ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

CO-LOCALIZATION, CO-SUPPORT AND LOCAL HOMOLOGY

ANDREW S. RICHARDSON

ABSTRACT. We propose a definition of co-support for modules over commutative, Noetherian rings that we hope will relate to the local homology functors of Greenlees and May in the same way ordinary support relates to local cohomology. By expressing local homology in terms of the Koszul complex, we prove some vanishing theorems involving this co-support. We also investigate the co-localization functor which gives rise to our definition of co-support. This functor treats Artinian-type constructions the same way the ordinary localization functor treats Noetherian-type constructions, and this duality extends to one between co-support and ordinary support.

Introduction. Given a module M over a commutative, Noetherian ring A and an ideal I of A, if we attempt to restrict the support of M to the variety $V(I)$, that amounts to taking the I-adic torsion of M. The right derived functors of this process are the local cohomology functors $H_I^{\bullet}$. The concept of support is thus part of local cohomology from the beginning. It also shows up in certain well-known vanishing results: If M is torsion to begin with, which is to say if $\operatorname{Supp} M \subset V(I)$, then all the higher local cohomology modules of M vanish. Furthermore, the local cohomology of any module M will always vanish past the dimension of the support of M.

In [**8**], Matlis defined the local homology functors to be the left derived functors of the I-adic completion functor. Since torsion and completion are dual, one expects these functors to live up to their name and behave in a manner dual to local cohomology. In particular, it is natural to expect there to be vanishing theorems for local homology dual to the ones that relate the local cohomology of a module to the module's support. Before we can prove or even state such theorems, however, we have to decide how to dualize the notion of support. That is the primary objective of this paper.

AMS *Mathematics Subject Classification.* Primary 13D07, 13E10.
Received by the editors on June 20, 2003, and in revised form on January 6, 2004.

To define the co-support of a module, we first create a co-localization functor; the co-support of a module will then be the set of primes at which the module's co-localization is nonzero. So, given a commutative ring A and a multiplicatively closed subset $S \subset A$, we would like to have a functor $S_{-1}(-)$ from the category of A-modules to the category of $S^{-1}A$-modules which is dual to the ordinary localization functor $S^{-1}(-)$. In particular, $S_{-1}(-)$ should preserve secondary representations and attached primes (the duals of primary decompositions and associated primes; see [**6**] or [**2**, Section 7.2]) and the co-localization of an Artinian A-module should be an Artinian $S^{-1}A$-module. Most importantly, from our point of view, this co-localization functor should define a sensible co-support. In particular, the co-support of a nonzero module should be nonempty, the co-support of an Artinian module M should be $V(\operatorname{ann} M)$, and the co-support of a finitely generated module should, like the ordinary support of an Artinian module, consist solely of maximal ideals.

Since $S^{-1}(-) = S^{-1}A \otimes -$, the obvious choice for $S_{-1}(-)$ would be $\operatorname{Hom}(S^{-1}A, -)$. Melkersson and Schenzel [**9**] tried this approach and were able to show that this definition works well when restricted to the class of Artinian modules, with the exception that it almost never takes an Artinian module to an Artinian module. However, this definition does not work at all well for non-Artinian modules. For example, if S is a multiplicatively closed set of integers which includes a nonunit, then $\operatorname{Hom}(S^{-1}\mathbf{Z}, \mathbf{Z}) = 0$, which says that the co-support of $\mathbf{Z}$, under this definition, is empty, which is definitely not what we want. Beyond our belief that nonzero modules should have nonempty co-support, this would violate our prospective vanishing theorem for local homology: For any prime integer p, $H_0^{(p)}(\mathbf{Z}) = \hat{\mathbf{Z}}_{(p)} \neq 0$, so we need the dimension of the co-support of $\mathbf{Z}$ to be at least 0, which is to say the co-support should be nonempty.

To motivate our alternative co-localization functor, we look at why the Melkersson-Schenzel version does not preserve Artinian modules; for simplicity, we consider the case of an Artinian module M over a complete, local ring A. Using the fact that M is reflexive relative to the Matlis duality functor $(-)^\vee$, one can show that $\operatorname{Hom}(S^{-1}A, M) \cong \big(S^{-1}(M^\vee)\big)^\vee$. Using this formula, one might try to "prove" that $\operatorname{Hom}(S^{-1}A, M)$ is Artinian by pointing out that $M^\vee$ is finitely generated, so $S^{-1}(M^\vee)$ is finitely generated, and thus its Matlis dual must be Artinian. However, $S^{-1}(M^{\vee})$ is only finitely generated as an $S^{-1}A$-module, so this argument would only work if we took the second Matlis dual with respect to the new ring $S^{-1}A$.

Therefore, we propose the following: For any A-module M, we let $S_{-1}M = D_{S^{-1}A}\left(S^{-1}D_A(M)\right)$, where D_B is the generalized Matlis duality functor for the ring B as defined in [**1**]. This co-localization functor will preserve Artinian modules when A is complete and $S^{-1}A$ is semi-local (but rarely otherwise, unfortunately), and it will also preserve secondary representations and attached primes, even for non-Artinian modules. More importantly, this co-localization gives a much better co-support for non-Artinian modules.

There is a price to pay for these improvements, however. For one thing, while there is a natural transformation $\mathrm{Hom}\,(S^{-1}A, -\,) \to \mathrm{id}$ dual to the familiar map $\mathrm{id} \to S^{-1}(-)$, there is no such natural transformation $S_{-1}(-) \to \mathrm{id}$. Also, and this may perhaps explain the previous problem, if $S = \{1\}$, one would expect $S_{-1}(-)$ to be the identity functor, but it is instead D_A^2. Finally, our co-localization is not transitive—that is, $(S_1)_{-1}\left((S_2)_{-1}(-)\right)$ and $(S_1 \cup S_2)_{-1}(-)$ need not be the same functor—but at least the co-support of a module is always closed under specialization.

We will generally assume that our rings are commutative and Noetherian. While Melkersson and Schenzel obtained their results for arbitrary commutative rings, they restricted their results to Artinian modules which, by [**15**, Theorem 3.2], can always be viewed over a Noetherian ring with little loss of information, so our Noetherian assumption is not a significant drawback. The assumption can also be omitted from those results which depend solely on the exactness and faithfulness of the Matlis duality functor, such as Lemma 2.1, Theorem 2.2 and Theorem 2.7, except part (vi).

In the first section of this paper we list some facts about the functors D_B which will be useful in later sections. In the second section we prove that our co-localization and co-support have properties dual to ordinary localization and support. We also show that our co-localization behaves well relative to some standard homological constructions; in particular, we show that Sharp's "shifted localization principle" for local cohomology modules [**14**, Theorem 3.7] can be expressed in terms of our co-localization functor. In the third section we show how the local homology modules relative to an ideal $I = (x_1, \dots, x_n)$ can be described using the Koszul homologies on powers of the sequence $\mathbf{x} = x_1, \dots, x_n$; namely, they fit into a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 H_{j+1}(\mathbf{x}^k; M) \longrightarrow H_j^I(M) \longrightarrow \varprojlim H_j(\mathbf{x}^k; M) \longrightarrow 0.$$

(For the definition and some basic properties of the functor $\varprojlim{}^1$, see [**19**, Section 3.5].) In the final section, we use this to show that the higher local homology modules vanish whenever the co-support of the module is contained in $V(I)$. We also show that $H_j^I(M) = 0$ for $j > \dim \operatorname{coSupp} M$ whenever M is Artinian or finitely generated. By reducing to the Artinian case, we can show that for a general module M, at least the $\varprojlim H_j(\mathbf{x}^k; M)$ part will vanish for $j > \dim \operatorname{coSupp} M$.

1. Generalized Matlis duality.

Let B be a commutative ring.

Definition. Let E_B be the injective hull of $\oplus B/\mathfrak{m}$, the sum running over all maximal ideals $\mathfrak{m}$ of B, and let D_B be the functor $\operatorname{Hom}(-, E_B)$.

This module E_B is the minimal injective cogenerator of the category of B-modules; that is, it is the smallest injective module with the property that, for every module M and nonzero $x \in M$, there is a homomorphism $\phi: M \to E_B$ with $\phi(x) \neq 0$. Two variations on this property are sometimes useful: If N is a submodule of M which does not contain x, then applying this property to $x + N \in M/N$, we can construct a $\phi: M \to E_B$ which vanishes on N but not on x; thus, maps from M to E_B can be used to separate points from submodules. Also, since we are generally assuming the ring B is Noetherian, we can split E_B into the direct sum of all $E(B/\mathfrak{m})$, and each of these modules is Artinian. So, by composing our ϕ with one of the projections, we get:

Lemma 1.1. *Assume that B is Noetherian. Then, given any B-module M, submodule $N \subset M$, and element $x \in M \setminus N$, there is an Artinian module Ξ and a homomorphism $\phi: M \to \Xi$ such that $\phi(x) \neq 0$ but $\phi(N) = 0$.*

One can think of this as dualizing the fact that any module is a direct limit of finitely generated modules; we will use this property later to reduce a question about general modules to a question about Artinian modules.

One can use the cogenerator property of E_B to prove the following facts about our duality functor, see [**2**, 10.2.2] for the local case; the proof of the general case is essentially the same.

Lemma 1.2. *Let M be a module over the ring B.*

(i) $D_B(M) = 0$ *if and only if* $M = 0$.

(ii) $\operatorname{ann} D_B(M) = \operatorname{ann} M$.

The following generalization of Matlis' original duality theorem was proven by Ooishi [**10**, Theorem 1.6]:

Proposition 1.3. *Assume B is semi-local and Noetherian.*

(i) *If M is a finitely generated B-module, then $D_B(M)$ is Artinian.*

(ii) *If M is an Artinian B-module then $D_B(M)$ is finitely generated over the completion of B.*

(iii) *If B is complete and M is either finitely generated or Artinian, then $M \cong D_B^2(M)$.*

We will also need the following homological facts, which follow from the adjoint isomorphism for Hom and $\otimes$ and from Sharp's isomorphism ([**14**, Lemma 2.7]; see also [**10**, Corollary 1.5]). In fact, most of this proposition will work for any functor of the form $\operatorname{Hom}(-, E)$ with E an injective module; the only exception is the "if" part of (i), which requires the faithfulness of Matlis duality.

Proposition 1.4. Assume B is Noetherian. Let M and N be B-modules and i any integer.

(i) *M is flat, respectively injective, if and only if $D_B(M)$ is injective, respectively flat.*

(ii) $D_B(\operatorname{Tor}_i(M, N)) \cong \operatorname{Ext}^i(M, D_B(N))$.

(iii) *If M is finitely generated, then*

$$D_B\left(\operatorname{Ext}^i(M,N)\right) \cong \operatorname{Tor}_i\left(M, D_B(N)\right).$$

We need the next lemma in order to prove a nonvanishing criterion for our co-localization functor. That criterion is sufficient but not necessary, which will show that the surjection in Lemma 1.5 need not be an isomorphism. (See [**11**, Lemma 1.2.13] for variations on this lemma which do involve isomorphisms.)

Lemma 1.5. *Let $\{f_\lambda : M \to M_\lambda | \lambda \in \Lambda\}$ be a collection of module homomorphisms with common domain. Then there is a surjection*

$$\frac{D_B(M)}{\sum_{\lambda\in\Lambda} \operatorname{im} D_B(f_\lambda)} \to D_B\Big(\bigcap\nolimits_{\lambda\in\Lambda} \ker f_\lambda\Big).$$

Proof. Let $K = \bigcap_\lambda \ker f_\lambda$ and $Q = M/K$. For each $\lambda \in \Lambda$, $K \subset \ker f_\lambda$ so that f_λ may be written as a composite $M \to Q \to M_\lambda$. This means that $D_B(f_\lambda)$ factors through $D_B(Q)$, and thus $\operatorname{im} D_B(f_\lambda)$ is contained in the image of $D_B(Q) \to D_B(M)$, which is the kernel of the surjection $D_B(M) \to D_B(K)$. $\square$

2. Co-localization. Let A be a commutative ring and S a multiplicatively closed subset of A.

Definition. For any A-module M, the *co-localization* of M relative to S is the $S^{-1}A$-module $S_{-1}M = D_{S^{-1}A}\left(S^{-1}D_A(M)\right)$. If $S = A \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} A$, we write ${}^{\mathfrak{p}}M$ for $S_{-1}M$.

Note that, as a composite of exact, additive functors, $S_{-1}(-)$ is exact and additive.

We start with a pair of simple results about the vanishing and nonvanishing of $S_{-1}(-)$:

Lemma 2.1. *Let M be an A-module.*

(i) *If $sM = 0$ for some $s \in S$ then $S_{-1}M = 0$.*

(ii) *If $\bigcap_{s\in S} sM \neq 0$, then $S_{-1}M \neq 0$.*

Proof. (i) follows from the analogous result for ordinary localization. For (ii), note that the kernel $(0 :_{D_A(M)} s)$ of the map $D_A(M) \xrightarrow{s} D_A(M)$ is the image of the map $D_A(M/sM) \to D_A(M)$. Lemma 1.5 gives a surjection

$$\frac{D_A(M)}{\sum_{s\in S}(0 :_{D_A(M)} s)} \longrightarrow D_A\Big(\bigcap\nolimits_{s\in S} sM\Big),$$

which means that $\cup(0 :_{D_A(M)} s) \neq D_A(M)$. That in turn means that $S^{-1}D_A(M) \neq 0$, so $S_{-1}M \neq 0$. $\square$

If M is Artinian, then by [**9**, Corollary 3.3], the submodule $\cap_{s\in S} sM$ is precisely the image of the natural map $\operatorname{Hom}(S^{-1}A, M) \to M$, so one may view (ii) as correcting for the lack of a natural map $S_{-1}M \to M$. Also, note that (ii) is not a necessary condition: Let $A = M = \mathbf{Z}$ and $S = \{1, p, p^2, \dots\}$ for some prime number p. Then $\cap_{n=0}^{\infty} p^n\mathbf{Z} = 0$, but $S^{-1}D_{\mathbf{Z}}(\mathbf{Z}) \cong \oplus_{q\neq p}\, \mathfrak{q}/\mathbf{Z}_{(q)}$, so $S_{-1}\mathbf{Z} \cong \prod_{q\neq p} \hat{\mathbf{Z}}_{(q)}$, which is nonzero.

We can use these results and the exactness of $S_{-1}(-)$ to show that our co-localization functor preserves secondary representations and attached primes:

Theorem 2.2. *Let M be an A-module.*

(i) *If M is $\mathfrak{p}$-secondary, then $S_{-1}M$ is either zero, if $S \cap \mathfrak{p} \neq$, or $S^{-1}\mathfrak{p}$-secondary, if $S \cap \mathfrak{p} = \varnothing$.*

(ii) *If M is representable then so is $S_{-1}M$ and* $\operatorname{Att} S_{-1}M = \{S^{-1}\mathfrak{p} | \mathfrak{p} \in \operatorname{Att} M$ *and* $S \cap \mathfrak{p} = \varnothing\}$.

Proof. (i) If $S \cap \mathfrak{p} \neq \varnothing$, then $s^n M = 0$ for some $s \in S$, so $S_{-1}M = 0$ by Lemma 2.1 (i). If $S \cap \mathfrak{p} = \varnothing$, then $sM = M$ for all $s \in S$, so $S_{-1}M \neq 0$ by Lemma (2.1) (ii). In this case, exactness of $S_{-1}(-)$ shows that for any $x/s \in S^{-1}A$, multiplication by x/s is either surjective on $S_{-1}M$ (if $x/s \notin S^{-1}\mathfrak{p}$) or nilpotent (if $x/s \in S^{-1}\mathfrak{p}$).

(ii) We need to show that if $M = T_1 + \cdots + T_n$ is an irredundant secondary representation, say with T_i a $\mathfrak{p}_i$-secondary submodule, then the $S_{-1}T_i$'s with $S \cap \mathfrak{p}_i = \varnothing$ give an irredundant representation of $S_{-1}M$. For any $X \subset \{1, \dots, n\}$, let $N_X = \sum_{i\in X} T_i$. Suppose that X and Y partition $\{1, \dots, n\}$, so that $M = N_X + N_Y$, and consider the exact sequence

$$\bigoplus_{i \in X} S_{-1}T_i \longrightarrow S_{-1}M \to S_{-1}\left(\frac{N_Y}{N_Y \cap N_X}\right);$$

this shows that the $S_{-1}T_i$, $i \in X$, give a representation of $S_{-1}M$ if and only if the module $S_{-1}(N_Y/N_X \cap N_Y)$ is zero. Now, co-localizing the surjection $\oplus_{i \in Y} T_i \to (N_Y/N_Y \cap N_X)$ shows that $S_{-1}(N_Y/N_X \cap N_Y)$ will be zero if $Y \subset \{i | S \cap \mathfrak{p}_i \neq \varnothing\}$, so that the $S_{-1}T_i$'s with $S \cap \mathfrak{p}_i = \varnothing$ do give a representation. On the other hand, for each $i \in Y$, $N_Y/(N_Y \cap N_X)$ surjects onto a $\mathfrak{p}_i$-secondary module, so $S_{-1}(N_Y/(N_Y \cap N_X))$ will be nonzero if Y includes some i with $S \cap \mathfrak{p}_i =$, which proves the minimality of our representation. □

We assume henceforth that our ring A is Noetherian.

Theorem 2.3. Suppose A is semi-local and complete. If $S^{-1}A$ is also semi-local, but not necessarily complete, then $S_{-1}(-)$ takes Artinian A-modules to Artinian $S^{-1}A$-modules.

Proof. This follows from Proposition 1.3 and the fact that $S^{-1}(-)$ takes finitely generated A-modules to finitely generated $S^{-1}A$-modules. □

This is essentially the best result possible: If A is complete and semi-local, then the co-localization of the Artinian module E_A is $E_{S^{-1}A}$, which will only be Artinian if $S^{-1}A$ is semi-local. On the other hand, suppose A is local but not complete and consider the co-localization of E_A at some $\mathfrak{p} \in \operatorname{Spec} A$:

$$^{\mathfrak{p}}E_A \cong \operatorname{Hom}_{A_{\mathfrak{p}}}((\hat{A})_{\mathfrak{p}}, E(A/\mathfrak{p})) \cong \operatorname{Hom}(\hat{A}, E(A/\mathfrak{p})).$$

Since $\hat{A}$ is a flat A-module, this is injective as an A-module; it will be Artinian over $A_{\mathfrak{p}}$ if and only if it is a direct sum of finitely many copies of $E(A/\mathfrak{p})$. That is to say, we would need for the $\kappa(\mathfrak{q})$-dimension of

$$\operatorname{Hom}\Big(\kappa(\mathfrak{q}), \operatorname{Hom}(\hat{A}, E(A/\mathfrak{p}))_{\mathfrak{q}}\Big) \cong \operatorname{Hom}\Big((\hat{A}/\mathfrak{q}\hat{A})_{\mathfrak{p}}, E(A/\mathfrak{p})\Big)_{\mathfrak{q}}$$

to be finite if $\mathfrak{q} = \mathfrak{p}$ and zero if $\mathfrak{q} \neq \mathfrak{p}$. However, both of these may fail if the ring is not complete: For one thing, we see that the number of times $E(A/\mathfrak{p})$ occurs as a summand of ${}^{\mathfrak{p}}E_A$ is determined by the dimension of $\kappa(\mathfrak{p})\otimes\hat{A}$ over $\kappa(\mathfrak{p})$; as this extension is generally transcendental, there will usually be infinitely many such summands. On the other hand, if for some $\mathfrak{q} \subset \mathfrak{p}$, the ring $A/\mathfrak{q}$ is not complete in the $A/\mathfrak{q}$-topology—that is, the linear topology defined by letting all the nonzero ideals form a subbase for the neighborhoods of zero; see [**7**, Section 6]—then $E(A/\mathfrak{q})$ will also occur as a summand of ${}^{\mathfrak{p}}E_A$. This is so because $\hat{A}/\mathfrak{q}\hat{A}$ will have a nonzero subquotient M which is divisible and torsion-free as an $A/\mathfrak{q}$-module, namely $\widetilde{A/\mathfrak{q}}/A/\mathfrak{q}$, where $\widetilde{A/\mathfrak{q}}$ is the completion of $A/\mathfrak{q}$ in the $A/\mathfrak{q}$-topology, see [**7**, Theorem 6.10]. Since M will then be an $A_{\mathfrak{q}}$-module, $\operatorname{Hom}(M_{\mathfrak{p}}, E(A/\mathfrak{p}))_{\mathfrak{q}} \cong \operatorname{Hom}(M_{\mathfrak{p}}, E(A/\mathfrak{p}))$ will be nonzero, which will mean that $\operatorname{Hom}((\hat{A}/\mathfrak{q}\hat{A})_{\mathfrak{p}}, E(A/\mathfrak{p}))_{\mathfrak{q}}$ will also be nonzero. (For a concrete example of this phenomenon, consider $A = \mathbf{Z}_{(p)}$, $\mathfrak{p} = pA$, and $\mathfrak{q} = 0$; thus, not even ${}^{\mathfrak{m}}E_A$ need be Artinian.)

Our co-localization functor also works well with standard homological concepts. In particular, applying Proposition 1.4, we get

Proposition 2.4. *The functor $S_{-1}(-)$ preserves flats and injectives*

and

Proposition 2.5. *Let M and N be A-modules with M finitely generated, and let i be any integer.*

(i) $S_{-1}\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_i^{S^{-1}A}(S^{-1}M, S_{-1}N)$.

(ii) $S_{-1}\operatorname{Ext}_A^i(M,N) \cong \operatorname{Ext}_{S^{-1}A}^i(S^{-1}M, S_{-1}N)$.

We will see in Theorem 2.7 below that a module is zero if and only if its co-localizations are all zero. Combining that fact with part (i) of the preceding proposition, we find that the flat dimension of an A-module M is the supremum of the flat dimensions of the co-localizations of M. Similarly, part (ii) tells us that the injective dimension of M is $\sup\{\mathrm{id}_{A_{\mathfrak{p}}}^{\mathfrak{p}} M | \mathfrak{p} \in \operatorname{Spec} A\}$.

Theorem 2.6. *Let $(A,\mathfrak{m})$ be a complete local ring, let $\mathfrak{p}$ be a prime of A, let $t = \dim A/\mathfrak{p}$, and let M be a finitely generated A-module. Then, for any integer i we have ${}^{\mathfrak{p}}H^{i+t}_{\mathfrak{m}}(M) = H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.*

Proof. Let $(B,\mathfrak{n})$ be a complete Gorenstein ring mapping onto A, and let $\tilde{\mathfrak{p}}$ be the inverse image of $\mathfrak{p}$ in B. Since $H^{i+t}_{\mathfrak{n}}(M) \cong H^{i+t}_{\mathfrak{m}}(M)$, $H^i_{\tilde{\mathfrak{p}}B_{\tilde{\mathfrak{p}}}}(M_{\tilde{\mathfrak{p}}}) \cong H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, and the functors ${}^{\tilde{\mathfrak{p}}}(-)$ and ${}^{\mathfrak{p}}(-)$ agree on A-modules, we may assume that A is itself Gorenstein.

Letting $n = \dim A$, we now apply the apply the formula

$$H^j_{\mathfrak{m}} \cong \operatorname{Tor}_{n-j}(\,-\,, E(A/\mathfrak{m}))$$

[**14**, Proposition 3.3]:

$${}^{\mathfrak{p}}H^{i+t}_{\mathfrak{m}}(M) \cong {}^{\mathfrak{p}}\operatorname{Tor}^A_{n-(i+t)}(M, E_A) \cong \operatorname{Tor}^{A_{\mathfrak{p}}}_{(n-t)-i}(M_{\mathfrak{p}}, {}^{\mathfrak{p}}E_A).$$

Since A is complete, ${}^{\mathfrak{p}}E_A \cong E(A/\mathfrak{p})$, and the result now follows from a second application of the Tor formula for local cohomology. $\square$

The discussion following Theorem 2.3 shows that this theorem fails dramatically if A is an incomplete Gorenstein ring. However, if we combine the formula ${}^{\mathfrak{p}}H^{i+t}_{\mathfrak{m}}(M) = H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ with Theorem 2.2, we get that $\operatorname{Att} H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{\mathfrak{q}A_{\mathfrak{p}} | \mathfrak{q} \subset \mathfrak{p} \text{ and } \mathfrak{q} \in \operatorname{Att} H^{i+t}_{\mathfrak{m}}(M)\}$; this is true for any ring which is a homomorphic image of a Gorenstein ring (see [**2**, 11.3.2]; this is Sharp's "shifted localization principle").

Definition. For any A-module M, the *co-support* of M is $\operatorname{coSupp} M = \{\mathfrak{p} \in \operatorname{Spec} A | {}^{\mathfrak{p}}M \neq 0\}$.

Theorem 2.7. *Let M be an A-module.*

(i) $\operatorname{coSupp} M = \operatorname{Supp} D_A(M)$.

(ii) $\operatorname{coSupp} M = \varnothing$ *if and only if* $M = 0$.

(iii) $\operatorname{coSupp} M \subset V(\operatorname{ann} M)$.

(iv) *If M is representable, then* $\operatorname{coSupp} M = \{\mathfrak{p} \in \operatorname{Spec} A | \mathfrak{p} \supset \mathfrak{q}$ *for some* $\mathfrak{q} \in \operatorname{Att} M\} = V(\operatorname{ann} M)$.

(v) *If* $0 \to M' \to M \to M'' \to 0$ *is exact, then* $\operatorname{coSupp} M = \operatorname{coSupp} M' \cup \operatorname{coSupp} M'''$.

(vi) *If* M *is finitely generated, then* $\operatorname{coSupp} M = V(\operatorname{ann} M) \cap \mathfrak{m}\text{-}\operatorname{Spec} A$ (*and* ${}^{\mathfrak{m}}M \cong \widehat{M_{\mathfrak{m}}}$ *for any* $\mathfrak{m} \in \operatorname{coSupp} M$).

Proof. (i) follows from the fact that the dual of a nonzero module is nonzero; (ii) follows from (i) together with the analogous result for ordinary localization; (iii) follows from Lemma 2.1 (i); (iv) follows from Theorem 2.2 together with the fact that any prime containing the annihilator of a representable module must contain an attached prime; and (v) follows from the exactness of co-localization.

To prove (vi), we note that, since M is finitely generated, we have $D_A(M)_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, (E_A)_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} A$, so $\operatorname{Supp} D_A(M) \subset \operatorname{Supp} M \cap \operatorname{Supp} E_A = V(\operatorname{ann} M) \cap \mathfrak{m}\text{-}\operatorname{Spec} A$. On the other hand, if $\mathfrak{m}$ is maximal, then $(E_A)_{\mathfrak{m}} \cong E_{A_{\mathfrak{m}}}$, so

$$\begin{aligned} {}^{\mathfrak{m}}M &\cong \operatorname{Hom}_{A_{\mathfrak{m}}}(\operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, E_{A_{\mathfrak{m}}}), E_{A_{\mathfrak{m}}}) \\ &\cong M_{\mathfrak{m}} \otimes \operatorname{Hom}(E_{A_{\mathfrak{m}}}, E_{A_{\mathfrak{m}}}) \\ &\cong \widehat{M_{\mathfrak{m}}}, \end{aligned}$$

which is nonzero if and only if $\mathfrak{m} \in V(\operatorname{ann} M)$. $\square$

Since we are assuming that our rings are Noetherian, saying that $\operatorname{Supp} M \subset V(I)$ for some ideal I is the same as saying that M is I-adically torsion. We finish this section by proving a similar result for co-support. This result will be useful when we prove that the higher local homology modules of M vanish when $\operatorname{coSupp} M \subset V(I)$.

Definition. Let M be an A-module and I an ideal of A. We say that M has *bounded I-adic torsion* if there is a fixed integer k such that any element of M annihilated by a power of I is annihilated by I^k.

Proposition 2.8. *Let* M *be an* A*-module and* I *an ideal of* A*. If* $\operatorname{coSupp} M \subset V(I)$*, then* M *has bounded* I*-adic torsion.*

Proof. For any integer $r > 0$, let $T(r) = \{m \in M | I^r m = 0\}$. Since $\operatorname{coSupp} \cup_{r=1}^{\infty} T(r) \subset \operatorname{coSupp} M$, we may assume that $M = \cup_{r=1}^{\infty} T(r)$.

We will show that, if every $T(r)$ is a proper submodule of M, then there must be a homomorphism $f{:}M \to E_A$ such that, for every positive integer l, we have $f(I^l M) \neq 0$. Such an f could not be annihilated by a power of I, thus contradicting $\operatorname{Supp}\operatorname{Hom}(M, E_A) = \operatorname{coSupp} M \subset V(I)$.

To construct this f, we first inductively define a sequence of functions $f_j{:}M \to E_A$ and integers r_j and l_j such that for every integer $s > 0$:

(i) $f_s(I^{l_s}T(r_s)) \neq 0$.

(ii) For every $0 < j < s$, $f_s(T(r_j)) = 0$

(iii) For every $0 < j < s$, $f_j(I^{l_s}M) = 0$.

Note that these conditions guarantee that $\{r_j\}$ and $\{l_j\}$ are strictly increasing sequences. On the one hand, that means that when constructing f_s and choosing l_s, we need only verify (ii) and (iii) for $j = s-1$. On the other hand, it also means that $\lim_{j\to\infty} r_j = \infty = \lim_{j\to\infty} l_j$, which will be useful when working with our eventual function f.

For $s = 1$, (ii) and (iii) are vacuous, so we need only pick an $m \in M \setminus T(1)$ and a y in I with $ym \neq 0$; then let f_1 be a map $M \to E_A$ which doesn't vanish on ym, let $l_1 = 1$, and let $r_1 = \min\{r \in \mathfrak{n} | m \in T(r)\}$. Now suppose we have defined $f_1, \dots, f_{s-1}$ (and the concomitant r_j's and l_j's). Let $l_s = 1 + \sup\{l \in \mathfrak{n} | f_{s-1}(I^l M) \neq 0\}$. If the supremum were infinite, then f_{s-1} would have the property we want our f to have, so we may assume that l_s is finite, and this choice makes (iii) hold. We are assuming that $T(r_{s-1} + l_s) \neq M$, so there is an $m \in M$ with $I^{l_s}m \notin T(r_{s-1})$; pick a specific $y \in I^{l_s}$ with $ym \notin T(r_{s-1})$. Then there is an $f_s{:}M \to E_A$ with $f_s(ym) \neq 0$ and $f_s(T(r_{s-1})) = 0$, and such a map satisfies (i), (ii) and (iii), where we again set $r_s = \min\{r \in \mathfrak{n} | m \in T(r)\}$.

Finally, once we have our sequences of functions and integers, we let $f = \sum_{j=1}^{\infty} f_j$: for any specific element of M, this sum will be finite because we are assuming $M = \cup T(r)$ and $f_j(T(r)) = 0$ for j sufficiently large. To see that this f is not annihilated by any power of I, we let l be any positive integer and then pick an integer s such that $l_s \geq l$. For any $j < s$ we have $f_j(I^{l_s}T(r_s)) = 0$ by (iii) while, for $j > s$, we have $f_j(I^{l_s}T(r_s)) = 0$ by (ii); that is to say, that $f(I^l M) \supset f(I^{l_s}T(r_s)) = f_s(I^{l_s}T(r_s)) \neq 0$, as desired. $\square$

This gives us a necessary condition for $\operatorname{coSupp} M$ to be contained in $V(I)$; to get a sufficient condition, we need a couple more assumptions:

Proposition 2.9. *Let M be an A-module, and let I be an ideal of A which is contained in the Jacobson radical of A. If every quotient of M has bounded I-adic torsion, then* $\operatorname{coSupp} M \subset V(I)$.

Proof. By Theorem 2.7 (i), it is enough to show that $D(M)$ is I-adically torsion. Given $f \in D(M)$, the image of f is a submodule of $E_A = \oplus E(A/\mathfrak{m})$. Since I is contained in the Jacobson radical of A, E_A is I-adically torsion, which means $\operatorname{im} f$ is I-adically torsion. However, our assumption on M tells us that $\operatorname{im} f$ has bounded I-adic torsion, so all of $\operatorname{im} f$ is annihilated by a single power of I. This power of I then annihilates f. $\square$

3. Local homology and the Koszul complex. Let A be a commutative, Noetherian ring.

We start this section by reminding the reader of the definition of local homology, see also [**4, 8**]:

Definition. Given an ideal I of A, we let Λ_I be the I-adic completion functor: $\Lambda_I = \varprojlim (A/I^k \otimes -)$. For every integer j we define the jth local homology functor H_j^I to be the jth left derived functor of Λ_I.

Since Λ_I is not, in general, a right exact functor, we cannot expect Λ_I to be the same as H_0^I; we do, however, get a surjection $H_0^I \to \Lambda_I$, see Theorem 3.2 below.

If M is a module which can be expressed as the Matlis dual of some module N, then the local homology of M is just the dual of the local cohomology of N. More generally, we have:

Proposition 3.1. *Let E be an injective A-module, and let $(-)^*$ be the functor* $\operatorname{Hom}(-, E)$. *Then, for any A-module M, any ideal I and any integer j, we have $H_j^I(M^*) = H_I^j(M)^*$.*

Proof. Let Γ_I be the I-adic torsion functor $\varinjlim \operatorname{Hom}(A/I^k, -)$. Using Sharp's isomorphism, as in Proposition 1.4 (iii), and the fact that $(-)^*$ converts direct limits to inverse limits, we get $\Gamma_I(N)^* \cong \Lambda_I(N^*)$ for any A-module N.

Now let $E^\bullet$ be an injective resolution of M. As in Proposition 1.4 (i), $(-)^*$ will convert $E^\bullet$ to a flat resolution of M^*. Since flat modules are acyclic relative to local homology, see Lemma 3.5 below, we can use this flat resolution to compute the local homology of M^*:

$$H_j^I(M^*) \cong H_j\left(\Lambda_I\left((E^\bullet)^*\right)\right) \cong \left(H^j(\Gamma_I(E^\bullet))\right)^* \cong H_I^j(M)^*. \qquad \square$$

This dual relationship is one-sided; however, it is not generally true that the local cohomology of a dual is the dual of the local homology:

Example 3.1. Let $A = \mathbf{Z}$, $I = 2\mathbf{Z}$, and $M = \oplus_{n=1}^{\infty} \mathbf{Z}/2^n\mathbf{Z}$. If $D_A(H_0^I(M))$ were isomorphic to $H_I^0(D_A(M))$ then in particular $D_A(H_0^I(M))$ would have to be 2-adically torsion. However, using the surjection $H_0^I(M) \to \Lambda_I(M)$ and the injection $M \to \Lambda_I(M)$—note that $\cap I^k M = 0$—we can see that $D_A(M)$ is a subquotient of $D_A(H_0^I(M))$. Since $D_A(M)$ is just $\prod_{n=1}^{\infty} \mathbf{Z}/2^n\mathbf{Z}$, which is not 2-adically torsion, we can conclude that $D_A(H_0^I(M))$ is also not 2-adically torsion, so it cannot be H_I^0 of anything.

One of the nice things about local cohomology modules is that they can be computed as direct limits of Koszul cohomologies; this description is more explicit and often more convenient to use than the usual approach to derived functors via injective resolutions. The goal for this section is to dualize this fact to get a description of local homology in terms of inverse systems of Koszul homologies. Schenzel [**13**, Theorem 1.1(v)] has done something along these lines using a derived category approach, but we will want a more concrete version.

We will at times need to work with individual elements of the Koszul complex, and for that I find a nonstandard notation to be more convenient (see [**11**, Section 2.1] for a more detailed description of this approach to the Koszul complex). Given an A-module M and a sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of A, the number of copies of M in the module $K_j(\mathbf{x};M)$ is the same as the number of subsets S of $\{1,\dots,n\}$ with cardinality j. Thus, we have a one-to-one correspondence between summands of $K_j(\mathbf{x};M)$ and these subsets; we will write an element of the summand corresponding to S as $m[S]$ where $m \in M$. In terms of the standard exterior algebra notation, one may think of my $[S]$ as representing the exterior product $e_{i_1}\wedge\cdots\wedge e_{i_j}$ where $S=\{i_1,\dots,i_j\}$ with $i_1<\cdots<i_j$. In addition to the advantage of brevity, my notation makes it easier to specify the sequence or module when there is some possibility of confusion; we just add a subscript to the $[S]$. For example, we will want to work with different powers $\mathbf{x}^k = x_1^k,\dots,x_n^k$ of our sequence; in this situation we write $[S]_k$ to refer specifically to a summand of $K_j(\mathbf{x}^k;M)$.

Given a sequence $\mathbf{x}=x_1,\dots,x_n$ of elements of A, an A-module M, and two positive integers $r>k$, there is a well-known map of complexes $\phi_{\mathbf{x}}^{r,k} : K_\bullet(\mathbf{x}^r;M)\to K_\bullet(\mathbf{x}^k;M)$ which is the identity on $K_0=M$ and multiplication by $(x_1\dots x_n)^{r-k}$ on $K_n=M$, see [**8**, Lemma 3.2], for example. In our notation, these maps are given explicitly by the formula $\phi_{\mathbf{x}}^{r,k}([S]_r)=x^{(r-k)S}[S]_k$ (where $x^{tS}=\prod_{i\in S}x_i^t$). The induced maps on homology make $\{H_j(\mathbf{x}^k;M)|k>0\}$ into an inverse system for each j, and we have:

Theorem 3.2. *Let $\mathbf{x}=x_1,\dots,x_n$ be a sequence of elements of A, and let I be the ideal $(\mathbf{x})$. Then, for any A-module M and any integer j, there is a short exact sequence*

$$0\longrightarrow \varprojlim{}^1 H_{j+1}(\mathbf{x}^k;M)\longrightarrow H_j^I(M)\longrightarrow \varprojlim H_j(\mathbf{x}^k;M)\longrightarrow 0.$$

Proof. By replacing A by the polynomial ring $A[X_1,\dots,X_n]$ and $\mathbf{x}$ by the sequence $X_1,\dots,X_n$, if necessary, we may assume that $\mathbf{x}$ is a regular sequence on A, see [**17**, 3.3 and 3.4].

For each $k>0$, let $I_k=(\mathbf{x}^k)$; since $\{I_k\}$ is cofinal with $\{I^k\}$, we can compute Λ_I as $\varprojlim(A/I_k\otimes -)$. Adapting the proof of [**4**, Proposition 1.1], we get a short exact sequence

$$0\to \varprojlim{}^1 \operatorname{Tor}_{j+1}(A/I_k,M)\to H_j^I(M)\to \varprojlim \operatorname{Tor}_j(A/I_k,M)\to 0.$$

However, since $\mathbf{x}^k$ is a regular sequence on A, we can compute $\operatorname{Tor}_p(A/I_k, M)$ as $H_p(\mathbf{x}^k; M)$ with the natural map $\operatorname{Tor}_p(A/I_r, M) \to \operatorname{Tor}_p(A/I_k, M)$ corresponding to $H_p(\phi_{\mathbf{x}}^{r,k})$. $\square$

Ideally, we would like to have just $H_j^I(M) \cong \varprojlim H_j(\mathbf{x}^k; M)$. Obviously one way to get that would be to have $\varprojlim^1 H_{j+1}(\mathbf{x}^k; M) = 0$. However, this is not always the case:

Example 3.2. Let $A = \mathbf{Z}$, $S = \oplus_{n=1}^{\infty} \mathbf{Z}/2^n\mathbf{Z}$, $P = \prod_{n=1}^{\infty} \mathbf{Z}/2^n\mathbf{Z}$, and consider the one-element sequence $x = 2$. We can compute $\varprojlim^1 H_1(2^k; S)$ as the cokernel of the map $\Theta_S : \prod_{k=1}^{\infty} H_1(2^k; S) \to \prod_{k=1}^{\infty} H_1(2^k; S)$ defined by $\Theta_S(\{m_k\})_r = m_r - 2m_{r+1}$, see [**19**, Section 3.5]. This is just the restriction of the analogously defined map $\Theta_P : \prod_{k=1}^{\infty} H_1(2^k; P) \to \prod_{k=1}^{\infty} H_1(2^k; P)$. The kernel of Θ_P is

$$\varprojlim H_1(2^k; P) \cong \varprojlim \prod_n H_1(2^k; \mathbf{Z}/2^n\mathbf{Z}) \cong \prod_n \varprojlim H_1(2^k; \mathbf{Z}/2^n\mathbf{Z})$$

which is zero since, for any fixed k, the map

$$H_1(2^{k+n}; \mathbf{Z}/2^n\mathbf{Z}) \xrightarrow{2^n} H_1(2^k; \mathbf{Z}/2^n\mathbf{Z})$$

is zero. Thus, to show Θ_S is not surjective, it suffices to produce an element of $\prod_{k=1}^{\infty} H_1(2^k; P)$ not in $\prod_{k=1}^{\infty} H_1(2^k; S)$ which Θ_P takes to $\prod_{k=1}^{\infty} H_1(2^k; S)$.

We write elements of $\prod_k H_1(2^k; P) \cong \prod_{k,n} H_1(2^k, \mathbf{Z}/2^n\mathbf{Z})$ as doubly indexed sequences $\{a_{n,k} | n, k \in \mathbf{Z}^+\}$ where $a_{n,k} \in H_1(2^k; \mathbf{Z}/2^n\mathbf{Z})$. Consider the sequence given by

$$a_{n,k} = \begin{cases} 1 & \text{if } n \leq k \\ 2^{n-k} & \text{if } n > k. \end{cases}$$

For each fixed k there are infinitely many nonzero $a_{n,k}$, so this sequence is not in $\prod_k H_1(2^k; S)$; however, for $m > l$, we have $\Theta_P(\{a_{n,k}\})_{m,l} = 2^{m-l} - 2 \cdot 2^{m-(l+1)} = 0$, so $\Theta_P(\{a_{n,k}\})$ is an element of $\prod_k H_1(2^k; S)$, as desired.

We do have $H_j^I(M) \cong \varprojlim H_j(\mathbf{x}^k; M)$ under some circumstances, however:

Proposition 3.3. *Let M be an A-module and suppose that either M is Artinian or $M \cong \operatorname{Hom}(N, E)$ where E is injective. Then, for any sequence* $\mathbf{x}$ *of elements of A and any integer j, we have $H_j^I(M) \cong \varprojlim H_j(\mathbf{x}^k; M)$, where I is the ideal generated by* $\mathbf{x}$.

Proof. First suppose M is Artinian. That means that, for every fixed k, the module $H_j(\mathbf{x}^k; M)$ is also Artinian, so the descending chain

$$\operatorname{im} H_j\left(\phi_{\mathbf{x}}^{k+1,k}\right) \supset \operatorname{im} H_j\left(\phi_{\mathbf{x}}^{k+2,k}\right) \supset \cdots,$$

eventually stabilizes. By [**19**, Proposition 3.5.7], this makes $\varprojlim^1 H_j(\mathbf{x}^k; M) = 0$, so $H_j^I(M) \cong \varprojlim H_j(\mathbf{x}^k; M)$ by Theorem 3.2.

On the other hand, if $M \cong \operatorname{Hom}(N, E)$ with E injective, then we just combine the fact that local cohomology is given by a direct limit of Koszul homologies with Proposition 3.1 to get (writing $(-)^*$ for $\operatorname{Hom}(-E)$):

$$\begin{aligned} H_j^I(M) &\cong (H_I^j(N))^* \cong \left(\varinjlim H^j(\mathbf{x}^k; N)\right)^* \\ &\cong \varprojlim H^j(\mathbf{x}^k; N)^* \cong \varprojlim H_j(\mathbf{x}^k; M). \qquad \square \end{aligned}$$

Some authors whose work is almost exclusively focused on Artinian modules simply define local homology this way; that is, instead of using derived functors of completion, they define local homology to be the inverse limit of simpler homology modules (Koszul homology in the case of [**18**] and Tor in the case of [**3**]).

To use our Koszul formula to prove vanishing results, we will make use of some additional concepts (see also [**13**] and [**4**]):

Definitions. (i) We say an inverse system $\{M_k | k > 0\}$ is *prozero* if, for each $k > 0$, there is an $r > k$ such that $M_r \to M_k$ is zero. (In [**19**] this is the "trivial Mittag-Leffler condition".)

(ii) Given a module M and a sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of A, for any pair $i \in \{1, \ldots, n\}$ and $k > 0$ we let

$$M(i,k) = \frac{\left((x_1^k, \ldots, x_{i-1}^k)M : x_i^k\right)}{(x_1^k, \ldots, x_{i-1}^k)M}.$$

For $r > k$ multiplication by x_i^{r-k} induces a map $M(i,r) \to M(i,k)$, and we say that our sequence $\mathbf{x}$ is *proregular* on M if for every i the inverse system $\{M(i,k)|k > 0\}$ is prozero.

(iii) Given a module M and a sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of A, we say that $\mathbf{x}$ is *weakly proregular* on M if, for every $j > 0$, the inverse system $\{H_j(\mathbf{x}^k; M)|k > 0\}$ is prozero.

Schenzel only discusses (weakly) proregular sequences on the ring itself, but his proof [**13**, Lemma 2.7] that a proregular sequence is weakly proregular carries over to modules, see also [**11**, Lemma 2.3.2]. Since $\varprojlim M_k = 0 = \varprojlim^1 M_k$ whenever $\{M_k\}$ is a prozero inverse system, Theorem 3.2 tells us that, whenever I is generated by a weakly M-proregular sequence, $H_j^I(M) = 0$ for all $j > 0$ and $H_0^I(M) \cong \Lambda_I(M)$. As with local cohomology, we do not actually need our weakly proregular sequence $\mathbf{x}$ to generate I; it is enough just to assume that $\sqrt{(\mathbf{x}) + \operatorname{ann} M} = \sqrt{I + \operatorname{ann} M}$, see [**17**, 3.5].

Since completion is exact on finitely generated modules, we expect their higher local homology modules to vanish. A slightly stronger statement is true:

Proposition 3.4. *Let M be a finitely generated module. Every sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of A is proregular on M.*

Proof. See [**13**, 2.6] or [**11**, Proposition 2.3.3]. □

In proving our Koszul formula, we used a change of rings result for local homology from [**17**]; her result applies only to ring homomorphisms $A \to B$ where B is finitely generated over A. We can use proregular sequences to drop this condition; to this end we need a lemma about flat modules:

Lemma 3.5. *Let F be a flat A-module, let M be any A-module, and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of A which is proregular on M. Then $\mathbf{x}$ is also proregular on $F \otimes M$.*

Proof. Because F is flat, tensoring the exact sequence

$$0 \longrightarrow M(i,k) \longrightarrow \frac{M}{(x_1^k,\dots,x_{i-1}^k)M} \xrightarrow{x_i^k} \frac{M}{(x_1^k,\dots,x_{i-1}^k)M}$$

with F shows that $(F\otimes M)(i,k) \cong F\otimes M(i,k)$. Since $\{M(i,k)\}$ is prozero, so is $\{F\otimes M(i,k)\}$. $\square$

Proposition 3.6. *Let $A \to B$ be a homomorphism of commutative, Noetherian rings. For any ideal I of A and any B-module M, we have $H_j^I(M) \cong H_j^{IB}(M)$ for all j.*

Proof. Let $\mathbf{x} = x_1,\dots,x_n$ be a sequence generating I. Given any B-module N, saying that $\mathbf{x}$ is proregular on N does not depend on whether we view $x_1,\dots,x_n$ as elements of A or as elements of B. Therefore, since $\mathbf{x}$ is proregular on B (because B is a Noetherian B-module), $\mathbf{x}$ will be proregular on any flat B-module.

So, if we let $F_\bullet$ be a B-flat resolution of M, then even though $F_\bullet$ may not be an A-flat resolution of M, it will still be a resolution of M by modules which are acyclic relative to the local homology functors ${H_\bullet}^I$. We can therefore use it to compute ${H_\bullet}^I(M)$:

$$H_j^I(M) \cong H_j(\Lambda_I(F_\bullet)) \cong H_j(\Lambda_{IB}(F_\bullet)) \cong H_j^{IB}(M). \qquad \square$$

As one application, this allows us to compute the local homology of an injective indecomposable module in terms of better understood local cohomology modules:

Example 3.3. Let A be a ring, I any ideal of A and $\mathfrak{p}$ any prime of A. Let E be the injective hull of $A/\mathfrak{p}$. Our change of rings result says that $H_j^I(E)$ is the same as $H_j^{IA_\mathfrak{p}}(E)$. Since $E = D_{A_\mathfrak{p}}(A_\mathfrak{p})$, this means $H_j^I(E) \cong D_{A_\mathfrak{p}}(H^j_{IA_\mathfrak{p}}(A_\mathfrak{p}))$. In particular, $H_j^I(E)$ vanishes whenever $j > \operatorname{ht}\mathfrak{p}$ or $j < \operatorname{depth}(IA_\mathfrak{p}, A_\mathfrak{p})$, see [**2**, Theorems 6.1.2 and 6.2.7].

We can also use this to show that, for an Artinian module, having finite length does not depend on the ring. That is:

Proposition 3.7. *Let $A \to B$ be a homomorphism of commutative rings, not necessarily Noetherian, and let M be a B-module of finite length. If M is Artinian as an A-module, then it also has finite length as an A-module.*

Proof. Using a result of Sharp [**15**, Theorem 3.2], we can replace A and B by Noetherian rings A' and B' without changing the submodule structure of M. We construct this A' by first letting $\hat{A}$ be the completion of A with respect to the intersection of the maximal ideals of A associated to M and then letting $A' = \hat{A}/\mathrm{ann}_{\hat{A}} M$. B' is formed similarly and one can easily show that the ring homomorphism $A \to B$ induces a homomorphism $A' \to B'$. Thus we may assume A and B are Noetherian and semi-local.

Let J be the Jacobson radical of A. Since M is finitely generated over B, we have $H_j^{JB}(M) = 0$ for all $j > 0$. By Proposition 3.6, that makes $H_j^J(M) = 0$ for all $j > 0$. Now, A is complete in the J-adic topology, so $M \cong D_A^2(M)$, which means $H_j^J(M) \cong D_A(H_J^j(D_A(M)))$, so we have $H_J^j(D_A(M)) = 0$ for $j > 0$. However, $D_A(M)$ is a finitely generated module, so $H_J^d(D_A(M)) \neq 0$ when d is the Krull dimension of $D_A(M)$ [**2**, 6.1.4]. Thus $D_A(M)$ has finite length, so M has finite length. $\square$

One way to compute local cohomology modules (in characteristic 0, at least) is to pass to a ring of differential operators, where the local cohomology modules will have finite length [**4**]. Since there are plenty of local cohomology modules which are Artinian but do not have finite length [**2**, 7.3.3], Proposition 3.7 shows that the ring of differential operators cannot be replaced with a commutative ring.

4. Local homology and co-support. Let A be a commutative, Noetherian ring.

If the support of a module M is contained in $V(I)$ for some ideal I, then all of M's higher local cohomology modules relative to I are known to vanish, see [**2**, Corollary 2.1.7], for example. We get a dual result for local homology:

Theorem 4.1. *Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of A, I the ideal they generate, and M an A-module. If* $\operatorname{coSupp} M \subset V(I)$, *then $\mathbf{x}$ is proregular on M. In particular, $H_j^I(M) = 0$ for $j > 0$ and $H_0^I(M) \cong \Lambda_I(M)$.*

Proof. Let i be an integer between 1 and n, and let k be any positive integer. Since $\operatorname{coSupp} M/(x_1^k, \dots, x_{i-1}^k)M \subset \operatorname{coSupp} M$ and $V(I) \subset V(x_i)$, Lemma 2.8 tells us that the module $M/(x_1^k, \dots, x_{i-1}^k)M$ has bounded x_i-adic torsion. Thus there is an $r > k$ such that any element of $M/(x_1^k, \dots, x_{i-1}^k)M$ annihilated by a power of x_i is annihilated by x_i^{r-k}; this implies that the map $M(i, r) \to M(i, k)$ is zero. $\square$

For local cohomology, it is also known that $H_I^j(M) = 0$ for all $j > \dim \operatorname{Supp} M$, see [**2**, Theorem 6.1.2], for example; we would like to have $H_j^I(M) = 0$ for all $j > \dim \operatorname{coSupp} M$. We cannot as yet prove that result in full generality, but we can at least prove some results in that direction. We start with the case of "small" modules:

Proposition 4.2. *Let I be an ideal of A and M an A-module. If M is either finitely generated or Artinian, then $H_j^I(M) = 0$ for all $j > \dim \operatorname{coSupp} M$.*

Proof. If M is finitely generated, any sequence is proregular on M, so $H_j^I(M) = 0$ for any I and any $j > 0$. Thus, the only way the proposition could fail in this case would be for $\dim \operatorname{coSupp} M$ to be negative, which would mean $M = 0$, but then $H_j^I(M) = 0$ for all j.

Now suppose M is Artinian. We prove the result using induction on the dimension of $\operatorname{coSupp} M$—if $\dim \operatorname{coSupp} M = 0$, then the attached primes of M are all maximal, whence M has finite length and the result follows from the finitely generated case.

Suppose M has one attached prime $\mathfrak{p}$, so $\operatorname{coSupp} M = V(\mathfrak{p})$. If $I \subset \mathfrak{p}$, then $H_j^I(M) = 0$ for all $j > 0$ by Theorem 4.1; thus, we may assume that there is an $x \in I \setminus \mathfrak{p}$. Then $\operatorname{coSupp}(0 :_M x) \subset \operatorname{coSupp} M \cap V(x) = V(\mathfrak{p}, x)$ which has dimension less than $\dim \operatorname{coSupp} M$. For any j, the exact sequence $0 \to (0 :_M x) \to M \xrightarrow{x} M \to 0$ yields

$$H_j^I(M) \xrightarrow{x} H_j^I(M) \longrightarrow H_{j-1}^I((0 :_M x)).$$

If $j > \dim \operatorname{coSupp} M$, we may assume by induction that the last module is zero, so $xH_j^I(M) = H_j^I(M)$. However, by [**16**, Lemma 5.1 (iii)], that means $H_j^I(M) = 0$.

The result follows for a general Artinian module by induction on the number of attached primes, using the long exact sequence in homology and the fact that for any submodule N of M, $\operatorname{coSupp} N$ and $\operatorname{coSupp} M/N$ are both contained in $\operatorname{coSupp} M$. $\square$

([**3**, Proposition 4.8] and [**18**, Theorem 3.3] give similar results relating the vanishing of local homology to the "Krull dimension" of Artinian modules as defined in [**12**].)

To prove the full version of the local cohomology vanishing result, one reduces the general case to the finitely generated case by noting that any module is the direct limit of its finitely generated submodules. Dualizing, we would like to say that every module is the inverse limit of its Artinian quotients, but this is not the case (that would say, for example, that every finitely generated module is complete). The most we can say is that the Artinian quotients "separate points from submodules" as in Lemma 1.1. In the next theorem we adapt that fact to Koszul homologies; that is enough to reduce the vanishing of $\varprojlim H_j(\mathbf{x}^k; M)$ to the case where M is Artinian.

Lemma 4.3. *Let A be a ring, $\mathbf{x} = x_1, \dots, x_n$ a sequence of elements of A, and M an A-module. If z is a nonzero element of $H_j(\mathbf{x}; M)$, then there exists a surjective homomorphism $f : M \to N$ with N Artinian such that z is not in the kernel of the induced map $\tilde{f} : H_j(\mathbf{x}; M) \to H_j(\mathbf{x}; N)$.*

Proof. Abusing notation, we view z as an element of $K_j(\mathbf{x}; M)$ which is not in the submodule of boundaries $B_j(\mathbf{x}; M)$. By Lemma 1.1 there is then a homomorphism $g : K_j(\mathbf{x}; M) \to \Xi$ with Ξ Artinian, $g(B_j(\mathbf{x}; M)) = 0$, and $g(z) \neq 0$. Indexing the Koszul complex using subsets of $\{1, \dots, n\}$, we think of this g as a homomorphism

$\oplus_{|S|=j} M \to \Xi$. For each S we let $g_S : M \to \Xi$ be the composition of g with the inclusion $M \to \oplus M$ corresponding to S. Putting these maps g_S together one way gives us our original function: $g\left(\sum m_s[S]_M\right) = \sum g_S(m_S)$. Putting them together in the other gives a map $M \to \oplus_{|S|=j}\Xi$; we let N be the image of this map, and we let $f{:}M \to N$ be the corestriction to N.

Let f_j be the induced map $K_j(\mathbf{x}; M) \to K_j(\mathbf{x}; N)$; we need to show that $f_j(z) \notin B_j(\mathbf{x}; N)$. Viewing N as a submodule of $\oplus_{|T|=j}\Xi$ makes $K_j(\mathbf{x}; N)$ a submodule of $\oplus_{|S|=j} \oplus_{|T|=j} \Xi$; we will write an element of $K_j(\mathbf{x}; N)$ as $\sum_{S,T} n_{S,T}[S, T]_N$. With this notation, the map f_j is defined by

$$f_j\left(\sum_S m_S[S]_M\right) = \sum_{S,T} g_T(m_S)[S,T]_N.$$

We next define a homomorphism of modules $\Delta : K_j(\mathbf{x}; N) \to \Xi$ by adding up diagonal terms: $\Delta(\sum n_{S,T}[S, T]_N) = \sum n_{S,S}$. Composing this Δ with f_j gives us back our original g, since for any $\sum m_S[S]_M \in K_j(\mathbf{x}; M)$ we have

$$\begin{aligned}\Delta\Big(f_j\Big(\sum m_S[S]_M\Big)\Big) &= \Delta\Big(\sum g_T(m_S)[S,T]_N\Big)\\ &= \sum g_S(m_S) = g\Big(\sum m_S[S]_M\Big).\end{aligned}$$

In particular, this says that $f_j(z) \notin f_j(B_j(\mathbf{x}; M))$, but since f is surjective the boundaries of $K_\bullet(\mathbf{x}; N)$ all come from boundaries of $K_\bullet(\mathbf{x}; M)$. Thus $f_j(z)$ represents a nonzero element of $H_j(\mathbf{x}; N)$. $\square$

Using this we get a partial vanishing result for local homology:

Theorem 4.4. *Let A be a ring, $\mathbf{x} = x_1, \dots, x_n$ any sequence of elements of A and M any A-module. For any $j > \dim \operatorname{coSupp} M$ we have $\varprojlim H_j(\mathbf{x}^k; M) = 0$.*

Proof. Suppose $\varprojlim H_j(\mathbf{x}^k; M) \neq 0$. Then we can find a sequence $\{z_k \in H_j(\mathbf{x}^k; M)\}$ with one of the $z_r \neq 0$. Let $f{:}M \to N$ be a surjective homomorphism with N Artinian and $\tilde{f}(z_r) \neq 0$ as provided in the previous lemma. Then the induced map $\varprojlim H_j(\mathbf{x}^k; M) \to \varprojlim H_j(\mathbf{x}^k; N)$

takes $\{z_k\}$ to a nonzero sequence, showing that $H_j^{(\mathbf{x})}(N) \neq 0$. Since N is Artinian, we can conclude that $j \leq \dim \operatorname{coSupp} N \leq \dim \operatorname{coSupp} M$. $\square$

This means that if $j > \dim \operatorname{coSupp} M$, then any nonzero element of the local homology module $H_j^I(M)$ would have to come from the more exotic $\varprojlim^1 H_{j+1}(\mathbf{x}^k; M)$ part. In particular, if M is one of the modules described in Proposition 3.3, so only the $\varprojlim H_j(\mathbf{x}^k; M)$ part has any relevance, then we do in fact have $H_j^I(M) = 0$ for all $j > \dim \operatorname{coSupp} M$. Also, the index shift in the $\varprojlim^1$ part means that if the sequence $\mathbf{x}$ has length n and $\dim \operatorname{coSupp} M < n$, then at least $H_n^I(M) = 0$. In particular, $H_j^I(M) = 0$ for all $j > \dim \operatorname{coSupp} M$ whenever I is a principal ideal.

REFERENCES

1. R. Belshoff, E. Enochs and J. Garcia Rosas, *Generalized Matlis duality*, Proc. Amer. Math. Soc. **128** (2000), 1307–1312.

2. M.P. Brodmann and R.Y. Sharp, *Local cohomology*: *An algebraic introduction with geometric applications*, Cambridge Univ. Press, Cambridge, 1998.

3. N.T. Cuong and T.T. Nam, *The I-adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 61–72.

4. J.P.C. Greenlees and J.P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), 438–453.

5. G. Lyubeznik, *Finiteness properties of local cohomology modules* (*an application of D-modules to commutative algebra*), Inv. Math. **113** (1993), 41–55.

6. I.G. MacDonald, *Secondary representations of modules over a commutative ring*, in Symposia Mat. **11**, Istituto Nazionale di alta Matematica, Roma, 1973, pp. 23–43.

7. E. Matlis, *Cotorsion Modules*, Mem. Amer. Math. Soc. **49** (1964).

8. ———, *The Koszul complex and duality*, Comm. Alg. **1** (1974), 87–144.

9. L. Melkersson and P. Schenzel, *The co-localization of an Artinian module*, Proc. Edinburgh Math. Soc. **38** (1995), 121–131.

10. A. Ooishi, *Matlis duality and the width of a module*, Hiroshima Math. J. **6** (1976), 573–587.

11. A. Richardson, *Some duality results in homological algebra*, Ph.D. Dissertation, Univ. of Illinois, 2002.

12. R.N. Roberts, *Krull dimension for Artinian modules over quasi-local commutative rings*, Quart. J. Math. Oxford **26** (1975), 269–273.

13. P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand., to appear.

14. R.Y. Sharp, *Some results on the vanishing of local cohomology modules*, Proc. London Math. Soc. **30** (1975), 177–195.

15. ———, *Artinian modules over commutative rings*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 25–33.

16. A.-M. Simon, *Some homological properties of complete modules*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 231–246.

17. ———, *Adic completion and some dual homological results*, Publ. Mat. **36** (1992), 965–979.

18. Z.M. Tang, *Local homology theory for Artinian modules*, Comm. Algebra **22** (1994), 1675–1684.

19. C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, Cambridge, 1994.

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7
E-mail address: asrichar@gmail.com