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ASYMPTOTIC APPROXIMATIONS OF EIGENVALUES AND EIGENFUNCTIONS FOR REGULAR STURM-LIOUVILLE PROBLEMS

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ABSTRACT. In this paper, we obtain estimates for the eigenvalues and eigenfunctions associated with Sturm-Liouville equation $y'' + (\lambda - q)y = 0$ having regular endpoints under the condition that $q \in L^1[0, \pi]$.

1. Introduction. We consider the differential equation

$$y''(t) + (\lambda - q(t))y(t) = 0, \tag{1}$$

defined on the interval $[0, \pi]$ where $q \in L^1[0, \pi]$. We impose the following boundary conditions

$$y(0)\sin\alpha + y'(0)\cos\alpha = 0 \tag{2}$$

$$y(\pi)\sin\beta + y'(\pi)\cos\beta = 0 \tag{3}$$

where α and β are real numbers. In the case $\beta = \pi/2$, the boundary condition at π is known as a Dirichlet boundary condition, and the case $\beta = 0$ known as a Neumann boundary condition. We make the point that a more general second-order differential equation

$$\{p(t)y'(t)\}' + \{\lambda s(t) - q(t)\}y(t) = 0 \tag{4}$$

can be reduced to an equation of type (1) by using the Liouville transformation if p'' and s'' exist and are piecewise continuous [**3**]. In this case, the boundary conditions (2)–(3) do not change their form.

The derivation of asymptotic approximations of eigenvalues for regular Sturm-Liouville problems has a long history. Motivation for studying eigenvalues and eigenfunctions arise from different types of problems

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including reconstructions of potentials from a knowledge of spectrum, the general theory of periodic potentials and Hill's equation. We mention in particular the results of [**1, 4**–**6, 8**–**11**].

In this paper we derive asymptotic approximations for the eigenvalues of (1)–(3) to within an error term of arbitrary order under the sole condition that q is a member of $L^1[0,\pi]$.

We suppose without loss of generality that q has a mean value zero, i.e.,

$$\int_0^\pi q(t)\,dt = 0, \tag{5}$$

and denote the real Fourier coefficients of q by A_n and B_n where

$$A_n = \frac{1}{2\pi}\int_0^\pi q(t)\cos(nt)\,dt, \quad B_n = \frac{1}{2\pi}\int_0^\pi q(t)\sin(nt)\,dt. \tag{6}$$

2. The method. We associate with (1) the Riccati equation

$$v' = -\lambda + q - v^2. \tag{7}$$

It was shown in [**7**] that if $v(x,\lambda)$ is a complex-valued solution of (7) and

$$S(x,\lambda) := \mathrm{Re}\,\{v(x,\lambda)\}\,, \tag{8}$$

$$T(x,\lambda) := \mathrm{Im}\,\{v(x,\lambda)\}\,, \tag{9}$$

then any real-valued solution of (1) may be written as

$$y(x,\lambda) = R(x,\lambda)\cos(\Psi(x,\lambda)), \tag{10}$$

where

$$\frac{R'}{R} = S(x,\lambda), \quad \Psi' = T(x,\lambda). \tag{11}$$

Our approach to approximating the eigenvalues λ_n of (1)–(3) is to approximate those λ which are such that

$$\Psi(\pi,\lambda) - \Psi(0,\lambda) = \int_0^\pi T(x,\lambda)\,dx. \tag{12}$$

We suppose that there exist functions $A(x)$ and $\eta(\lambda)$ so that

$$\left|\int_x^\pi e^{2i\lambda^{(1/2)t}} q(t)\,dt\right| \le A(x)\eta(\lambda), \quad x \in [0,\pi] \tag{13}$$

where $A(x)$ is a decreasing function of x, $\eta(\lambda) \to 0$ as $\lambda \to \infty$ and $A(x) \in L^1[0,\pi]$. The existence of these functions are established in [**2**]. We restate them for completeness. Avoiding the trivial case $\int_x^\pi |q(t)|\,dt = 0$, we define

(14)

$$F(x,\lambda) := \begin{cases} \left|\int_x^\pi e^{2i\lambda^{(1/2)t}} q(t)\,dt\right| \Big/ \int_x^\pi |q(t)|\,dt & \text{if } \int_x^\pi |q(t)|\,dt \ne 0, \\ 0 & \text{if } \int_x^\pi |q(t)|\,dt = 0, \end{cases}$$

then $0 \le F(x,\lambda) \le 1$, and we set

$$\eta(\lambda) := \begin{cases} \sup \\ 0 \le x \le \pi \end{cases} F(x,\lambda), \tag{15}$$

$\eta(\lambda)$ is well defined by (14) and goes to zero as $\lambda \to \infty$ [**2**]. We now approximate to a solution of (7) on $[0,\pi]$. We set

$$v(x,\lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x,\lambda), \tag{16}$$

and choose v_n so that

$$v_1' + 2i\lambda^{1/2} v_1 = q,$$
$$v_2' + 2i\lambda^{1/2} v_2 = -v_1^2,$$

and for $n = 3, 4, \ldots,$

$$v_n' + 2i\lambda^{1/2} v_n = -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right).$$

Solving v_ns for $n = 1, 2, 3, \ldots,$

$$v_1(x,\lambda) = -e^{-2i\lambda^{(1/2)x}} \int_x^\pi e^{2i\lambda^{(1/2)t}} q(t)\,dt, \tag{17}$$

$$v_2(x,\lambda) = -e^{-2i\lambda^{(1/2)x}} \int_x^\pi e^{2i\lambda^{(1/2)t}} v_1(t,\lambda)^2\,dt, \tag{18}$$

(19) $$v_n(x,\lambda) = -e^{-2i\lambda^{(1/2)x}} \int_x^{\pi} e^{2i\lambda^{(1/2)t}} \left(v_{n-1}^2 + 2v_{n-1} \sum_{m=1}^{n-2} v_m \right) dt.$$

The following lemma is also going to be used in the proof of the results.

Lemma 1 [**2**]. *There exists a sequence* $\{k_n\}$ *of real numbers with*

$$|v_n(x,\lambda)| \le k_n \eta(\lambda)^n \quad \text{for} \quad \lambda \text{ sufficiently large.}$$

It is shown in [**2**] that $\sum_{n=1}^{\infty} v_n(x,\lambda)$ is uniformly absolutely convergent for all $\lambda \ge \lambda_0$ and for all $x \in [0,\pi]$. It also follows from the choice of v_ns that $\sum_{n=1}^{\infty} v_n'(x,\lambda)$ is uniformly absolutely convergent. The series $i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x,\lambda)$ is therefore a solution of (7) and

(20) $$T(x,\lambda) = \operatorname{Im}\{v(x,\lambda)\} = \lambda^{1/2} + \operatorname{Im}\left(\sum_{n=1}^{\infty} v_n(x,\lambda) \right),$$

(21) $$S(x,\lambda) = \operatorname{Re}\{v(x,\lambda)\} = \operatorname{Re}\left(\sum_{n=1}^{\infty} v_n(x,\lambda) \right).$$

Using (5), (19), and changing the order of integration one can easily obtain that

(22) $$\int_0^{\pi} v_1(t,\lambda)\,dt = \frac{i}{2\lambda^{1/2}} \int_0^{\pi} q(t) e^{2i\lambda^{1/2}t}\,dt,$$

(23) $$\int_0^{\pi} v_2(t,\lambda)\,dt = \frac{i}{2\lambda^{1/2}} \int_0^{\pi} v_1^2(t,\lambda)(1 - e^{2i\lambda^{1/2}t})\,dt,$$

and, for $n \ge 3$,

(24)

$$\int_0^{\pi} v_n(t,\lambda)\,dt = \frac{i}{2\lambda^{1/2}} \int_0^{\pi} \left(v_{n-1}^2 + 2v_{n-1} \sum_{m=1}^{n-2} v_m \right)(1 - e^{2i\lambda^{1/2}t})\,dt.$$

3. Approximations for the eigenvalues. In this section, we derive asymptotic approximations for λ_n, the eigenvalues of regular Sturm-Liouville problem, to within an error term of arbitrary order under the condition that $q \in L^1[0,\pi]$.

Theorem 2. *The eigenvalues* λ_n *of* (1)–(3) *satisfy as* $\lambda \to \infty$

i) *for* $\alpha \neq \pi/2$, $\beta \neq \pi/2$,

$$(25)\quad \begin{aligned}(n+1)\pi = \lambda^{1/2}\pi + \sum_{n=1}^{\infty} \operatorname{Im}\left\{\int_0^{\pi} v_n(x,\lambda)\,dx\right\} \\ - \tan^{-1}\left(\frac{1}{T(\pi,\lambda)}\,(S(\pi,\lambda)+\tan\beta)\right) \\ + \tan^{-1}\left(\frac{1}{T(0,\lambda)}\,(S(0,\lambda)+\tan\alpha)\right)\end{aligned}$$

ii) *for* $\alpha = \beta = \pi/2$

$$(26)\quad (n+1)\pi = \lambda^{1/2}\pi + \sum_{n=1}^{\infty} \operatorname{Im}\left\{\int_0^{\pi} v_n(x,\lambda)\,dx\right\}$$

where $T(x,\lambda)$ *and* $S(x,\lambda)$ *are defined in* (20) *and* (21), *respectively.*

Corollary 3. *In the case of the Dirichlet boundary condition, eigenvalues of* (1)–(3) *satisfy as* $n \to \infty$

$$(27)\quad \lambda_n^{1/2} = (n+1) - \frac{1}{(n+1)}\,A_{2(n+1)} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$

where $\eta(n)$, *defined by* (15), *is the order of the Fourier coefficients of* q, *and* $A_{2(n+1)}$ *is given by* (6).

Corollary 4. *In the case of the Dirichlet boundary condition, eigenvalues of* (1)–(3) *satisfy as* $n \to \infty$

$$(28)\quad \begin{aligned}\lambda_n^{1/2} = (n+1) - \frac{1}{(n+1)}\left[A_{2(n+1)} - \frac{1}{\pi}\,F(n)\right] \\ + O(n^{-1}\eta(n)^3) + O(n^{-2}\eta(n)^2) + O(n^{-3}\eta(n)),\end{aligned}$$

where

$$F(n) = \int_0^\pi \sin((n+1)x)\sin(3(n+1)x)(F_c^2(n,x) - F_s^2(n,x))\,dx \tag{29}$$
$$- 2\int_0^\pi \sin((n+1)x)\cos(3(n+1)x)F_c(n,x)F_s(n,x\,)dx,$$

and

$$F_c(n,x) = \int_x^\pi q(t)\cos 2(n+1)t\,dt \tag{30}$$
$$F_s(n,x) = \int_x^\pi q(t)\sin 2(n+1)t\,dt.$$

Corollary 5. *In the case of the Neumann boundary condition, eigenvalues of* (1)–(3) *satisfy as* $n \to \infty$

$$\lambda_n^{1/2} = (n+1) + \frac{1}{(n+1)}\,A_{2(n+1)} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)). \tag{31}$$

Proof of Theorem 2. i) Using (10) we find that for $\alpha \neq \pi/2$, $\beta \neq \pi/2$

$$\begin{aligned} y(0)\sin\alpha + y'(0)\cos\alpha &= R(0,\lambda)\cos(\Psi(0,\lambda))\sin\alpha + R'(0,\lambda)\cos(\Psi(0,\lambda))\cos\alpha \\ &\quad - \Psi'(0,\lambda)\sin(\Psi(0,\lambda))R(0,\lambda)\cos\alpha \\ &= R(0,\lambda)\sin(\gamma - \Psi(0,\lambda)), \end{aligned} \tag{32}$$

where

$$\sin\gamma = \sin\alpha + \frac{R'(0,\lambda)}{R(0,\lambda)}\cos\alpha, \quad \text{and} \quad \cos\gamma = \Psi'(0,\lambda)\cos\alpha. \tag{33}$$

Therefore,

$$\tan\gamma = \left(\frac{\tan\alpha}{\Psi'(0,\lambda)} + \frac{R'(0,\lambda)}{R(0,\lambda)\Psi'(0,\lambda)}\right). \tag{34}$$

Considering (11) and (34) we see that

$$\gamma = \tan^{-1}\left(\frac{1}{T(0,\lambda)}\,(S(0,\lambda)+\tan\alpha)\right). \tag{35}$$

Hence from (32) the first boundary condition (2) is satisfied if

$$\Psi(0,\lambda) = \gamma = \tan^{-1}\left(\frac{1}{T(0,\lambda)}\,(S(0,\lambda)+\tan\alpha)\right). \tag{36}$$

Similarly, the second boundary condition (3) is satisfied if

$$\Psi(\pi,\lambda) = (n+1)\pi + \tan^{-1}\left(\frac{1}{T(\pi,\lambda)}\,(S(\pi,\lambda)+\tan\beta)\right) \tag{37}$$

for the integer n. The result now follows from (12), (20), (36) and (37).

ii) when $\alpha=\beta=\pi/2$ the first boundary condition (2) is satisfied if

$$\Psi(0,\lambda) = \frac{\pi}{2}, \tag{38}$$

and the second boundary condition (3) is satisfied if

$$\Psi(\pi,\lambda) = (n+1)\pi + \frac{\pi}{2} \tag{39}$$

for integer n. Again the result follows from (12), (20), (38) and (39).

Proof of Corollary 3. Using Theorem 2-ii, (5), (17)–(19) and Lemma 1, we see that the first approximation for $\lambda_n^{1/2}$ is

$$\lambda_n^{1/2} = (n+1) + O(n^{-1}\eta(n))$$

where we have written $\eta(n)$ for $\eta(n+1)$. Also

$$\lambda_n^{-1/2} = \frac{1}{(n+1)(1+O(n^{-2}\eta(n)))} = \frac{1}{(n+1)} + O(n^{-3}\eta(n)),$$

and

$$\begin{aligned} e^{2i\lambda_n^{1/2}t} &= \exp\left\{2i(n+1)t + O(n^{-1}\eta(n))\right\} \\ &= \exp\left\{2i(n+1)t)\right\} + O(n^{-1}\eta(n)). \end{aligned} \tag{40}$$

Thus by Lemma 1, Theorem 2 ii), (22) and (24)
(41)

$$\begin{aligned}\lambda_n^{1/2}\pi &= (n+1)\pi - \operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} + O(n^{-1}\eta(\lambda)^2)\\ &= (n+1)\pi - \operatorname{Im}\left\{\frac{i}{2\lambda_n^{1/2}}\int_0^\pi e^{2i\lambda_n^{1/2}t}q(t)\,dt\right\} + O(n^{-1}\eta(\lambda_n)^2)\\ &= (n+1)\pi - \operatorname{Im}\left\{\frac{i}{2(n+1)}\int_0^\pi \exp(2i(n+1)t)q(t)\,dt\right\}\\ &\quad + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n))\\ &= (n+1)\pi - \frac{1}{2(n+1)}\int_0^\pi \cos(2(n+1)t)q(t)\,dt\\ &\quad + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)).\end{aligned}$$

The result follows from (6) and (41).

Proof of Corollary 4. From (27) one obtains that

$$\begin{aligned}\lambda_n^{-1/2} &= \frac{1}{(n{+}1)\,\left[1-(1/(n{+}1)^2)A_{2(n+1)} + O(n^{-2}\eta(n)^2) + O(n^{-3}\eta(n))\right]}\\ &= \frac{1}{(n{+}1)} + \frac{1}{(n{+}1)^3}\,A_{2(n+1)} + O(n^{-3}\eta(n)^2) + O(n^{-4}\eta(n)).\end{aligned}$$

Also from Theorem 2 ii), Lemma 1, (22) and (24)

$$\begin{aligned}\lambda_n^{1/2}\pi &= (n+1)\pi - \operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} - \operatorname{Im}\left\{\int_0^\pi v_2(t,\lambda)\,dt\right\}\\ &\quad + O(n^{-1}\eta(\lambda)^3).\end{aligned} \tag{42}$$

We now evaluate the integral terms in (42). From (22)

$$\operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} = \frac{1}{2\lambda_n^{1/2}}\int_0^\pi q(t)\cos(2\lambda_n^{1/2}t)\,dt. \tag{43}$$

By using the second approximation for $\lambda_n^{1/2}$ in (27), we see that after some calculations

$$
\begin{aligned}
\operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} &= \frac{1}{2(n+1)}\int_0^\pi q(t)\cos(2(n+1)t)\,dt \\
&\quad + \frac{A_{2(n+1)}}{(n+1)^2}\int_0^\pi tq(t)\sin(2(n+1)t)\,dt \\
&\quad + O(n^{-2}\eta(n)^2) + O(n^{-3}\eta(n)) \\
&= \frac{1}{2(n+1)}\int_0^\pi q(t)\cos(2(n+1)t)\,dt \\
&\quad + O(n^{-2}\eta(n)^2) + O(n^{-3}\eta(n)).
\end{aligned} \tag{44}
$$

The last equality holds since

$$\int_0^\pi tq(t)\sin(2(n+1)t)\,dt = O(\eta(n)).$$

Similarly, from (23),

$$
\begin{aligned}
&\operatorname{Im}\left\{\int_0^\pi v_2(t,\lambda)\,dt\right\} \\
&\quad = \operatorname{Im}\left\{\frac{i}{2\lambda^{1/2}}\int_0^\pi v_1^2(t,\lambda)\,dt - \frac{i}{2\lambda^{1/2}}\int_0^\pi v_1^2(t,\lambda)e^{2i\lambda^{1/2}t}\,dt\right\}.
\end{aligned}
$$

After some calculations we find that

$$
\begin{aligned}
&\operatorname{Im}\left\{\int_0^\pi v_2(t,\lambda)\,dt\right\} \\
&\quad = \frac{1}{2\lambda^{1/2}}\left[\int_0^\pi \cos(4\lambda^{1/2}t)\left(\int_t^\pi q(x)\cos(2\lambda^{1/2}x)\,dx\right)^2 dt\right. \\
&\quad - \int_0^\pi \cos(4\lambda^{1/2}t)\left(\int_t^\pi q(x)\sin(2\lambda^{1/2}x)\,dx\right)^2 dt \\
&\quad + 2\int_0^\pi \sin(4\lambda^{1/2}t)\left(\int_t^\pi q(x)\sin(2\lambda^{1/2}x)\,dx\right) \\
&\quad \times\left(\int_t^\pi q(x)\cos(2\lambda^{1/2}x)\,dx\right)dt \\
&\quad - \int_0^\pi \cos(2\lambda^{1/2}t)\left(\int_t^\pi q(x)\cos(2\lambda^{1/2}x)\,dx\right)^2
\end{aligned} \tag{45}
$$

$$+\int_0^\pi \cos(2\lambda^{1/2}t)\left(\int_t^\pi q(x)\sin(2\lambda^{1/2}x)\,dx\right)^2$$
$$-2\int_0^\pi \sin(2\lambda^{1/2}t)\left(\int_t^\pi q(x)\sin(2\lambda^{1/2}x)\,dx\right)$$
$$\times\left(\int_t^\pi q(x)\cos(2\lambda^{1/2}x)\,dx\right)dt.$$

Combining the similar terms in (45), we get

$$\begin{aligned}(46)\quad &\operatorname{Im}\left\{\int_0^\pi v_2(t,\lambda)\,dt\right\}\\ =\ &\frac{1}{2\lambda_n^{1/2}}\left\{\int_0^\pi \left[\cos(4\lambda_n^{1/2}t)-\cos(2\lambda_n^{1/2}t)\right]\left(\int_t^\pi q(x)\cos(2\lambda_n^{1/2}x)\,dx\right)^2 dt\right.\\ &+\int_0^\pi \left[\cos(2\lambda_n^{1/2}t)-\cos(4\lambda_n^{1/2}t)\right]\left(\int_t^\pi q(x)\sin(2\lambda_n^{1/2}x)\,dx\right)^2 dt\\ &+2\int_0^\pi \left[\sin(4\lambda_n^{1/2}t)-\sin(2\lambda_n^{1/2}t)\right]\left(\int_t^\pi q(x)\sin(2\lambda_n^{1/2}x)\,dx\right)\\ &\times\left(\int_t^\pi q(x)\cos(2\lambda_n^{1/2}x)\,dx\right)dt.\end{aligned}$$

Using (27) and the trigonometric formulas

$$\sin 2A-\sin 2B = 2\sin(A-B)\cos(A+B),$$
$$\cos 2A-\cos 2B = -2\sin(A-B)\sin(A+B),$$

we see that (46) becomes

$$(47)\qquad \operatorname{Im}\left\{\int_0^\pi v_2(t,\lambda)\,dt\right\} = -\frac{1}{(n+1)}\,F(n)+O(n^{-2}\eta(n)^3),$$

where $F(n)$ is given by (29). Finally, substituting (44) and (47) into (42), we prove the corollary.

Proof of Corollary 5. From Theorem 2 i) for $\alpha=\beta=0$ we have

$$(48)\qquad \begin{aligned}\lambda^{1/2}\pi = (n+1)\pi - \operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} + \tan^{-1}\left(\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right)\\ -\tan^{-1}\left(\frac{S(0,\lambda)}{T(0,\lambda)}\right)+O(n^{-1}\eta(\lambda)^2).\end{aligned}$$

Also from (17)–(19), (20), (21) and Lemma 1,

$$
\begin{aligned}
S(x,\lambda) &= -\bigg(\cos(2\lambda^{1/2}x)\bigg(\int_x^\pi q(t)\cos(2\lambda^{1/2}t)\,dt\bigg) \\
&\qquad\qquad + \sin(2\lambda^{1/2}x)\bigg(\int_x^\pi q(t)\sin(2\lambda^{1/2}t)\,dt\bigg)\bigg) \\
&\quad + O(\eta(\lambda)^2) \\
&= -\sin(2\lambda^{1/2}x+\xi_x) + O(\eta(\lambda)^2),
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
T(x,\lambda) &= \lambda^{1/2} - \cos(2\lambda^{1/2}x)\bigg(\int_x^\pi q(t)\sin(2\lambda^{1/2}t)\,dt\bigg) \\
&\quad + \sin(2\lambda^{1/2}x)\bigg(\int_x^\pi q(t)\cos(2\lambda^{1/2}t)\,dt\bigg) + O(\eta(\lambda)^2) \\
&= \lambda^{1/2} - \cos(2\lambda^{1/2}x+\xi_x) + O(\eta(\lambda)^2),
\end{aligned} \tag{50}
$$

where

$$
\sin\xi_x = \int_x^\pi q(t)\cos(2\lambda^{1/2}t)\,dt, \cos\xi_x = \int_x^\pi q(t)\sin(2\lambda^{1/2}t)\,dt. \tag{51}
$$

Hence,

$$
\begin{aligned}
\frac{S(x,\lambda)}{T(x,\lambda)} &= -\frac{\sin(2\lambda^{1/2}x+\xi_x) + O(\eta(\lambda)^2)}{\lambda^{1/2}} \\
&\quad \times [1 + \lambda^{-1/2}\cos(2\lambda^{1/2}x+\xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)] \\
&= -\lambda^{-1/2}\sin(2\lambda^{1/2}x+\xi_x) - \lambda^{-1}\sin(2\lambda^{1/2}x+\xi_x) \\
&\quad \times \cos(2\lambda^{1/2}x+\xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2) \\
&= -\lambda^{-1/2}\sin(2\lambda^{1/2}x+\xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2).
\end{aligned} \tag{52}
$$

The last equality holds since

$$
\sin(2\lambda^{1/2}x+\xi_x) = O(\eta(\lambda)), \quad \cos(2\lambda^{1/2}x+\xi_x) = O(\eta(\lambda)).
$$

From (52)

$$
\tan^{-1}\left(\frac{S(x,\lambda)}{T(x,\lambda)}\right) = -\lambda^{-1/2}\sin(2\lambda^{1/2}x+\xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2), \tag{53}
$$

and finally, using the first approximation $\lambda_n^{1/2} = (n+1) + O(n^{-1}\eta(n))$, one observes that

(54)
$$\tan^{-1}\left(\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right) = O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$
$$\tan^{-1}\left(\frac{S(0,\lambda)}{T(0,\lambda)}\right) = -\frac{2\pi}{(n+1)}\,A_{2(n+1)} + O(n^{-1}\eta(n)^2) + O(n^{-2}\eta(n)),$$

and from (43)

(55)
$$\begin{aligned}\operatorname{Im}\left\{\int_0^\pi v_1(t,\lambda)\,dt\right\} &= \frac{1}{2(n+1)}\int_0^\pi q(t)\cos 2(n+1)t\,dt \\ &\quad + O(n^{-2}\eta(n)) \\ &= \frac{\pi}{(n+1)}\,A_{2(n+1)} + O(n^{-2}\eta(n)).\end{aligned}$$

Substituting (54) and (55) into (48), we prove the corollary.

4. Approximations for the eigenfunctions. In this section we obtain approximations for the solution of (1)–(3). It is shown in [**7**] that any nontrivial real-valued solution, z, of (1) can be expressed as

$$z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)\,dt\right)\cos\left\{c_2 + \int_0^x T(t,\lambda)\,dt\right\}, \tag{56}$$

with

(57)
$$\begin{aligned}z'(x,\lambda) &= c_1 S(x,\lambda)\exp\left(\int_0^x S(t,\lambda)\,dt\right)\cos\left\{c_2 + \int_0^x T(t,\lambda)\,dt\right\} \\ &\quad - c_1\exp\left(\int_0^x S(t,\lambda)\,dt\right)\left(\sin\left\{c_2 + \int_0^x T(t,\lambda)\,dt\right\}\right)T(x,\lambda),\end{aligned}$$

where $T(x,\lambda)$ and $S(x,\lambda)$ are given by (20) and (21), respectively.

Lemma 6. *Let* $\varphi(x,\lambda)$ *be the solution of* (1) *satisfying*

$$\varphi(0,\lambda) = 0, \varphi'(0,\lambda) = -1, \tag{58}$$

then

$$\varphi(x,\lambda) = \frac{-1}{T(0,\lambda)} \exp\left(\int_0^x S(t,\lambda)\,dt\right)\left(\sin \int_0^x T(t,\lambda)\,dt\right). \tag{59}$$

Proof. From (56), (57) and (58)

$$\begin{aligned} &\varphi(0,\lambda) = c_1 \cos c_2 = 0,\\ &\varphi'(0,\lambda) = c_1 S(0,\lambda)\cos c_2 - c_1 \sin c_2 T(0,\lambda) = -1, \end{aligned}$$

and it follows that a choice of $c_2 = \pi/2$ and $c_1 = 1/T(0,\lambda)$ satisfy the last equalities. Substitution of these constants in (56) will prove the lemma.

Lemma 7. *Let* $\psi(x,\lambda)$ *be the solution of* (1) *satisfying*

$$\psi(0,\lambda) = 1, \psi'(0,\lambda) = 0, \tag{60}$$

then

$$\psi(x,\lambda) = \frac{1}{\cos c_2} \exp\left(\int_0^x S(t,\lambda)\,dt\right)\cos\left(c_2 + \int_0^x T(t,\lambda)\,dt\right), \tag{61}$$

where

$$c_2 = \tan^{-1}\left(\frac{S(0,\lambda)}{T(0,\lambda)}\right). \tag{62}$$

Proof. From (56), (57) and (60)

$$\psi(0,\lambda) = c_1 \cos c_2 = 1 \tag{63}$$

$$\psi'(0,\lambda) = c_1 S(0,\lambda)\cos c_2 - c_1 \sin c_2 T(0,\lambda) = 0, \tag{64}$$

hence

$$\frac{S(0,\lambda)}{T(0,\lambda)} = \tan c_2.$$

Therefore $c_2 = \tan^{-1}(S(0,\lambda)/T(0,\lambda))$ and, from (63), $c_1 = 1/\cos c_2$. Substitution of these constants in (56) will prove the lemma.

Theorem 8. *Solution of* (1) *corresponding to the Dirichlet boundary condition satisfies, as* $n \to \infty$*,*

(65)
$$\varphi(x,n) = -\frac{1}{(n+1)} \sin\left[(n+1)x - \frac{1}{2(n+1)} \int_0^x q(t)\,dt\right] + O(n^{-2}\eta(n)).$$

Proof. We evaluate each term in (59). From (50),

$$T(0,\lambda) = \lambda^{1/2} + O(\eta(\lambda)),$$

and

$$\frac{1}{T(0,\lambda)} = \frac{1}{\lambda^{1/2} + O(\eta(\lambda))} = \lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda)). \tag{66}$$

We now evaluate $\int_0^x S(t,\lambda)\,dt = \int_0^x \mathrm{Re}\,(v(t,\lambda))\,dt$. Using (17)–(19) and Lemma 1, we see that

$$\begin{aligned} \int_0^x S(t,\lambda)\,dt = &-\int_0^x \cos 2\lambda^{1/2} t\left(\int_t^\pi \cos 2\lambda^{1/2} x q(x)\,dx\right) dt \\ &- \int_0^x \sin 2\lambda^{1/2} t\left(\int_t^\pi \sin 2\lambda^{1/2} x q(x)\,dx\right) dt \\ &+ O(\lambda^{-1/2}\eta(\lambda)^2), \end{aligned} \tag{67}$$

and after some calculations

(68)
$$\begin{aligned} \int_0^x S(t,\lambda)\,dt = \frac{1}{2\lambda^{1/2}}&\left[\cos(2\lambda^{1/2}x + \xi_x) - \int_0^\pi \sin(2\lambda^{1/2}x) q(x)\,dx\right] \\ &+ O(\lambda^{-1/2}\eta(\lambda)^2), \end{aligned}$$

where $\sin\xi_x$ and $\cos\xi_x$ are defined by (51). From (68)

$$\begin{aligned} &\exp\left(\int_0^x S(t,\lambda)\,dt\right) \\ &\quad = 1 + \frac{1}{2\lambda^{1/2}}\left[\cos(2\lambda^{1/2}x + \xi_x) - \int_0^\pi \sin(2\lambda^{1/2}x) q(x)\,dx\right] \\ &\qquad + O(\lambda^{-1/2}\eta(\lambda)^2). \end{aligned} \tag{69}$$

Similarly

(70)
$$\int_0^x T(t,\lambda)\,dt = \lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\left[\sin(2\lambda^{1/2}x+\xi_x) - \int_0^\pi \cos(2\lambda^{1/2}x)q(x)\,dx + \int_0^x q(t)\,dt\right] + O(\lambda^{-1/2}\eta(\lambda)^2),$$

and

(71)
$$\sin\left(\int_0^x T(t,\lambda)\,dt\right) = \sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)\,dt\right) + O(\lambda^{-1/2}\eta(\lambda)).$$

Substituting (66), (69) and (71) into (59) we get

$$\varphi(x,\lambda) = -\lambda^{-1/2}\sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)\,dt\right) + O(\lambda^{-1}\eta(\lambda)). \tag{72}$$

Using the second approximation for λ_n given by (27) we prove the theorem.

Theorem 9. *Solution of* (1) *corresponding to the Neumann boundary condition satisfies, as* $n \to \infty$,

$$\psi(x,n) = \cos\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)\,dt\right) + O(n^{-1}\eta(n)).$$

Proof. We evaluate each term in (61). From (53)

$$\begin{aligned} c_2 = \tan^{-1}\left(\frac{S(0,\lambda)}{T(0,\lambda)}\right) &= -\lambda^{-1/2}\int_0^\pi q(t)\cos 2\lambda^{1/2}t\,dt \\ &\quad + O(\lambda^{-1/2}\eta(\lambda)^2) \\ &= -\frac{2\pi}{(n+1)}A_{2(n+1)} + O(n^{-2}\eta(n)) + O(n^{-1}\eta(n)^2), \end{aligned} \tag{73}$$

$$\cos c_2 = 1 + O(n^{-2}\eta(n)) + O(n^{-1}\eta(n)^2),$$
$$\frac{1}{\cos c_2} = 1 + O(n^{-2}\eta(n)) + O(n^{-1}\eta(n)^2),$$
$$\sin c_2 = -\frac{2\pi}{(n+1)} A_{2(n+1)} + O(n^{-2}\eta(n)) + O(n^{-1}\eta(n)^2).$$

Also, from (71) and (31),

$$\sin\left(\int_0^x T(t,\lambda)\,dt\right) = \sin\left[(n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)\,dt\right] + O(n^{-1}\eta(n)),$$
$$\cos\left(\int_0^x T(t,\lambda)\,dt\right) = \cos\left[(n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)\,dt\right] + O(n^{-1}\eta(n)),$$

$$\begin{aligned}\cos\left(c_2 + \int_0^x T(t,\lambda)\,dt\right) &= \cos\left(\int_0^x T(t,\lambda)\,dt\right) + O(n^{-1}\eta(n)) \\ &= \cos\left[(n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)\,dt\right] \\ &\quad + O(n^{-1}\eta(n)),\end{aligned} \tag{74}$$

and, from (69),

$$\exp\left(\int_0^x S(t,\lambda)\,dt\right) = 1 + O(n^{-1}\eta(n)). \tag{75}$$

Substitution of the values of (73), (74) and (75) into (61) proves the theorem.

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REFERENCES

1. H. Coskun, *Some inverse results for Hill's equation*, J. Math. Anal. Appl. **276** (2002), 833–844.

2. H. Coskun and B.J. Harris, *Estimates for the periodic and semi-periodic eigenvalues of Hill's equation*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 991–998.

3. M.S.P. Eastham, *The spectral theory of periodic differential equations*, Scottish Academic Press, Edinburgh, 1973.

4. G. Fix, *Asymptotic eigenvalues of Sturm-Liouville systems*, J. Math. Anal. Appl. **19** (1967), 519–525.

5. C.T. Fulton, *An integral equation iterative scheme for asymptotic expansions of spectral quantities of regular Sturm-Liouville problems*, J. Integral Equations **4** (1982), 163–172.

6. B.J. Harris, *Asymptotics of eigenvalues for regular Sturm-Liouville problems*, J. Math. Anal. Appl. **183** (1994), 25–36.

7. ———, *The Form of the spectral functions associated with Sturm-Liouville problems with continuous spectrum*, Mathematika **44** (1997), 162–194.

8. E.W. Hobson, *On a general convergence theorem and the theory of the representation of a function by series of normal functions*, Proc. London Math. Soc. **6** (1908), 349–395.

9. H. Hochstadt, *Asymptotics estimates for the regular Sturm-Liouville spectrum*, Comm. Pure Appl. Math. **14** (1961), 749–764.

10. E.C. Titchmarsh, *The theory of functions*, Oxford University Press, Amen House, London.

11. J.L. Walsh, *On the Convergence of the Sturm-Liouville series*, Ann. Math. **24** (1923), 109–120.

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