na_1 + sa_t}{a_1a_t}.$$

Hence,

$$\rho(n) = \frac{L(n)}{l(n)} > \frac{na_t - sa_t}{na_1 + sa_t}.$$

Let N be an integer greater than $(2sa_t)/(a_t - a_1)$ and define $m = (Na_t - sa_t)/(Na_1 + sa_t)$. Note by the choice of N that $m > 1$. If $n > N$ then

$$\rho(n) > \frac{na_t - sa_t}{na_1 + sa_t} > \frac{Na_t - sa_t}{Na_1 + sa_t} = m$$

since $\{(na_t - sa_t)/(na_1 + sa_t)\}_{n \in \mathbf{N}}$ is an increasing sequence. Thus there are at most N elements of S which have elasticity m or less. Since there are infinitely many rationals in $[1, m]$, this implies S is not fully elastic. □

The proof of Theorem 2.2 can be used to prove a result which is of its own interest.

Corollary 2.3. *Let $a_1 < a_2 < \cdots < a_t$ be a minimal set of generators for the numerical monoid $S = \langle a_1, \ldots, a_t \rangle$, where $t \geq 2$. Then the only limit point of $\mathcal{R}(S)$ is $\frac{a_t}{a_1}$.*

Proof. First, if $n = k(a_1 a_t) + a_1$ for $k \in \mathbf{N}_0$, then $\rho(n) = \frac{L(n)}{l(n)} = \frac{ka_t + 1}{ka_1 + 1}$. It follows that $\rho(n) < a_t/a_1$ for all $k \in \mathbf{N}$ and $\lim_{k\to\infty} \rho(n) = a_t/a_1$. Thus, a_t/a_1 is a limit point of the set $\mathcal{R}(S)$.

We now show a_t/a_1 is the only limit point of this set. Let $r \in [1, (a_t/a_1))$, let $s = a_1 a_2 + \cdots + a_{t-1} a_t$ and take N to be an integer greater than $((r+1)sa_t)/(a_t - ra_1)$, which is positive by the restrictions on r. By the proof of Theorem 2.2, for all $n > N$,

$$\rho(n) > \frac{Na_t - sa_t}{Na_1 + sa_t}.$$

The reader can verify that

$$N > ((r+1)sa_t)/(a_t - ra_1)$$

implies

$$(Na_t - sa_t)/(Na_1 + sa_t) > r$$

and so $\rho(n) > r$. Thus there are at most N elements of S which have elasticity r or less. Since there are a finite number of elasticities less than r there can be no limit points less than r. Since this is true of any $r \in [1, (a_t)/(a_1))$, there are no limit points other than $(a_t)/(a_1)$. □

We summarize our results concerning numerical monoids in the following corollary, whose proof is now obvious.

Corollary 2.4. *Let $S = \langle a_1, \dots, a_t \rangle$ be a numerical monoid, where $a_1, \dots, a_t \in \mathbf{N}$ minimally generate S. The following statements are equivalent*:

(a) *S is a factorial monoid.*

(b) *S is a half-factorial monoid.*

(c) *S is fully elastic.*

(d) *S is cyclic (i.e., $S = \langle a_1 \rangle$).*

Example 2.5. Theorem 2.2 can also be used to construct a class of integral domains which are not fully elastic. Let K be any field, $n \geq 2$ a positive integer, and consider

$$D = K[[X^n, X^{n+1}, \dots, X^{2n-1}]] = K[[X; S]]$$

where S is the numerical monoid $\langle n, n+1, \dots, 2n-1 \rangle$. If $f(X) \in D$, then let $\operatorname{ord}(f(X))$ represent the smallest nonnegative integer k such that the coefficient of X^k in $f(X)$ is nonzero. It is relatively easy to argue that

(i) each nonzero element with $\operatorname{ord}(f(X)) = 0$ is a unit of D,

(ii) each element with $\operatorname{ord}(f(X)) = n, n+1, \dots, 2n-1$ is irreducible in D and

(iii) each element with $\operatorname{ord}(f(X)) > 2n-1$ is not irreducible.

Let $g(X)$ be a nonunit of D. Then in the UFD $K[[X]]$, $g(X) = X^m \cdot u_1(X)$ where $m \geq n$ and $u_1(X)$ is a unit of $K[[X]]$. If $g(X) = u_2(X) \cdot f_1(X) \dots f_t(X)$ where the $f_i(X)$'s are irreducible in D and $u_2(X)$ is a unit of D, then $m = \sum_{i=1}^{t} \operatorname{ord}(f_i(X))$. By regrouping, this can be rewritten as $m = \sum_{j=0}^{n-1} a_i \cdot (n+i)$ where a_i represents the number of elements $f_j(X)$ with $\operatorname{ord}(f_j(X)) = n+i$. Hence, $m = \sum_{j=0}^{n-1} a_i \cdot (n+i)$ represents an irreducible factorization of m in the numerical monoid S with the same length as the given factorization of $g(X)$ in D. Thus $\mathcal{L}(g(X)) \subseteq \mathcal{L}(m)$. On the other hand, let $m = \sum_{j=0}^{n-1} c_i \cdot (n+i)$ be an irreducible factorization of m in S and $\varpi_0(X), \varpi_1(X), \dots, \varpi_{n-1}(X)$

any units of D. Assume that s is the first index with $c_s \neq 0$. Since $g(X) = X^m \cdot u_1(X)$ in $K[[X]]$, we can write $g(X)$ in D as

$$g(X) = (X^{n+s}u_1(X))(X^{n+s}\varpi_s(X))^{c_s-1}(X^{n+s+1}\varpi_{s+1}(X))^{c_{s+1}}\cdots \\ (X^{2n-1}\varpi_{n-1}(X))^{c_{n-1}} \cdot (\varpi_s^{-1}(X))^{c_s-1}\prod_{j=s+1}^{n-1}(\varpi_j^{-1}(X))^{c_j}.$$

Since $(\varpi_s^{-1}(X))^{c_s-1}\prod_{j=s+1}^{n-1}(\varpi_j^{-1}(X))^{c_j}$ is a unit in D, this irreducible factorization of $g(X)$ in D has the same length of the original factorization of m in S. Thus $\mathcal{L}(m) \subseteq \mathcal{L}(g(X))$ and we have shown that $\mathcal{L}(m) = \mathcal{L}(g(X))$. From this it easily follows that $\mathcal{L}(S) = \mathcal{L}(D)$ and $\mathcal{R}(S) = \mathcal{R}(D)$. Since S is not fully elastic, neither is D.

3. Full elasticity in Krull monoids. Throughout this section $G \cong \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_k}$ will denote a nontrivial finite abelian group of rank k with $1 < n_1|\cdots|n_k$. A *zero-sequence* of G is a sequence $\{g_1, \ldots, g_t\}$ of, not necessarily distinct, nonzero elements of G such that $\sum_{i=1}^t g_i = 0$. A zero-sequence is called *minimal* if it contains no proper zero-subsequence. The *length* of the zero-sequence $B = \{g_1, \ldots, g_t\}$, denoted by $|B|$, is defined to be t.

Define $\mathcal{B}(G)$ to be the set of zero-sequences of G, without regard to the ordering of the sequence elements. Under the operation

$$\{g_1, \ldots, g_n\} \cdot \{h_1, \ldots, h_m\} = \{g_1, \ldots, g_n, h_1, \ldots, h_m\}$$

$\mathcal{B}(G)$ forms a commutative atomic monoid called the *block monoid* of G. $\mathcal{B}(G)$ is a Krull monoid with divisor class group G. Let $\mathcal{U}(G)$ denote the subset of $\mathcal{B}(G)$ consisting of minimal zero-sequences of G. Then the elements of $\mathcal{U}(G)$ are precisely the irreducibles of $\mathcal{B}(G)$. For ease of notation, we write our blocks multiplicatively in the form $g_1^{x_1}\cdots g_t^{x_t}$, where the x_i are nonnegative integers and $g_1, \ldots, g_t$ are distinct group elements. If S is a nonempty subset of G, then define

$$\mathcal{B}(G, S) = \{g_1^{x_1}\cdots g_t^{x_t} \in \mathcal{B}(G) \mid g_i \in S \text{ for all } 1 \le i \le t\}.$$

Then $\mathcal{B}(G, S)$ is a submonoid of $\mathcal{B}(G)$ with atoms $\mathcal{A}(\mathcal{B}(G, S)) = \mathcal{B}(G, S) \cap \mathcal{U}(G)$. The importance of block monoids in our current discussion is reflected in the following result, which first appeared in [**13**] (an alternate proof can be found in [**8**, Lemma 3.2]).

Theorem 3.1 (Geroldinger [**13**]). *If M is a Krull monoid with divisor class group $\mathfrak{C}(M)$ such that S is the set of divisor classes in $\mathfrak{C}(M)$ which contain prime divisors of M, then $\mathcal{L}(M) = \mathcal{L}(\mathcal{B}(\mathfrak{C}(M), S))$. Thus, M is fully elastic if and only if $\mathcal{B}(\mathfrak{C}(M), S)$ is fully elastic.*

Theorem 3.1 is especially applicable to Krull and Dedekind domains, since their multiplicative monoids are Krull monoids.

The *Davenport constant* of G, denoted by $D(G)$, is defined to be the maximum length of an irreducible in $\mathcal{B}(G)$. It is easy to argue that $D(G) \leq |G|$ and if $G \cong \mathbf{Z}_n$ then $D(G) = n$. For $G \cong \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_k}$ we define

$$\mathcal{M}(G) = 1 + \sum_{i=1}^{k}(n_i - 1).$$

In general, we have $D(G) \geq \mathcal{M}(G)$, and $D(G) = \mathcal{M}(G)$ if G is a p-group, a group of rank less than 3, or $|G| < 96$. It is also known that $\rho(\mathcal{B}(G)) = (D(G))/2$. For a survey of known results concerning the Davenport constant and their relation to factorization theory, consult [**7**].

Let S be a nonempty subset of $G - \{0\}$. Following [**4**], we define $D_S(G)$ to be the maximum length of an irreducible in $\mathcal{B}(G, S)$. It is easy to verify that $D_S(G) \leq D(G)$ and $D_G(G) = D(G)$. We will subsequently use the next result and omit the simple proof.

Lemma 3.2. *Let G be an abelian group, H a subgroup of G and S a nonempty subset of H. Then $\mathcal{R}(\mathcal{B}(H, S)) \subseteq \mathcal{R}(\mathcal{B}(G))$.*

Example 3.3. Consider the following interesting contrast to Example 2.5. Let K be a field of characteristic 0. Then by the main result of [**5**], $\rho(K[X^2, X^3]) = \infty$. For a set S, let $\mathcal{P}_F(S)$ represent the set of all *finite* subsets of S. The irreducible polynomials in $K[X]$ can be broken into three classes.

(i) Those irreducibles $f(X)$ prime in $K[X]$ which remain prime in $K[X^2, X^3]$. We will denote such irreducibles with the notation $p_i(X)$.

(ii) The polynomial $f(X) = X$ and its associates.

(iii) Irreducible polynomials of the form $f(X) = f_0 + f_1X + f_2X^2 + \cdots + f_tX^t$ where both f_0 and f_1 are not zero. We will denote such irreducibles with the notation $g_i(X)$.

If $f(X)$ is a nonunit nonzero element of $K[X^2, X^3]$, then in the UFD $K[X]$ the polynomial $f(X)$ factors uniquely in the form

$$(*) \qquad f(X) = u \cdot X^k \cdot g_1(X) \cdots g_t(X) \cdot p_1(X) \cdots p_r(X)$$

where u is a unit of K and each $g_i(X)$ and $p_j(X)$ has constant term 1. When using the representation in $(*)$, for each $1 \leq i \leq t$ we will set

$$g_i(X) = g_{i,0} + g_{i,1}X + \cdots + g_{i,j_i}X^{j_i}.$$

If $f(X)$ is an irreducible element of $K[X^2, X^3]$, then $f(X)$ has exactly one of the following $(*)$ forms.

(i) $f(X) = u \cdot p_1(X)$ is a prime element of $K[X]$.

(ii) $f(X) = u \cdot X^2$ or $f(X) = u \cdot X^3$.

(iii) $f(X) = u \cdot g_1(X) \cdots g_t(X)$ where $S = \{g_{1,1}, g_{2,1}, \ldots, g_{t,1}\}$ is a minimal zero-sequence of the abelian group K.

(iv) $f(X) = u \cdot X^n \cdot g_1(X) \cdots g_t(X)$ where $n = 2$ or 3 and $S = \{g_{1,1}, g_{2,1}, \ldots, g_{t,1}\}$ is a zero-free sequence of the abelian group K.

We claim that

$$\mathcal{L}(K[X^2, X^3]) = \mathcal{P}_F(\mathbf{N} - 1) \cup \{\{1\}\}.$$

To see this, note that since $K[X^2, X^3]$ is bounded factorization domain we have $\mathcal{L}(K[X^2, X^3]) \subseteq \mathcal{P}_F(\mathbf{N} - 1) \cup \{\{1\}\}$. Let $S = \{n_1, \ldots, n_t\}$ be an element of $\mathcal{P}_F(\mathbf{N} - 1) \cup \{\{1\}\}$. By Theorem 1.1 there is a block $B = \{m_1, \ldots, m_r\}$ in $\mathcal{B}(\mathbf{Z})$ such that $\mathcal{L}(B) = S$. Let n_i be in S. Then there is a decomposition of B as $B_1 \cdots B_{n_i}$ where each B_j is irreducible in $\mathcal{B}(\mathbf{Z})$. Write $B_j = \{b_{j,1}, \ldots, b_{j,t_j}\}$ where each $b_{s,r}$ is a nonzero element of $\mathbf{Z}$. By our comments above,

$$f_j(X) = \prod_{c=1}^{t_j} (X + b_{j,c})$$

is an irreducible element of $K[X^2, X^3]$. If $f(X) = \prod_{d=1}^{n_i} f_d(X)$, then $n_i \in \mathcal{L}(f(X))$. Hence, $\mathcal{L}(f(X)) \supseteq S$ and equality easily follows. Thus, $K[X^2, X^3]$ is fully elastic and not root closed as a monoid nor integrally closed as an integral domain.

Example 3.4. For each rational $\beta \geq 1$, we construct an example of a fully elastic block monoid $\mathcal{B}(G, S)$ such that $\rho(\mathcal{B}(G, S)) = \beta$. Let $G = \mathbf{Z}_n \oplus \cdots \oplus \mathbf{Z}_n = (\mathbf{Z}_n)^k$, with n and $k \geq 2$. Let $e_1, \dots, e_k$ be the canonical basis vectors of G, let $e^* = (n-1)(e_1 + \cdots + e_k)$, and take $S = \{e_1, \dots, e_k, e^*\}$. It's easy to see that the irreducibles of $\mathcal{B}(G, S)$ are $\{e_1^n, \dots, e_k^n, (e^*)^n, e_1 \cdots e_k e^*\}$. If $n = k+1$, then all irreducibles have the same length. In this case, $\mathcal{B}(G, S)$ is an HFM and it's trivially fully elastic, so assume $n \neq k+1$. By [**9**, Corollary 1.11], $\rho(\mathcal{B}(G, S)) = \max\{n/(k+1), (k+1)/n\}$.

Suppose $k+1 > n$. Let $p, q \in \mathbf{N}$ with $1 < p/q \leq (k+1)/n$. Consider the block

$$B = (e_1 \cdots e_k e^*)^{\alpha n} (e_1^n)^{\beta},$$

where $\alpha, \beta \in \mathbf{N}_0$. Since the only lengths of irreducibles in $\mathcal{B}(G, S)$ are n and $k+1$, and this factorization of B clearly has the greatest possible number of irreducible factors of length $k+1$, it follows that $l(B) = \alpha n + \beta$. Also, the factorization $B = (e_1^n)^{\alpha+\beta} (e_2^n)^{\alpha} \cdots (e_k^n)^{\alpha} ((e^*)^n)^{\alpha}$ clearly has maximal length since all irreducible factors have minimal length. Thus, $L(B) = (k+1)\alpha + \beta$ and so

$$\rho(B) = \frac{(k+1)\alpha + \beta}{\alpha n + \beta}.$$

One readily verifies that, setting $\alpha = p - q$ and $\beta = (k+1)q - np$ (which are both nonnegative integers), yields

$$\rho(B) = \frac{(k+1)(p-q) + ((k+1)q - np)}{(p-q)n + ((k+1)q - np)} = \frac{(k+1-n)p}{(k+1-n)q} = \frac{p}{q}.$$

Thus, we have shown that every rational number in $[1, \rho(\mathcal{B}(G, S))]$ is obtained as an elasticity and $\mathcal{B}(G, S)$ is fully elastic. The argument for the case where $n > k+1$ is nearly identical.

Using the last example and a construction from [**1**], we can produce a fully elastic Dedekind domain of every possible elasticity.

Proposition 3.5. *If $\alpha > 1$ is any real number or $\alpha = \infty$, then there exists a fully elastic Dedekind domain D with $\rho(D) = \alpha$.*

Proof. If $\alpha = \infty$, then by Theorem 1.1 D can be chosen to be any Dedekind domain with class group $\mathbf{Z}$ such that every ideal class of D contains a prime ideal (such a D exists by [**14**, Corollary 1.6]).

If $1 \leq \alpha < \infty$, then the construction parallels the construction of [**2**, Theorem 3.2]. First, if α is rational, then let G_α and S_α be a group and subset from Example 3.4 such that $\rho(\mathcal{B}(G_\alpha, S_\alpha)) = \alpha$ and $\mathcal{B}(G_\alpha, S_\alpha)$ is fully elastic. Since $\langle S_\alpha \rangle = G_\alpha$, by [**14**, Corollary 1.5] there exists a Dedekind domain D with class group G_α whose prime ideals lie in the ideal classes S_α of G_α. By Theorem 3.1, $\rho(D) = \alpha$ and is fully elastic.

If α is irrational, then let $\{q_i\}_{i=1}^{\infty}$ be an increasing sequence of rationals greater than one which converge to α. Let $G_\alpha = \sum_{i=1}^{\infty} G_{q_i}$ and for each q_i, let $\imath_{q_i}$ be the projection which maps an element $g \in G_{q_i}$ to the element $\hat{g} \in G_\alpha$ which equals g in the q_i coordinate and zero elsewhere. Set $S_\alpha = \cup_{i \in \mathbf{N}} \{\imath_{q_i}(x) \mid x \in S_{q_i}\}$. Since each S_{q_i} generates G_{q_i}, it follows that $\langle S_\alpha \rangle = G_\alpha$. Again, by [**14**, Corollary 1.5] there exists a Dedekind domain D with class group G_α whose prime ideals lie in the ideal classes S_α of G_α and $\rho(D) = \rho(\mathcal{B}(G_\alpha, S_\alpha))$. That $\rho(D) = \alpha$ follows in a manner analogous to the proof of [**2**, Theorem 3.2, part (VIII)]. We complete the argument by asserting that $\mathcal{B}(G_\alpha, S_\alpha)$ is fully elastic. Choose a rational q' with $1 \leq q' \leq \alpha$. Let i be a positive integer such that $q_i > q'$. Then, in the block monoid $\mathcal{B}(G_{q_i}, S_{q_i})$, there is a block $B = g_1 \dots g_t$ with $\rho(B) = q'$ and $g_1, \dots, g_t$ not necessarily distinct elements from S_{q_i}. It clearly follows that $B^* = \imath_{q_i}(g_1) \cdots \imath_{q_i}(g_t)$ in $\mathcal{B}(G_\alpha, S_\alpha)$ has $\mathcal{L}(B^*) = \mathcal{L}(B)$. By Theorem 3.1, there is a nonunit $x \in D^\bullet$ with $\mathcal{L}(x) = \mathcal{L}(B^*)$, which yields that $\rho(x) = q'$. □

We first approach the general case by considering Krull monoids M where $\mathfrak{C}(M)$ is cyclic.

Lemma 3.6. *Let M be a Krull monoid with divisor class group $\mathfrak{C}(M) = \mathbf{Z}_n$ where $n \geq 2$ such that S is the set of divisor classes in $\mathfrak{C}(M)$ which contain prime divisors of M. If there is an element $g \in S$ of order n such that $-g \in S$, then M is fully elastic. In particular, if $S = \mathbf{Z}_n$, then M is fully elastic.*

Proof. By Theorem 3.1, we can consider the block monoid $\mathcal{B}(\mathbf{Z}_n, S)$. If $n = 2$ then $\mathcal{B}(\mathbf{Z}_n, S)$ is an HFM and hence is fully elastic. Now suppose $n > 2$. By [**4**, Proposition 3] and the remarks preceding it, $\rho(\mathcal{B}(\mathbf{Z}_n, S)) = (D_S(G))/2 = n/2$. Let $x \in \mathbf{Q} \cap [1, (n/2)]$. Then it suffices to show that $\rho(B) = x$ for some $B \in \mathcal{B}(\mathbf{Z}_n, S)$. Suppose $u \leq v$ and consider the block $B = (g^n)^u((-g)^n)^v$. The only possible irreducible divisors of B are g^n, $(-g)^n$, and $g \cdot (-g)$, which have lengths n or 2. Since the given factorization of B contains only maximal length irreducibles, it has minimal length and $l(B) = u + v$. Since the only irreducible of length 2 contains the element g, of which there are nu total in B, it follows that the factorization $B = (g \cdot (-g))^{nu}((-g)^n)^{v-u}$ has maximal length and $L(B) = (n-1)u + v$. Now let $p/q = x$ for $p, q \in \mathbf{N}$. Take $u = p - q$ and $v = (n-1)q - p$. That $u \leq v$ follows from $p/q \leq n/2$. With these choices of u and v we have

$$\rho(B) = \frac{(n-1)u + v}{u + v} = \frac{(n-1)(p-q) + ((n-1)q - p)}{(p-q) + ((n-1)q - p)} = \frac{p}{q} = x,$$

so $\mathcal{B}(\mathbf{Z}_n, S)$ is fully elastic.

To see that $\mathcal{B}(\mathbf{Z}_n)$ is fully elastic, take $S = \{1, n-1\}$. By the above result, $\mathcal{B}(\mathbf{Z}_n, S)$ is fully elastic, and by Lemma 3.2, $\mathcal{B}(\mathbf{Z}_n)$ is also. $\square$

The next lemma will allow us to prove a general result concerning the set S.

Lemma 3.7. *Let G be a finite abelian group and S a nonempty subset of G. Let $\alpha = g_1^{x_1} \cdots g_t^{x_t}$ be an irreducible in $\mathcal{B}(G, S)$ of length $D_S(G)$, where the g_i are all distinct. Suppose $-g_1, \ldots, -g_t \in S$ and $x_1 = |g_1| - 1$. Then $\mathcal{R}(\mathcal{B}(G, S)) = \mathcal{Q} \cap [1, (D_S(G))/2]$ so $\mathcal{B}(G, S)$ is fully elastic.*

Proof. First note that if $D_S(G) = |g_1|$ and $g_1^{|g_1|-1} g_2$ is an irreducible of length $D_S(G)$, then we must have $g_2 = -(g_1^{|g_1|-1}) = g_1$ so g_1 and g_2 are not distinct. Thus, the hypotheses imply that $D_S(G) > |g_1|$.

Let $\bar{\alpha} = (-g_1)^{x_1} \cdots (-g_t)^{x_t}$, $\beta = g_1^{|g_1|}$, and $\bar{\beta} = (-g_1)^{|g_1|}$. Then $\alpha, \bar{\alpha}, \beta$, and $\bar{\beta}$ are irreducible in $\mathcal{B}(G, S)$. Let u, v be nonnegative integers not both zero and consider the block

$$B = \alpha^u \bar{\alpha}^u \beta^v \bar{\beta}^v.$$

We claim the given factorization of B has minimal length. Suppose F is a factorization of B of length at most $2u+2v$. For all $\gamma \in \mathcal{B}(G,S)$, let $v_{\hat{g_1}}(\gamma)$ denote the total number of the elements g_1 and $-g_1$ in γ. Note that if γ is irreducible, then $v_{\hat{g_1}}(\gamma) \leq |g_1|$. Let δ be the number of irreducible factors γ in F with $v_{\hat{g_1}}(\gamma) < |g_1|$, and let σ be the number of such factors with $v_{\hat{g_1}}(\gamma) = |g_1|$. Then

$$\delta(|g_1|-1) + \sigma|g_1| \geq v_{\hat{g_1}}(B) = 2u(|g_1|-1) + 2v|g_1|.$$

Since the length of the factorization F is $\delta + \sigma \leq 2u+2v$, it follows that $\sigma \geq 2v$. Note that all factors counted by σ have length $|g_1|$, and those counted by δ have length at most $D_S(G)$. Thus,

$$\delta D_S(G) + \sigma|g_1| \geq |B| = 2uD_S(G) + 2v|g_1|.$$

Since $\delta+\sigma \leq 2u+2v$, it follows that $\delta(D_S(G)-|g_1|) \geq 2u(D_S(G)-|g_1|)$, whence $\delta \geq 2u$. Hence, $\delta+\sigma \geq 2u+2v$, and the given factorization of B is minimal as claimed (i.e., $l(B) = 2u+2v$).

The factorization $B = (g_1(-g_1))^{ux_1+v|g_1|}(g_2(-g_2))^{ux_2}\cdots(g_t(-g_t))^{ux_t}$ has maximal length since all factors have minimal length 2. Thus, $L(B) = u(x_1+\cdots+x_t) + v|g_1| = uD_S(G) + v|g_1|$.

Now let $p/q \in \mathcal{Q} \cap [|g_1|/2, (D_S(G))/2]$, and take $u = 2p - |g_1|q$ and $v = D_S(G)q - 2p$. Note that the restrictions on p/q ensure that u, v are nonnegative and not both zero. With this choice of u and v, we have

$$\begin{aligned}\rho(B) = \frac{uD_S(G)+v|g_1|}{2u+2v} &= \frac{(2p-|g_1|q)D_S(G) + (D_S(G)q-2p)|g_1|}{2(2p-|g_1|q)+2(D_S(G)q-2p)}\\ &= \frac{2(D_S(G)-|g_1|)p}{2(D_S(G)-|g_1|)q} = \frac{p}{q}.\end{aligned}$$

Thus, $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathbf{Q} \cap [|g_1|/2, (D_S(G))/2]$.

Now let $n = |g_1|$. Then $\langle g_1 \rangle \cong \mathbf{Z}_n$ is a subgroup of G. Using Lemmas 3.2 and 3.6, $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathcal{Q} \cap [1, n/2] = \mathcal{Q} \cap [1, |g_1|/2]$, and so $\mathcal{R}(\mathcal{B}(G,S)) \supseteq \mathcal{Q} \cap [1, (D_S(G)/2)]$. By [**4**, Proposition 3], $\rho(\mathcal{B}(G,S)) = (D_S(G))/2$ so $\mathcal{R}(\mathcal{B}(G,S)) = \mathcal{Q} \cap [1, \rho(\mathcal{B}(G,S))]$ and $\mathcal{B}(G,S)$ is fully elastic. □

Theorem 3.8 will imply that a large class of Krull monoids is fully elastic.

Theorem 3.8. *Let M be a Krull monoid with finite divisor class group $\mathfrak{C}(M)$ such that each divisor class of $\mathfrak{C}(M)$ contains a prime divisor of M. Then*

$$\mathcal{R}(M) \supseteq \mathcal{Q} \cap \Big[1, \frac{\mathcal{M}(\mathfrak{C}(M))}{2}\Big].$$

Proof. By Theorem 3.1, we can consider the block monoid $\mathcal{B}(\mathfrak{C}(M))$. For convenience, set $\mathcal{C}(M) = G$. Let $G = \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_k}$, where $1 < n_1 | \cdots | n_k$. Let

$$e_1 = (1, 0, \ldots, 0), \ldots, e_k = (0, 0, \ldots, 1)$$

be the standard basis elements of G, and let $e_* = (1, 1, \ldots, 1)$. Let

$$S = \{e_1, \ldots, e_k, e_*, -e_1, \ldots, -e_k, -e_*\}.$$

We next show $D_S(G) = \mathcal{M}(G)$.

First note that if G is a finite cyclic group then $D_S(G) = D(G) = \mathcal{M}(G)$, so suppose G has rank $k \geq 2$. Since the block $e_1^{n_1-1} \cdots e_k^{n_k-1} e_*$ has length $\mathcal{M}(G)$, we know $D_S(G) \geq \mathcal{M}(G)$. We show $D_S(G) \leq \mathcal{M}(G)$. Suppose B is an irreducible of length $D_S(G)$. Then B cannot contain both e and $-e$ for any $e \in S$. Without loss of generality, we can assume B contains e_* but not $-e_*$. So suppose $B = \hat{e_1}^{x_1} \cdots \hat{e_k}^{x_k} e_*^{x_*}$, where $\hat{e_i} \in \{e_i, -e_i\}$ for all $1 \leq i \leq k$, and $x_1, \ldots, x_k, x_* \in \mathbf{N}_0$.

Now let j be the smallest index such that $n_j = n_k$. If $\hat{e_i} = e_i$ for any $i \geq j$, then we can assume by some isomorphism that $\hat{e_k} = e_k$. It follows that $x_k = n_k - x_*$, and so

$$\begin{aligned}|B| &= x_1 + \cdots + x_k + x_* = x_1 + \cdots + x_{k-1} + (n_k - x_*) + x_* \\ &\leq (n_1 - 1) + \cdots + (n_k - 1) + 1 = \mathcal{M}(G).\end{aligned}$$

Now assume $\hat{e_i} = -e_i$ for all $i \geq j$. Then $x_i = x_*$ for all $i \geq j$. If $j = 1$ then we must have $x_1 = \cdots = x_k = x_* = 1$ since B is irreducible, so $|B| = k + 1 \leq \mathcal{M}(G)$. So suppose $j > 1$.

Since $x_k = x_*$,

$$|B| = x_1 + \cdots + x_{k-1} + 2x_* \leq (n_1 - 1) + \cdots + (n_{k-1} - 1) + 2x_*.$$

Suppose $|B| > \mathcal{M}(G)$. It follows from the last equation that $x_* > n_k/2 \geq n_{j-1}$. By the minimality of B, we must have $B = (-e_j)^{n_{j-1}} \cdots (-e_k)^{n_{j-1}} {e_*}^{n_{j-1}}$ since this is a zero-subsequence of B. Thus $|B| = (k-j+2)n_{j-1}$, and hence

$$\begin{aligned}\mathcal{M}(G) &= (n_1-1) + \cdots + (n_{j-1}-1) + (k-j+1)(n_k-1) + 1 \\ &\geq n_{j-1} + (k-j+1)(2n_{j-1}-1) \\ &= (k-j+2)n_{j-1} + (k-j+1)(n_{j-1}-1) \geq (k-j+2)n_{j-1} \\ &= |B|.\end{aligned}$$

This is a contradiction, so in all cases $D_S(G) = |B| \leq \mathcal{M}(G)$.

Now let $\alpha = e_1^{n_1-1} \cdots e_k^{n_k-1} e_*$. Then α is an irreducible in $\mathcal{B}(G,S)$ of length $D_S(G)$. By Lemma 3.7, $\mathcal{R}(\mathcal{B}(G)) \supseteq \mathcal{R}(\mathcal{B}(G,S)) = \mathcal{Q} \cap [1,(D_S(G))/2] = \mathcal{Q} \cap [1,(\mathcal{M}(G))/2]$. □

Our main result now follows directly from Lemmas 3.6, 3.7 and Theorem 3.8.

Theorem 3.9. *Let M be a Krull monoid with finite divisor class group $\mathfrak{C}(M)$ such that each divisor class of $\mathfrak{C}(M)$ contains a prime divisor of M. If*

(i) $D(\mathfrak{C}(M)) = \mathcal{M}(\mathfrak{C}(M))$, *or*

(ii) *there exists a maximal length irreducible in $\mathcal{B}(\mathfrak{C}(M))$ which contains $g^{|g|-1}$ for some $g \in \mathfrak{C}(M)$,*

then M is fully elastic.

Proof. By Theorem 3.1, we can consider the block monoid $\mathcal{B}(\mathfrak{C}(M))$. For (i), if $D(G) = \mathcal{M}(G)$, then it follows immediately from Theorem 3.8 that $\mathcal{B}(G)$ is fully elastic.

For (ii), suppose $G \cong \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_k}$, where $1 < n_1 | \cdots | n_k$. Suppose $\alpha \in \mathcal{A}(\mathcal{B}(G))$ has length $D(G)$ and contains $g^{|g|-1}$. If α contains $g^{|g|}$ then $D(G) = |g| \leq n_k$, and so $\mathcal{M}(G) = 1 + \sum_{i=1}^{k}(n_i - 1) \leq D(G) \leq n_k$. Thus, $\sum_{i=1}^{k-1}(n_i - 1) \leq 0$ so G is cyclic, and hence fully elastic by Lemma 3.6.

Now suppose α contains $g^{|g|-1}$ and no higher power of g. Write $\alpha = g^{|g|-1} g_2^{x_2} \cdots g_t^{x_t}$. Letting $S = G$ and applying Lemma 3.7, we have

$\mathcal{R}(\mathcal{B}(G)) = \mathcal{Q} \cap [1, (D_G(G))/2] = \mathcal{Q} \cap [1, (D(G))/2]$ so $\mathcal{B}(G)$ is fully elastic. □

Condition (ii) in Theorem 3.9 is satisfied by divisor class groups with "large exponent," see [**12**, Section 8]. We close our results by noting that Theorem 3.9 can be applied in several different ways.

Corollary 3.10. *Let M be a Krull monoid with divisor class group $\mathfrak{C}(M)$ such that each divisor class of $\mathfrak{C}(M)$ contains a prime divisor.*

1. *If* $\operatorname{rank}(\mathfrak{C}(M)) < 3$*, then M is fully elastic.*
2. *If* $|\mathfrak{C}(M)| = p^k$ *where p is a prime integer and $k \in \mathbf{N}$, then M is fully elastic.*
3. *If* $\mathfrak{C}(M) = G' \oplus \mathbf{Z}_{mn}$ *where G' is a direct summand with* $\exp(G')|m$ *and* $n \geq 4|G'| > 4(m-2)$*, then M is fully elastic.*

In particular, if D is a ring of integers in finite extension of $\mathbf{Q}$ *with class number p^k where p is prime, then D is fully elastic.*

Proof. 1 and 2 follow since groups of these types satisfy $D(G) = \mathcal{M}(G)$, see [**7**, Theorem 1.8]. For 3, groups of this form satisfy condition (ii) of Theorem 3.9 by [**12**, Theorem 8.2]. The remaining assertion follows directly from 1, 2 and 3 and Theorem 3.1. □

Our work in Section 3 clearly raises two interesting problems.

Problem 1. Let M be a Krull monoid with divisor class group $\mathfrak{C}(M)$ such that each divisor class of $\mathfrak{C}(M)$ contains a prime divisor. If $|\mathfrak{C}(M)| < \infty$, then must M be fully elastic?

Problem 2. Let M be a Krull monoid whose divisor class group $\mathfrak{C}(M)$ is finite and satisfies $D(\mathfrak{C}(M)) = \mathcal{M}(\mathfrak{C}(M))$. Must M be fully elastic?

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