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WEAKLY CLEAN RINGS AND ALMOST CLEAN RINGS

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ABSTRACT. Let R be a commutative ring with identity. Nicholson defined R to be clean if each element of R is the sum of a unit and an idempotent. In this paper we study two related classes of rings. We define a ring R to be weakly clean if each element of R can be written as either the sum or difference of a unit and an idempotent and following McGovern we say that R is almost clean if each element of R is the sum of a nonzero-divisor and an idempotent.

1. Introduction. All rings except for one fleeting instance will be commutative with identity. Following Nicholson [**5, 6**] we say that a ring R is *clean* if, for each $x \in R$, x can be written as $x = u + e$ where $u \in U(R)$, the group of units of R, and $e \in \text{Id}\,(R)$, the set of idempotents of R. Clean rings have also been studied in [**2**] and [**1**]. It is easy to see [**1**, Proposition 15] that a ring R is clean if and only if each $x \in R$ can be written in the form $x = u - e$ where $u \in U(R)$ and $e \in \text{Id}\,(R)$. This raises the question of whether a ring with the property that, for each $x \in R$, either $x = u + e$ or $x = u - e$ for some $u \in U(R)$ and $e \in \text{Id}\,(R)$ must be clean. Let us call rings with this property *weakly clean*. In [**1**, Proposition 16] it was shown that if R has exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \{0, 1\}$. Thus $\mathbf{Z}_{(3)} \cap \mathbf{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasilocal [**1**, Theorem 3].

In Section 1 we study weakly clean rings. We begin by proving the converse of the above mentioned [**1**, Proposition 16]: a ring R with the property that each $x \in R$ has the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \{0, 1\}$ that is not clean has exactly two maximal ideals and $2 \in U(R)$. We also give weakly clean analogs of several

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results for clean rings given in [**1**]. For example, we show that a direct product $\prod R_\alpha$ of rings $\{R_\alpha\}$ is weakly clean if and only if each R_α is weakly clean and at most one R_α is not clean.

In his study of when a ring $C(X)$ of continuous real-valued functions is clean, McGovern [**4**] introduced the notion of an almost clean ring. A ring R is *almost clean* if each $x \in R$ can be written in the form $x = r+e$ where $r \in \operatorname{reg}(R)$, the set of regular elements (= nonzero-divisors) of R, and $e \in \operatorname{Id}(R)$. Certainly a clean ring is almost clean, but an integral domain while always almost clean is clean if and only if it is quasilocal. In Section 2 a number of analogs of results concerning clean rings are given for almost clean rings. We also determine the indecomposable almost clean rings.

1. Weakly clean rings. In this section we study weakly clean rings and $\mathcal{S}$-weakly clean rings which are defined below. We also consider other classes of rings defined using idempotents and their negatives.

Definition 1.1. A ring R is *weakly clean* if each $x \in R$ can be written as $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \operatorname{Id}(R)$. More generally, let $\mathcal{S}$ be a nonempty set of idempotents of R; then R is $\mathcal{S}$-*weakly clean* if each $x \in R$ can be written in the form $x = u+e$ or $x = u - e$ where $u \in U(R)$ and $e \in \mathcal{S}$.

Like the case for a clean ring, the homomorphic image of a weakly clean ring is easily seen to be weakly clean. More generally, if R is $\mathcal{S}$-weakly clean and $\overline{R}$ is a homomorphic image of R, then $\bar{R}$ is $\overline{\mathcal{S}}$-weakly clean where $\overline{\mathcal{S}}$ is the image of $\mathcal{S}$.

In a similar fashion we could define an $\mathcal{S}$-clean ring R: each $x \in R$ can be written as $x = u+e$ where $u \in U(R)$ and $e \in \mathcal{S}$. However, according to [**1**, Lemma 7], if R is $\mathcal{S}$-clean, then $\mathcal{S} = \operatorname{Id}(R)$. We first consider $\{0, 1\}$-weakly clean rings, i.e., rings in which each element $x \in R$ has the form $x = u + 0 = u$, $x = u + 1$, or $x = u - 1$ where $u \in U(R)$. Equivalently, R is $\{0, 1\}$-weakly clean if at least one of $x - 1$, x, $x + 1$ is a unit. Note that a quasilocal ring is both clean and $\{0, 1\}$-weakly clean. The proof of [**1**, Proposition 16] shows that a ring R with exactly two maximal ideals in which 2 is a unit is $\{0, 1\}$-weakly clean. Our first goal is to prove the converse.

Lemma 1.2. (1) *If R is weakly clean or $\{0,1\}$-weakly clean, then so is every homomorphic image of R.*

(2) *If R is $\{0,1\}$-weakly clean, then R has at most two maximal ideals.*

(3) *Let K_1 and K_2 be fields. Then $K_1 \times K_2$ is $\{0,1\}$-weakly clean if and only if both K_1 and K_2 have characteristic not equal to 2.*

Proof. (1) Clear. Of course, here if $\overline{R}$ is a homomorphic image of R, we mean that $\overline{R}$ is $\{\bar{0},\bar{1}\}$-weakly clean.

(2) Suppose that R has more than two maximal ideals, say M_1, M_2 and M_3 are distinct maximal ideals of R. Let $\overline{R} = R/M_1M_2M_3 \cong R/M_1 \times R/M_2 \times R/M_3$. By (1), $\overline{R}$ is $\{0,1\}$-weakly clean. However, $(0,1,-1)$, $(0,1,-1)+(1,1,1) = (1,2,0)$, and $(0,1,-1)-(1,1,1) = (-1,0,-2)$ are all nonunits, a contradiction.

(3) $(\Leftarrow)$. Suppose that K_1 and K_2 are not of characteristic 2. Then $K_1 \times K_2$ has exactly two maximal ideals and $2 \in U(K_1 \times K_2)$. So by the proof of [**1**, Proposition 16], R is $\{0,1\}$-weakly clean. $(\Rightarrow)$. Suppose that, say $\operatorname{char} K_1 = 2$. Then $(1,0)-(1,1) = (0,-1)$, $(1,0)$, and $(1,0)+(1,1) = (0,1)$ are all nonunits. Hence $K_1 \times K_2$ is not $\{0,1\}$-weakly clean. □

Theorem 1.3. *A ring R is $\{0,1\}$-weakly clean if and only if either* (1) *R is quasilocal, or* (2) *R has exactly two maximal ideals and $2 \in U(R)$.*

Proof. $(\Leftarrow)$. A quasilocal ring is $\{0,1\}$-weakly clean by [**1**, Proposition 2(1)], and a ring with two maximal ideals in which 2 is a unit is $\{0,1\}$-weakly clean by the proof of [**1**, Proposition 16].

$(\Rightarrow)$. Let R be $\{0,1\}$-weakly clean. Suppose that R is not quasilocal. Then by Lemma 1.2 (2), R has exactly two maximal ideals, say M_1 and M_2. Then $R/M_1M_2 \cong R/M_1 \times R/M_2$ is $\{0,1\}$-weakly clean (Lemma 1.2 (1)), so both R/M_1 and R/M_2 have characteristic different from 2 by Lemma 1.2 (3). Hence $2 \notin M_1 \cup M_2$, so $2 \in U(R)$. □

Corollary 1.4. *An indecomposable weakly clean ring is either quasilocal or is an indecomposable ring with exactly two maximal ideals in which* 2 *is a unit.*

Proof. If R is indecomposable, $\mathrm{Id}\,(R) = \{0, 1\}$. Hence R is weakly clean if and only if it is $\{0, 1\}$-weakly clean. □

We next consider the case where R is $\mathcal{S}$-weakly clean with $|\mathcal{S}| < \infty$.

Theorem 1.5. *Let R be $\mathcal{S}$-weakly clean where $|\mathcal{S}| < \infty$. If $0 \notin \mathcal{S}$, then R has at most $2\,|\mathcal{S}| - 1$ maximal ideals. If $0 \in \mathcal{S}$, then R has at most $2\,|\mathcal{S}| - 2$ maximal ideals. In particular, $|Id\,(R)| < \infty$ and R is a finite direct product of indecomposable rings.*

Proof. Let $\mathcal{S} = \{e_1, \dots, e_n\}$. Suppose that R has more than $2n - 1$ maximal ideals, say $\mathcal{M}_1, \dots, \mathcal{M}_{2n}$ are distinct maximal ideals of R. Let $\bar{R} = R/\mathcal{M}_1 \cdots \mathcal{M}_{2n} \cong R/\mathcal{M}_1 \times \cdots \times R/\mathcal{M}_{2n}$. Let $\bar{e}_i$ be the image of e_i in $\bar{R}$ and $\overline{\mathcal{S}} = \{\bar{e}_1, \dots, \bar{e}_n\}$. Clearly $\bar{R}$ is $\overline{\mathcal{S}}$-weakly clean. Let $f_1, \dots, f_{2n}$ be idempotents of $\overline{R}$ corresponding to the standard basis for $R/\mathcal{M}_1 \times \cdots \times R/\mathcal{M}_{2n}$. Let $x = -\bar{e}_1 f_1 - \cdots - \bar{e}_n f_n + \bar{e}_1 f_{n+1} + \cdots + \bar{e}_n f_{2n}$. Then $x + \bar{e}_i$ has ith coordinate $-\bar{e}_i f_i + \bar{e}_i f_i = 0$ and $x - \bar{e}_i$ has $n + i$th coordinate $\bar{e}_i f_{n+i} - \bar{e}_i f_{n+i} = 0$. Thus no $x \pm \bar{e}_i$ is a unit in $\overline{R}$, a contradiction. If $0 \notin \mathcal{S}$, we are done. Suppose that $0 \in \mathcal{S}$. Let $e_1, \dots, e_{n-1}$ be the nonzero members of $\mathcal{S}$, so $e_n = 0$. Suppose that R has more than $2(n-1)$ maximal ideals, say $\mathcal{M}_1, \dots, \mathcal{M}_{2(n-1)+1}$ are distinct maximal ideals of R. Then in the notation as above, we see that for $x = -\bar{e}_1 f_1 - \cdots - \bar{e}_{(n-1)} f_{n-1} + \bar{e}_1 f_n + \cdots + \bar{e}_{n-1} f_{2(n-1)} + 0 f_{2n-1}$, $x \pm \bar{e}_i$ is a nonunit in $\overline{R}$ for each $i = 1, \dots, n$. (Note that in the case where $n = 1$, we have shown that no ring is $\{0\}$-weakly clean.) The last statement follows from the fact that $R_1 \times \cdots \times R_m$ has at least m maximal ideals. □

Unlike the case for an "$\mathcal{S}$-clean ring" where $\mathcal{S}$ must be $\mathrm{Id}\,(R)$, an $\mathcal{S}$-weakly clean ring may have $\mathcal{S} \subsetneq \mathrm{Id}\,(R)$. Indeed, by Lemma 1.2 (3), $\mathbf{Z}_3 \times \mathbf{Z}_3$ is $\{0, 1\}$-weakly clean. However, we must have $1 \in \mathcal{S}$. For $0 = u \pm e$ where $u \in U(R)$ and $e \in \mathrm{Id}\,(R)$ implies $e = \mp u \in U(R)$ and so $e = 1$. The rings that are $\{1\}$-weakly clean are easy to characterize.

Theorem 1.6. *A commutative ring R is $\{1\}$-weakly clean if and only if R is quasilocal and $2 \in U(R)$.*

Proof. ($\Rightarrow$). By Theorem 1.5, R is quasilocal. Since one of 1 ± 1 is a unit, $2 \in U(R)$.

($\Leftarrow$). Let M be the maximal ideal of R. If $x \in M$, then both $x \pm 1$ are units. Suppose that $x \notin M$. Suppose that both $x+1, x-1 \in M$. Then $2x = (x+1)+(x-1) \in M$ implies $x \in M$, a contradiction. □

A direct product $\prod R_\alpha$ of rings is clean if and only if each R_α is clean. We next determine when $\prod R_\alpha$ is weakly clean.

Theorem 1.7. *Let $\{R_\alpha\}$ be a family of commutative rings. Then the direct product $R = \prod R_\alpha$ is weakly clean if and only if each R_α is weakly clean and at most one R_α is not clean.*

Proof. ($\Rightarrow$). Suppose that R is weakly clean. Then each R_α being a homomorphic image of R is weakly clean. Suppose that, say R_{α_1} and R_{α_2}, $\alpha_1 \neq \alpha_2$, are not clean. So there is an $x_{\alpha_1} \in R_{\alpha_1}$ with $x_{\alpha_1} = u_{\alpha_1} + e_{\alpha_1}$, $u_{\alpha_1} \in U(R_{\alpha_1})$, $e_{\alpha_1} \in \mathrm{Id}\,(R_{\alpha_1})$, but $x_{\alpha_1} \neq u - e$ where $u \in U(R_{\alpha_1})$ and $e \in \mathrm{Id}\,(R_{\alpha_1})$. For R_{α_1} is not clean, so not every x is a $u-e$ where $u \in U(R_{\alpha_1})$ and $e \in \mathrm{Id}\,(R_{\alpha_1})$, say x_{α_1} is not. But then R_{α_1} weakly clean gives $x_{\alpha_1} = u_{\alpha_1} + e_{\alpha_1}$. Likewise, there is an $x_{\alpha_2} \in R_{\alpha_2}$ with $x_{\alpha_2} = u_{\alpha_2} - e_{\alpha_2}$ where $u_{\alpha_2} \in U(R_{\alpha_2})$ and $e_{\alpha_2} \in \mathrm{Id}\,(R_{\alpha_2})$, but $x_{\alpha_2} \neq u + e$ where $u \in U(R_{\alpha_2})$ and $e \in \mathrm{Id}\,(R_{\alpha_2})$. Define $x = (x_\alpha) \in R$ by

$$x_\alpha = \begin{cases} x_{\alpha_i} & \alpha \in \{\alpha_1, \alpha_2\} \\ 0 & \alpha \notin \{\alpha_1, \alpha_2\} \end{cases}.$$

Then $x \neq u \pm e$ where $u \in U(R)$ and $e \in \mathrm{Id}\,(R)$.

($\Leftarrow$).If each R_α is clean, then so is $R = \prod R_\alpha$. So assume some R_{α_0} is weakly clean but not clean and that all the other R_α's are clean. Let $x = (x_\alpha) \in R = \prod R_\alpha$. In R_{α_0}, we can write $x_\alpha = u_{\alpha_0} + e_{\alpha_0}$ or $x_\alpha = u_{\alpha_0} - e_{\alpha_0}$ where $u_{\alpha_0} \in U(R_{\alpha_0})$ and $e_{\alpha_0} \in \mathrm{Id}\,(R_{\alpha_0})$. If $x_{\alpha_0} = u_{\alpha_0} + e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = u_\alpha + e_\alpha$ ($u_\alpha \in U(R_\alpha)$, $e_\alpha \in \mathrm{Id}\,(R_\alpha)$); while if $x_{\alpha_0} = u_{\alpha_0} - e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = u_\alpha - e_\alpha$ ($u_\alpha \in U(R)$, $e_\alpha \in \mathrm{Id}\,(R)$). Then $u = (u_\alpha) \in U(R)$, $e = (e_\alpha) \in \mathrm{Id}\,(R)$, and $x = u + e$ or $x = u - e$. □

Theorem 1.8. *For a commutative ring R the following conditions are equivalent*:

(1) *R is an $\mathcal{S}$-weakly clean ring with $\mathcal{S}$ finite,*

(2) *R is a weakly clean ring that is a finite direct product of indecomposable rings* (*equivalently,* $\operatorname{Id}(R)$ *is finite*), *and*

(3) *R is a finite direct product of quasilocal rings or R is a finite direct product of quasilocal rings and an indecomposable ring with exactly two maximal ideals in which 2 is a unit.*

Proof. (1) $\Rightarrow$ (2). By Theorem 1.5, $R = R_1 \times \cdots \times R_n$ where each R_i is indecomposable.

(2) $\Rightarrow$ (3). Let $R = R_1 \times \cdots \times R_n$ where each R_i is indecomposable. By Theorem 1.7 each R_i is weakly clean with at most one not clean. Since an indecomposable clean ring is quasilocal and an indecomposable weakly clean ring is either quasilocal or has exactly two maximal ideals with 2 a unit (Corollary 1.4), the result follows.

(3) $\Rightarrow$ (1). This follows from Theorem 1.7. $\square$

A number of results concerning clean rings carry over mutatis mutandis to weakly clean rings. The next theorem lists several of these.

Theorem 1.9. *Let R be a commutative ring.*

(1) *R is weakly clean if and only if $R/\sqrt{0}$ is weakly clean.*

(2) *R is weakly clean if and only if $R[[X]]$ is weakly clean.*

(3) *$R[X]$ is never weakly clean.*

Proof. For (1), modify the proof of [**1**, Theorem 9] as necessary. For (2) and (3), modify the proof of [**1**, Proposition 12]. $\square$

For more examples of weakly clean rings, we consider the method of idealization. Let R be a commutative ring and M an R-module. The idealization of R and M is the ring $R(M) = R \oplus M$ with product $(r,m)\,(r',m') = (rr', rm' + r'm)$.

Theorem 1.10. *Let R be a commutative ring and M an R-module. Then the idealization $R(M)$ of R and M is clean, respectively, weakly clean, $\{0,1\}$-weakly clean, if and only if R is clean, respectively, weakly clean, $\{0,1\}$-weakly clean.*

Proof. ($\Rightarrow$). Note that $R \approx R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence if $R(M)$ is clean, respectively weakly clean, $\{0,1\}$-weakly clean, so is R.

($\Leftarrow$). Observe that if $u \in U(R)$, then $(u,m) \in U(R(M))$ for each $m \in M$ and if $e \in \mathrm{Id}\,(R)$, then $(e,0) \in \mathrm{Id}\,(R(M))$. Hence if $r \in R$ with $r = u \pm e$ where $u \in U(R)$ and $e \in \mathrm{Id}\,(R)$, then for $m \in M$, $(r,m) = (u,m) \pm (e,0)$ where $(u,m) \in U(R(M))$ and $(e,0) \in \mathrm{Id}\,(R(M))$. Thus if R is clean, respectively, weakly clean, $\{0,1\}$-weakly clean, so is $R(M)$. □

Suppose that R is an indecomposable weakly clean ring that is not clean. So by Corollary 1.4, R is an indecomposable ring with exactly two maximal ideals in which 2 is a unit. Then for any clean ring S, $R \times S$ is weakly clean, but not clean. We know of no examples of nonclean, weakly clean rings not of this form.

Question 1.11. If a commutative ring T is weakly clean but not clean, is $T \approx R \times S$ where R is an indecomposable nonclean, weakly clean ring and S is a clean ring?

By Corollary 1.4 and Theorem 1.7, Question 1.11 has an affirmative answer if T is a (possibly infinite) direct product of indecomposable rings. In particular, the answer is affirmative for T Noetherian. Note that Theorem 1.10 is not useful for settling the question. For if $T = R_1 \times R_2$ and M is a T-module, then $M = M_1 \times M_2$ where M_i is an R_i-module and $T(M) \approx R_1(M_1) \times R_2(M_2)$. Thus if $T(M)$ gives a negative answer to Question 1.11, T must already give a negative answer.

In [**1**, Theorem 14] it was shown that a commutative ring R satisfies $R = U(R) \cup \mathrm{Id}\,(R)$ if and only if R is a field or a Boolean ring. We next wish to characterize the commutative rings R with $R = U(R) \cup \mathrm{Id}\,(R) \cup -\mathrm{Id}\,(R)$. We begin with the following special case.

In the next result we do not need to assume that R is commutative.

Theorem 1.12. *For a ring R the following conditions are equivalent*:

(1) $R = \text{Id}\,(R) \cup -\text{Id}\,(R)$,

(2) *for $x \in R$, $x^2 = x$ or $x^2 = -x$, and*

(3) *R is isomorphic to one of the following*:

(a) $\mathbf{Z}_3$,

(b) *a Boolean ring, or*

(c) $\mathbf{Z}_3 \times B$ *where B is Boolean.*

Proof. (1) $\Rightarrow$ (2). $x \in \text{Id}\,(R) \Rightarrow x^2 = x$ while $-x \in \text{Id}\,(R) \Rightarrow x^2 = (-x)^2 = -x$.

(2) $\Rightarrow$ (1). Of course $x^2 = x \Rightarrow x \in \text{Id}\,(R)$ while $x^2 = -x \Rightarrow (-x)^2 = -x \Rightarrow -x \in \text{Id}\,(R)$. Note that $x^2 = -x \Rightarrow x^3 = -x^2 = x$. Thus for $x \in R$, either $x^2 = x$ or $x^3 = x$ and hence by a celebrated theorem of Jacobson [**3**, Theorem 1, p. 217], R is commutative.

(3) $\Rightarrow$ (1). Clear.

(1) $\Rightarrow$ (3). First, suppose that R is indecomposable, so $\text{Id}\,(R) = \{0, 1\}$. Then $R = \{0, \pm 1\}$ and hence R is isomorphic to either $\mathbf{Z}_2$ or $\mathbf{Z}_3$. Suppose that R is not indecomposable. Let $e \in \text{Id}\,(R) - \{0, 1\}$. Suppose that in the decomposition $R = Re \oplus R(1-e)$, Re is not a Boolean ring. We claim that then $R(1-e)$ is Boolean. Suppose re is not idempotent. So for any $s \in R$, $re + (-s)(1-e)$ is not idempotent. Hence $-(re + (-s)(1-e)) = -re + s(1-e)$ is idempotent. So each $s(1-e)$ is idempotent. Thus $R(1-e)$ is Boolean and hence $2R(1-e) = 0$. Hence for each idempotent $e \in R$, either $2e = 0$ or $2(1-e) = 0$.

If $(0:2) = \{x \in R \mid 2x = 0\} = R$, then $\text{char}\, R = 2$. Hence $R = \text{Id}\,(R)$ and so R is Boolean. So assume $(0:2) \neq R$. Note that $(0:2)$ is a maximal ideal. For suppose $M \supseteq (0:2)$, a maximal ideal. Suppose $x \in M - (0:2)$. Then x or $-x \in \text{Id}\,(R)$ and $-x \in M - (0:2)$ if and only if $x \in M - (0:2)$. So we may assume that x is idempotent. If $x \notin (0:2)$, then $2x \neq 0$, and hence by the previous paragraph $2(1-x) = 0$. Then $1 - x \in (0:2) \subseteq M$, a contradiction. Thus $(0:2)$ is a maximal ideal. Thus $\overline{R} = R/(0:2)$ is an indecomposable ring with

$\overline{R} = \mathrm{Id}\,(\overline{R}) \cup -\mathrm{Id}\,(\overline{R})$. By the previous paragraph, $\overline{R}$ is isomorphic to $\mathbf{Z}_2$ or $\mathbf{Z}_3$.

Now $2R \cap (0:2) = 0$. For, if $x \in 2R \cap (0:2)$, then $x = 2s$ and $2x = 0$. But then $x = x^2 = (2s)^2 = 4s^2 = 2xs = 0$. If $2R = 0$, then R is Boolean. So assume $2R \neq 0$. If $2R = R$, then $(0:2) = 0$ is a maximal ideal of R. Then R is a field and hence by the first paragraph of this proof is isomorphic to $\mathbf{Z}_3$. If $2R \neq R$, then $R = 2R \oplus (0:2)$ where $(0:2)$ is a Boolean ring and $2R \cong R/(0:2)$ is isomorphic to $\mathbf{Z}_3$. For $2R \cong \mathbf{Z}_2$ or $\mathbf{Z}_3$ by the above paragraph and $2R \nsubseteq (0:2)$. □

Corollary 1.13. *A commutative ring R satisfies $R = U(R) \cup \mathrm{Id}\,(R) \cup -\mathrm{Id}\,(R)$ if and only if R is isomorphic to one of the following*: (1) *a field,* (2) *a Boolean ring,* (3) $\mathbf{Z}_3 \times B$ *where B is a Boolean ring, or* (4) $\mathbf{Z}_3 \times \mathbf{Z}_3$.

Proof. ($\Leftarrow$). It is easily checked that any of these four types of rings satisfies the desired condition.

($\Rightarrow$). If R is indecomposable, then $\mathrm{Id}\,(R) = \{0, 1\}$. Hence each nonzero element of R is a unit, that is, R is a field. Next, suppose that R is decomposable, say $R = S \times T$. Let $t \in T$. Then since $(0, t)$ is not a unit, $(0, t) \in \mathrm{Id}\,(R) \cup -\mathrm{Id}\,(R)$. So $T = \mathrm{Id}\,(T) \cup -\mathrm{Id}\,(T)$. By Theorem 1.12, T is isomorphic to either $\mathbf{Z}_3$, a Boolean ring, or $\mathbf{Z}_3 \times B$, where B is a Boolean ring. The same applies to S. Since the direct product of two Boolean rings is again a Boolean ring, we get R is isomorphic to a Boolean ring, $\mathbf{Z}_3 \times \mathbf{Z}_3$, $\mathbf{Z}_3 \times B$, or $\mathbf{Z}_3 \times \mathbf{Z}_3 \times B$ where B is a Boolean ring. However, for $R = \mathbf{Z}_3 \times \mathbf{Z}_3 \times B$, $R \neq U(R) \cup \mathrm{Id}\,(R) \cup -\mathrm{Id}\,(R)$. Indeed, $(-1, 1, 0)$ is not in the union. □

2. Almost clean rings. In this section we study almost clean rings, defined below, which were introduced by McGovern [**4**].

Definition 2.1. A ring R is *almost clean* if each $x \in R$ can be written as $x = r + e$ where $r \in \mathrm{reg}\,(R)$, the set of regular elements of R, and $e \in \mathrm{Id}\,(R)$.

Of course, a clean ring is almost clean. However, as any integral domain is almost clean while an integral domain is clean if and only if it is quasilocal, an almost clean ring need not be clean. Let R be a commutative ring and let $\mathcal{S} \subseteq \mathrm{Id}\,(R)$. We could define R to be $\mathcal{S}$-almost clean if each $x \in R$ can be written as $x = r + e$ where $r \in \mathrm{reg}\,(R)$ and $e \in \mathcal{S}$. However, the proof given for [**1**, Lemma 6], that if an idempotent $e \in \mathrm{Id}\,(R)$ is represented as $e = u + f$ where $u \in U(R)$ and $f \in \mathrm{Id}\,(R)$, then $u = 2e - 1$ and $f = 1 - e$, only requires that u be regular. Thus as in the case for clean rings [**1**, Lemma 7], if R is $\mathcal{S}$-almost clean, then $\mathcal{S} = \mathrm{Id}\,(R)$. Hence a $\{0,1\}$-almost clean ring is the same thing as an indecomposable almost clean ring.

We next give a large class of indecomposable almost clean rings. For a ring R, $Z(R)$, respectively, $J(R)$, will denote the set of zero divisors, respectively, Jacobson radical, of R.

Proposition 2.2. *Suppose that R is présimplifiable, that is, $Z(R) \subseteq J(R)$. Then R is an indecomposable almost clean ring.*

Proof. Let $1 \neq e \in \mathrm{Id}\,(R)$; so $e \in Z(R) \subseteq J(R)$. Hence $e = 0$. So R is indecomposable. If $x \in \mathrm{reg}\,(R)$, then $x = x + 0$ where $x \in \mathrm{reg}\,(R)$ and $0 \in \mathrm{Id}\,(R)$. If $x \notin \mathrm{reg}\,(R)$, then $x \in Z(R) \subseteq J(R)$. So $x - 1 = u \in U(R)$. Hence $x = u + 1$ where $u \in U(R)$, $1 \in \mathrm{Id}\,(R)$. ◻

The converse of Proposition 2.2 is false. Let $R = K[X,Y]/(X)(X,Y)$ where K is a field. Then R is almost clean by Proposition 2.3 below and R is indecomposable since it has a unique minimal prime ideal $(X)/(X)(X,Y)$. But $Z(R) = (X,Y)/(X)(X,Y)$ while $J(R) = \sqrt{0} = (X)/(X)(X,Y)$ (as R is a Hilbert ring). Actually, the condition $Z(R) \subseteq J(R)$ is equivalent to the following condition: for $x \in R$, $x = x + 0$ where $x \in \mathrm{reg}\,(R)$ or $x = u + 1$ where $u \in U(R)$. (For, if $x \in Z(R)$, then for each $r \in R$, $rx \in Z(R)$ and so $rx - 1 \in U(R)$ and thus $x \in J(R)$.) We next characterize the indecomposable almost clean rings.

Theorem 2.3. *For a commutative ring R the following conditions are equivalent.*

(1) *R is an indecomposable almost clean ring.*

(2) *For $x \in R$, either x or $x-1$ is regular.*

(3) *For (prime) ideals I and J of R consisting of zero divisors, $I+J \neq R$.*

Proof. (1)⇔(2). Note that condition (2) is equivalent to R being $\{0,1\}$-almost clean. By the remarks in the paragraph after Definition 2.1, R is $\{0,1\}$-almost clean if and only if R is an indecomposable almost clean ring.

(2)⇒(3). Suppose that I and J are ideals consisting of zero divisors and that $I+J=R$. So $1=i+j$ where $i \in I$ and $j \in J$. Then i and $i-1=-j$ are both zero divisors, a contradiction.

(3)⇒(2). Let $x \in R$ and suppose that x and $x-1$ are both zero divisors. Then $(x)+(x-1) \neq R$, a contradiction. Note that since an ideal $I \subseteq Z(R)$ can be enlarged to a prime ideal $P \subseteq Z(R)$, we can assume that the I and J in (3) are actually prime, or even maximal primes of zero divisors. □

Corollary 4. *Suppose that R has a reduced primary decomposition $0=Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i-primary. Then R is an indecomposable almost clean ring if and only if $Q_i+Q_j \neq R$ (or equivalently, $P_i+P_j \neq R$).*

Note that condition (3) of Theorem 2.3 cannot be extended to: If $I_1, \ldots, I_n$ are ideals consisting of zero divisors, then $I_1+\cdots+I_n \neq R$ for $n \geq 2$. For, take $R=k[X,Y,Z]_S/I_S$ where k is a field, $I=(XYZ)$ and $S=k[X,Y,Z]-((X,Y)\cup(X,Z)\cup(Y,Z))$. Then $Z(R)=\overline{X}R \cup \overline{Y}R \cup \overline{Z}R$. So if $I_1, I_2 \subseteq Z(R)$ are ideals, I_1+I_2 is contained in either $\left(\overline{X},\overline{Y}\right)R$, $\left(\overline{X},\overline{Z}\right)R$, or $\left(\overline{Y},\overline{Z}\right)R$ and hence $I_1+I_2 \neq R$; so R is an indecomposable almost clean ring. However, $\overline{X}R, \overline{Y}R, \overline{Z}R \subseteq Z(R)$, but $\overline{X}R+\overline{Y}R+\overline{Z}R=R$.

The next result gives a characterization of Noetherian almost clean rings.

Theorem 2.5. *Suppose that the commutative ring R is a finite direct product of indecomposable rings, e.g., R is Noetherian. Then the* *following conditions are equivalent.*

(1) *R is almost clean.*

(2) *For prime ideals $P, Q \subseteq Z(R)$ with $P + Q = R$, there exists an idempotent e with $e \in P$ and $1 - e \in Q$.*

Proof. (1)⇒(2). Suppose that R is almost clean. So $R = R_1 \times \cdots \times R_n$ where each R_i is an indecomposable almost clean ring, Proposition 2.10. Let $P, Q \subseteq Z(R)$; so $P = R_1 \times \cdots \times R_{i-1} \times P_i \times R_{i+1} \times \cdots \times R_n$ and $Q = R_1 \times \cdots \times R_{j-1} \times Q_j \times R_{j+1} \times \cdots \times R_n$ where $P_i \subseteq Z(R_i)$ and $Q_j \subseteq Z(R_j)$. If $i = j$, then $P_i + Q_i = R_i$, contradicting Theorem 2.3. So assume $i \neq j$. Then $e = (1, \dots, 1, 0_{R_i}, 1, \dots, 1)$ is idempotent with $e \in P$ and $1 - e \in Q$.

(2)⇒(1). Suppose $R = R_1 \times \cdots \times R_n$ where each R_i is indecomposable. By Proposition 2.10 it suffices to show that each R_i is almost clean. By a change of notation we can take $i = 1$. Let $P_1, Q_1 \subseteq Z(R_1)$ and put $P = P_1 \times R_2 \times \cdots \times R_n$ and $Q = Q_1 \times R_2 \times \cdots \times R_n$. Suppose that $P_1 + Q_1 = R_1$. Then $P + Q = R$. So there is an idempotent $e \in R$ with $e \in P$ and $1 - e \in Q$. Thus there is an idempotent $e_1 \in R_1$ with $e_1 \in P_1$ and $1 - e_1 \in Q_1$. But R_1 is indecomposable so either $e_1 = 1$ or $1 - e_1 = 1$ and hence either $1 \in P_1$ or $1 \in Q_1$, a contradiction. Hence $P_1 + Q_1 \neq R_1$. By Theorem 2.3, R_1 is an indecomposable almost clean ring. □

While the polynomial ring $R[X]$ is never clean [**1, 2**], we next show that if R is almost clean, then $R[X]$ is almost clean and conversely.

Proposition 2.6. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, be a graded ring.*

(1) *If R is almost clean, then R_0 is almost clean.*

(2) *If R_0 is almost clean and each R_n is a torsion-less R_0-module, i.e., $r \notin Z(R_0) \Rightarrow r \notin Z(R_n)$, then R is almost clean.*

Proof. (1) Write $r_0 \in R_0$ as $r_0 = r + e$ where $r \in \operatorname{reg}(R)$ and $e \in \operatorname{Id}(R)$. Since $\operatorname{Id}(R) = \operatorname{Id}(R_0)$, $e \in R_0$, so $r \in R_0 \cap \operatorname{reg}(R) \subseteq \operatorname{reg}(R_0)$.

(2) Let $x = x_0 + x_1 + \cdots + x_n \in R$ where $x_i \in R_i$. Write $x_0 = r_0 + e_0$

where $r_0 \in \operatorname{reg}(R_0)$ and $e_0 \in \operatorname{Id}(R_0) = \operatorname{Id}(R)$; put $x' = r_0 + x_1 + \cdots + x_n$ so $x = x' + e_0$ where $e_0 \in \operatorname{Id}(R)$. If $x' \in Z(R)$, then there exists a nonzero homogeneous $t \in R_n$ (say) with $tx' = 0$. (The proof of this is similar to the proof of McCoy's theorem.) But then $r_0 t = 0$, a contradiction. □

Corollary 2.7. (1) *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, be a graded ring where R_0 is a field. Then R is almost clean.*

(2) *If R is almost clean, then $R[\{X_\alpha\}]$ is almost clean for any set $\{X_\alpha\}$ of indeterminates. Conversely, if some $R[\{X_\alpha\}]$ is almost clean, then R is almost clean.*

The condition in Proposition 2.6 (2) that each R_n be a torsion-less R_0-module is essential. For by Theorem 2.11 the ring $R = \mathbf{Z}(\mathbf{Z}/6\mathbf{Z})$ is a graded ring ($R_0 = \mathbf{Z} \oplus 0$, $R_1 = 0 \oplus \mathbf{Z}/6\mathbf{Z}$, $R_n = 0$ for $n \geq 2$) with R_0 weakly clean, but R is not weakly clean. We next note that for a commutative ring R, $R[[X]]$ is almost clean if and only if R is clean.

Theorem 2.8. *If a commutative ring R is almost clean, then the power series ring $R[[\{X_\alpha\}]]$ is also almost clean. Conversely, if $R[[\{X_\alpha\}]]$ is almost clean for some set of indeterminates, then R is almost clean.*

Proof. ($\Rightarrow$). Suppose that R is almost clean. Let $f \in R[[\{X_\alpha\}]]$, so $f = f_0 + f'$ where $f_0 \in R$ and $f' \in (\{X_\alpha\})$. Write $f_0 = r + e$ where $r \in \operatorname{reg}(R)$ and $e \in \operatorname{Id}(R)$. Then $f = (r + f') + e$ where $r + f' \in \operatorname{reg}(R[[\{X_\alpha\}]])$ and $e \in \operatorname{Id}(R) \subseteq \operatorname{Id}(R[[\{X_\alpha\}]])$.

($\Leftarrow$). Suppose that $R[[\{X_\alpha\}]]$ is almost clean. For $r \in R$, $r = f + e$ where $f \in \operatorname{reg}(R[[\{X_\alpha\}]])$ and $e \in \operatorname{Id}(R[[\{X_\alpha\}]])$. Observe that $\operatorname{Id}(R[[\{X_\alpha\}]]) \subseteq R$. (Write $e = e_0 + e_1 + e_2 + \cdots$, where $\deg e_i = i$. Then $e = e^2$ gives $e_0 = e_0^2$ and by induction each coefficient of e_i lies in Re_0. So $e = e_0 u$ where $u \in U(R[[\{X_\alpha\}]])$. Then $e_0 u = e = e^2 = e_0^2 u^2 = e_0 u^2$ implies $e_0 = e_0 u = e$.) Thus $f \in \operatorname{reg}(R[[\{X_\alpha\}]]) \cap R = \operatorname{reg}(R)$. □

If R is clean, then so is each homomorphic image R/I. We next show that this is not the case for almost clean rings. Note that since every commutative ring is the homomorphic image of $\mathbf{Z}[\{X_\alpha\}]$ for some set of indeterminates, every commutative ring is the homomorphic image of an almost clean ring.

Example 2.9. R almost clean does not imply that R/I is almost clean. Let $\overline{R} = K[X,Y]/(X(X-1)Y)$, K a field. Certainly $K[X,Y]$, being a domain, is almost clean. Note that $\overline{R}$ is indecomposable. Here $\overline{X}$ and $\overline{X} - \overline{1}$ are both zero divisors, so $\overline{R}$ is not almost clean.

The proof of the next result, being similar to the clean case, is left to the reader.

Proposition 2.10. *Let $\{R_\alpha\}$ be a nonempty collection of commutative rings. Then $R = \prod R_\alpha$ is almost clean if and only if each R_α is almost clean.*

We next determine when the idealization $R(M)$ is almost clean. Unlike the clean and weakly clean cases, R is almost clean does not necessarily give that $R(M)$ is almost clean. Indeed, it follows from the next theorem that $\mathbf{Z}(\mathbf{Z}/6\mathbf{Z})$ is not almost clean.

Theorem 2.11. *Let R be a commutative ring and M an R-module. Then the idealization $R(M)$ of R and M is almost clean if and only if each $x \in R$ can be written in the form $x = r + e$ where $r \in R - (Z(R) \cup Z(M))$ and $e \in \operatorname{Id}(R)$.*

Proof. We first observe that $\operatorname{Id}(R(M)) = \{(e,0) \in R(M) \mid e \in \operatorname{Id}(R)\}$. Suppose $(e,m) \in \operatorname{Id}(R(M))$; so $(e,m) = (e,m)^2 = (e^2, 2em)$. Hence $e = e^2$ and $m = 2em$. So $em = 2e^2m = 2em$ gives $em = 0$ and hence $m = 2em = 0$. As the other containment is obvious, we have equality. Also, if $(r,0) \in \operatorname{reg}(R(M))$, then $r \in R - (Z(R) \cup Z(M))$. For if $r \in Z(R)$, then $rs = 0$ where $s \neq 0$ and then $(r,0)(s,0) = (0,0)$; while if $r \in Z(M)$, then $rm = 0$ where $m \neq 0$ and then $(r,0)(0,m) = (0,0)$. Conversely, if $r \in R - (Z(R) \cup Z(M))$, then (r,m) is regular. For $(r,m)(s,n) = (0,0)$ gives $rs = 0$ and hence $s = 0$ and then $rn = 0$ and hence $n = 0$.

Suppose that $R(M)$ is almost clean. Let $x \in R$. Since $\mathrm{Id}\,(R(M)) = \{(e,0) \mid e \in \mathrm{Id}\,(R)\}$, $(x,0) = (r,0) + (e,0)$ where $r \in \mathrm{reg}\,(R(M))$ and $e \in \mathrm{Id}\,(R)$. Since $r \in \mathrm{reg}\,(R(M))$, $r \in R - (Z(R) \cup Z(M))$, and $x = r + e$. Conversely, let $x \in R$ and $m \in M$. Write $x = r + e$ where $r \in R - (Z(R) \cup Z(M))$ and $e \in \mathrm{Id}\,(R)$. Then $(x,m) = (r,m) + (e,0)$ where $(r,m) \in \mathrm{reg}\,(R(M))$ and $(e,0) \in \mathrm{Id}\,(R(M))$. Hence $R(M)$ is almost clean. □

We end with an almost clean analog of Corollary 1.13.

Theorem 2.12. *A commutative ring R satisfies $R = reg\,(R) \cup Id\,(R) \cup -Id\,(R)$ if and only if R is isomorphic to one of the following*: (1) *a domain,* (2) *a Boolean ring,* (3) $\mathbf{Z}_3 \times B$ *where B is a Boolean ring, or* (4) $\mathbf{Z}_3 \times \mathbf{Z}_3$.

Proof. ($\Leftarrow$). Clear.

($\Rightarrow$). First, suppose that R is indecomposable. Then $\mathrm{Id}\,(R) = \{0,1\}$, so $R = \mathrm{reg}\,(R) \cup \{0\}$. Hence R is an integral domain. Next, suppose that R is not indecomposable, say $R = S \times T$. For $s \in S$, $(s,0) \notin \mathrm{reg}\,(R)$; so s or $-s \in \mathrm{Id}\,(S)$. Hence $S = \mathrm{Id}\,(S) \cup -\mathrm{Id}\,(S)$. By Theorem 1.12, S is either a Boolean ring or is isomorphic to$\mathbf{Z}_3$ or $\mathbf{Z}_3 \times B$ where B is a Boolean ring. The same also applies to T. As in the proof of Corollary 1.13 we get that $R = S \times T$ has one of the forms (2), (3), or (4). □

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