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POINTS AT RATIONAL DISTANCES ON A PARABOLA

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ABSTRACT. This paper deals with the open problem of finding the maximum number of points on the parabola $y = x^2$ such that all of their mutual distances are rational. We obtain, in parametric terms, a set of seven points on this parabola such that four of the points have all of their six mutual distances rational, five of the points have all but one of their 10 mutual distances rational, six of the points have 12 mutual distances rational and the seven points have 15 of their mutual distances rational. By giving suitable numerical values to the parameters, we can obtain infinitely many examples of seven points in which the first four points, with all of their six mutual distances rational, have positive abscissae and are non-concyclic. Further, for any arbitrary positive integer n, we obtain in parametric terms the abscissae of $n+1$ pairs of points on the given parabola such that $5n+1$ of their mutual distances are rational. With a suitable choice of parameters, we get numerical examples with $5n+2$ of the mutual distances rational.

1. Introduction. There are several interesting diophantine problems concerning the existence, on a plane, of a set of points all of whose mutual distances are rational. For example, Guy [**5**, pp. 181–188] mentions the following open problems:

(i) Is there a point all of whose distances from the corners of the unit square are rational?

(ii) Are there more than six points in the plane, no three on a line, no four on a circle, all of whose mutual distances are rational?

This paper is concerned with an analogous open problem posed by Dean [**3**] who asks the following: "How many points can you find on the (half) parabola $y = x^2$, $x > 0$, so that the distance between any pair of them is rational?"

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Campbell [**1**] has given a method of generating infinitely many numerical examples of four non-concyclic rational points on the parabola $y = x^2$ such that their six mutual distances are rational. However, Dean's problem requires that the points have positive abscissae and except for two numerical examples, the solutions given by Campbell consist of points with either positive or negative abscissae. Georgieva and Dobrikov [**4**] have listed 18 examples of four points on this parabola, all of them having positive abscissae, such that all of their six mutual distances are rational. It appears that the number of points on the parabola $y = x^2$ such that all of their mutual distances are rational is finite. Thus, in addition to Dean's question, one may ask the following question: "For an arbitrary positive integer n, $n \geq 2$, how many of the $n(n+1)/2$ mutual distances of n points on the parabola $y = x^2$ can be rational?"

In this paper we shall determine, in parametric terms, seven points with rational co-ordinates on the parabola $y = x^2$ such that four of the points have all of their six mutual distances rational, five of the points have all but one of their 10 mutual distances rational, six of the points have 12 mutual distances rational and the seven points have 15 of their mutual distances rational. This solution generates infinitely many sets of four non-concyclic points on the given parabola such that all the four points have positive abscissae and all of their six mutual distances are rational. We also give, for an arbitrary positive integer n, the abscissae of $n + 1$ pairs of points in parametric terms such that $5n + 1$ of their mutual distances are rational. With a suitable choice of parameters, we get numerical examples with $5n + 2$ of the mutual distances rational. These $n + 1$ pairs of points together with their reflections across the y-axis give us $4n + 4$ points with $12n + 6$ of their mutual distances rational.

In Section 2 we prove two preliminary lemmas regarding points on a parabola and in Section 3 we obtain rational solutions of a diophantine chain. In Section 4 we apply the results of Sections 2 and 3 to obtain sets of rational points on the parabola $y = x^2$ with the properties mentioned above.

2. Preliminaries. We first define the function $\phi(x)$, which we shall use throughout this paper, as follows:

$$\phi(x) = \frac{x^2 - 1}{2x}. \tag{2.1}$$

Lemma 2.1. *The distance between any two rational points on the parabola $y = x^2$ is rational if and only if the sum of their abscissae is $\phi(\xi)$ where ξ is any arbitrary rational number.*

Proof. If the abscissae of any two rational points P_1 and P_2 on the parabola $y = x^2$ are x_1 and x_2, the distance between these two points is given by $(x_1 - x_2)\{(x_1 + x_2)^2 + 1\}^{1/2}$. If we take the sum $x_1 + x_2$ to be $\phi(\xi)$ where ξ is an arbitrary rational number, the distance between the points P_1 and P_2 is easily seen to be rational. Conversely, if the distance between P_1 and P_2 is rational, there must exist a rational number d such that $(x_1 + x_2)^2 + 1 = d^2$, and hence $(x_1 + x_2 - d)(x_1 + x_2 + d) = -1$. Writing $x_1 + x_2 - d = \xi$, we get $x_1 + x_2 + d = -\xi^{-1}$, and on eliminating d from these two equations we get $x_1 + x_2 = \phi(\xi)$ which proves the lemma. □

We note that a straight line cannot intersect the parabola $y = x^2$ in more than two points, hence three or more points on the parabola $y = x^2$ cannot be collinear. The next lemma gives a condition for any four points on the parabola $y = x^2$ to be concyclic.

Lemma 2.2. *Four points on the parabola $y = x^2$ are concyclic if and only if the sum of their abscissae is zero.*

Proof. The abscissae of the points of intersection of the parabola $y = x^2$ and an arbitrary circle $(x - h)^2 + (y - k)^2 = r^2$ are given by the roots of the equation

$$x^4 + (1 - 2k)x^2 - 2hx + h^2 + k^2 - r^2 = 0, \tag{2.2}$$

in which the coefficient of x^3 is 0, and hence the sum of the abscissae is seen to be zero. Conversely, given any four points on the parabola with abscissae x_1, x_2, x_3, x_4 such that $x_1+x_2+x_3+x_4=0$, it is easy to find h, k and r such that equation (2.2) has the roots x_1, x_2, x_3, x_4, and hence the four given points lie on the circle $(x-h)^2+(y-k)^2=r^2$ so determined.

This lemma has also been proved by Campbell [**1**].

3. A diophantine chain. We will now obtain a parametric rational solution of the diophantine chain

$$\phi(\alpha_1)+\phi(\beta_1)=\phi(\alpha_2)+\phi(\beta_2)=\cdots=\phi(\alpha_n)+\phi(\beta_n) \tag{3.1}$$

where n is any arbitrary positive integer. For this, we first solve the diophantine equation

$$\phi(\alpha_1)+\phi(\beta_1)=\phi(\alpha_2)+\phi(\beta_2). \tag{3.2}$$

We substitute $\alpha_2=p\theta+\alpha_1, \beta_2=q\theta+\beta_1$ in (3.2) which becomes a quadratic equation in θ. We choose p, q such that the term independent of θ becomes zero, and then we easily obtain a non-zero solution for θ which gives us the following solution of (3.2):

$$\alpha_2=\psi_1(\alpha_1,\beta_1), \qquad \beta_2=\psi_2(\alpha_1,\beta_1), \tag{3.3}$$

where

$$\begin{aligned} \psi_1(\alpha_1,\beta_1)&=-\frac{\alpha_1(\alpha_1\beta_1+1)(\beta_1^2+1)}{(\alpha_1^2+1)(\alpha_1-\beta_1)\beta_1},\\ \psi_2(\alpha_1,\beta_1)&=\frac{(\alpha_1^2+1)(\alpha_1\beta_1+1)\beta_1}{\alpha_1(\alpha_1-\beta_1)(\beta_1^2+1)}. \end{aligned} \tag{3.4}$$

In this solution α_1 and β_1 are arbitrary rational parameters (except that they have to be so chosen such that the numerators and denominators of both $\psi_1(\alpha_1,\beta_1)$ and $\psi_2(\alpha_1,\beta_1)$ are non-zero), and hence we can repeat the process, replacing α_1, β_1 by α_2, β_2 respectively to get

$$\phi(\alpha_2)+\phi(\beta_2)=\phi(\alpha_3)+\phi(\beta_3), \tag{3.5}$$

where

$$\alpha_3=\psi_1(\alpha_2,\beta_2), \qquad \beta_3=\psi_2(\alpha_2,\beta_2). \tag{3.6}$$

In fact, we can repeat the above process any number of times and since α_1 and β_1 are arbitrary, we obtain a solution of the diophantine chain (3.1) in terms of the parameters α_1 and β_1 such that for any i, $2 \le i \le n$, we have

$$\alpha_{i+1} = \psi_1(\alpha_i, \beta_i), \qquad \beta_{i+1} = \psi_2(\alpha_i, \beta_i). \tag{3.7}$$

We note that $\alpha_i,\ \beta_i,\ i = 1, 2, \ldots, n$ are rational functions of α_1 and β_1, and we will now show that these functions are all distinct. If we substitute $\alpha_1 = t,\ \beta_1 = -t$ in the functions $\alpha_i,\ \beta_i$, $i = 1, 2, \ldots, n$ and simplify them, for each i we get $\beta_i = -\alpha_i$, and it is easily proved by induction that the denominator of the function obtained by simplifying α_i is divisible by 2^{i-1} but not by 2^i. Thus the substitution $\alpha_1 = t,\ \beta_1 = -t$ leads to distinct simplified functions $\alpha_1(t)$, $\beta_1(t)$, $\alpha_2(t)$, $\beta_2(t)$, $\ldots$, $\alpha_n(t)$, $\beta_n(t)$, and hence the original functions α_1, β_1, α_2, β_2, $\ldots$, α_n, β_n, must necessarily be all distinct.

We have thus obtained a non-trivial solution of (3.1) in terms of the parameters α_1 and β_1. Since n is any arbitrary integer, we have obtained inter alia an arbitrarily large number of parametric solutions of the diophantine equation (3.2).

We also note that, by giving suitable rational values to the parameters α_1 and β_1, we can obtain non-trivial solutions in rational numbers of the arbitrarily long diophantine chain (3.1).

The solution of the diophantine chain (3.1) obtained above is not complete. Therefore we will now indicate a method of obtaining additional parametric solutions of (3.1). First we obtain additional parametric solutions of equation (3.2). To solve (3.2), we write

$$\begin{aligned} \alpha_1 &= x/\{p(x+y)\}, \qquad & \beta_1 &= y/\{q(x+y)\}, \\ \alpha_2 &= u/\{p(u+v)\}, \qquad & \beta_2 &= v/\{q(u+v)\}, \end{aligned} \tag{3.8}$$

when (3.2) reduces to the equation

$$\begin{aligned} &\frac{(qx+py)\{pqx^2+(2pq-1)xy+pqy^2\}}{xy(x+y)} \\ &\qquad\qquad = \frac{(qu+pv)\{pqu^2+(2pq-1)uv+pqv^2\}}{uv(u+v)}, \end{aligned} \tag{3.9}$$

and a solution of this equation can be obtained by solving the simultaneous cubic equations

$$\begin{aligned} xy(x+y) &= uv(u+v),\\ C(x,y) &= C(u,v), \end{aligned} \tag{3.10}$$

where

$$C(x,y) = (qx+py)\{pqx^2 + (2pq-1)xy + pqy^2\}.$$

Using a method described by Choudhry [**2**], we can obtain several parametric solutions of (3.10), and thus several parametric solutions of (3.2), the simplest solution of (3.2) obtained this way being as follows:

$$\begin{aligned} \alpha_1 &= \frac{p(p-q)}{q(p^2+pq-1)}, &\qquad \beta_1 &= \frac{q(-p+q)}{p(pq+q^2-1)},\\ \alpha_2 &= \frac{p(pq+q^2-1)}{p^2+pq-1}, &\qquad \beta_2 &= \frac{q(p^2+pq-1)}{pq+q^2-1}, \end{aligned} \tag{3.11}$$

where p, q are arbitrary rational parameters.

We can use any new parametric solution of (3.2) together with the solution (3.3) of (3.2) to generate a new parametric solution of the diophantine chain (3.1).

4. Sets of points at rational distances on the parabola $y = x^2$. In this section, we determine the sets of points on the parabola $y = x^2$ as mentioned in the Introduction.

4.1 A set of seven points on the given parabola. We will first determine seven points $P_1, P_2, \ldots, P_7$ with abscissae $x_1, x_2, \ldots, x_7$ such that all the six mutual distances of the first four points P_1, P_2, P_3, P_4 are rational and, in addition, the distances of each of the points P_5, P_6, P_7 from three of the four points P_1, P_2, P_3, P_4 are also rational.

Let there exist rational numbers α_i, β_i, $i = 1, 2, 3$ such that

$$\phi(\alpha_1) + \phi(\beta_1) = \phi(\alpha_2) + \phi(\beta_2) = \phi(\alpha_3) + \phi(\beta_3). \tag{4.1}$$

We write

$$
\begin{aligned}
x_1 &= \{\phi(\alpha_1) + \phi(\alpha_2) - \phi(\alpha_3)\}/2,\\
x_2 &= \{-\phi(\alpha_1) + \phi(\alpha_2) - \phi(\alpha_3) + 2\phi(\beta_2)\}/2,\\
x_3 &= \{-\phi(\alpha_1) + \phi(\alpha_2) + \phi(\alpha_3)\}/2,\\
x_4 &= \{\phi(\alpha_1) - \phi(\alpha_2) + \phi(\alpha_3)\}/2,\\
x_5 &= \{-\phi(\alpha_1) - \phi(\alpha_2) + \phi(\alpha_3) - 2\phi(\beta_1)\}/2,\\
x_6 &= \{-\phi(\alpha_1) + \phi(\alpha_2) - \phi(\alpha_3) - 2\phi(\beta_1)\}/2,\\
x_7 &= \{-\phi(\alpha_1) - \phi(\alpha_2) + \phi(\alpha_3) - 2\phi(\beta_2)\}/2,
\end{aligned}
\tag{4.2}
$$

so that

$$
\begin{aligned}
x_1 + x_2 &= \phi(\alpha_2) - \phi(\alpha_3) + \phi(\beta_2) = \phi(\beta_3),\\
x_1 + x_3 &= \phi(\alpha_2),\\
x_1 + x_4 &= \phi(\alpha_1),\\
x_2 + x_3 &= -\phi(\alpha_1) + \phi(\alpha_2) + \phi(\beta_2) = \phi(\beta_1),\\
x_2 + x_4 &= \phi(\beta_2),\\
x_3 + x_4 &= \phi(\alpha_3),
\end{aligned}
\tag{4.3}
$$

Thus, all the pairwise sums $x_i + x_j$, $1 \leq i,\ j \leq 4$, are of the type $\phi(\xi)$ and hence all the six mutual distances of the points $P_1,\ P_2,\ P_3,\ P_4$ will be rational. Further, we note that

$$
\begin{aligned}
&x_1 + x_5 = \phi(-\beta_1), x_2 + x_5 = \phi(-\alpha_2), x_3 + x_5 = \phi(-\beta_3),\\
&x_2 + x_6 = \phi(-\alpha_3), x_3 + x_6 = \phi(-\beta_2), x_4 + x_6 = \phi(-\beta_1),\\
&x_1 + x_7 = \phi(-\beta_2), x_2 + x_7 = \phi(-\alpha_1), x_4 + x_7 = \phi(-\beta_3).
\end{aligned}
\tag{4.4}
$$

This shows that the points $P_5,\ P_6,\ P_7$ are such that the distances of each of them from three of the four points $P_1,\ P_2,\ P_3,\ P_4$ are rational. Thus, the four points $P_1,\ P_2,\ P_3,\ P_4$ have all their six mutual distances rational, the five points $P_1, P_2, \ldots, P_5$ have 9 mutual distances rational, the six points $P_1, P_2, \ldots, P_6$ have 12 mutual distances rational, and the seven points $P_1, P_2, \ldots, P_7$ have 15 mutual distances rational.

It follows from Section 3 that a rational solution of the diophantine chain (4.1) in terms of the parameters α_1 and β_1 is given by

$$
\begin{aligned}
\alpha_2 &= \psi_1(\alpha_1, \beta_1),\\
\beta_2 &= \psi_2(\alpha_1, \beta_1),\\
\alpha_3 &= \psi_1(\alpha_2, \beta_2),\\
\beta_3 &= \psi_2(\alpha_2, \beta_2),
\end{aligned}
\tag{4.5}
$$

where the functions $\psi_1(\alpha_1, \beta_1)$ and $\psi_2(\alpha_1, \beta_1)$ are defined by (3.4). Using (4.2) and (4.5), we can obtain rational functions

$$x_i = x_i(\alpha_1, \beta_1), \quad i = 1, 2, \ldots, 7, \tag{4.6}$$

giving the abscissae of the seven points $P_1, P_2, \ldots, P_7$ in terms of the parameters α_1 and β_1. However, the functions $x_i(\alpha_i, \beta_i)$ are too cumbersome to write and are not being given explicitly. We also note that once we have obtained a diophantine chain of type (4.1) we may, in view of the symmetry, choose $\phi(\alpha_i)$, $\phi(\beta_i)$ in 8 different ways leading to 8 different sets of points $P_1, P_2, \ldots, P_7$.

We can obtain numerical examples by substituting rational values for α_1 and β_1 in the parametric solution (4.5) of (4.1), and then using (4.2). As an example, taking $\alpha_1 = 2$ and $\beta_1 = 10$, we get the following diophantine chain of type (4.1):

(4.7)

$$\begin{aligned}&\frac{3}{4} + \frac{99}{20} \\ &\quad = \frac{4458641}{848400} + \frac{377239}{848400} \\ &\quad = \frac{673594559903135279538913 19}{145919263095302993965032 00} + \frac{158145239740091786061769 21}{145919263095302993965032 00}.\end{aligned}$$

With a re-arrangement of this chain, we may take

$\phi(\alpha_1) = 3/4,$

$\phi(\beta_1) = 99/20,$

$\phi(\alpha_2) = 377239/848400,$

$\phi(\beta_2) = 4458641/848400,$

$\phi(\alpha_3) = 15814523974009178606176921/14591926309530299396503200,$

$\phi(\beta_3) = 67359455990313527953891319/14591926309530299396503200,$

which yields the following solution:

$$
\begin{aligned}
x_1 &= \frac{539228453671869790410167}{9727950873020199597668800}, \\
x_2 &= \frac{133101226619611446536552137}{29183852619060598793006400}, \\
x_3 &= \frac{11358843844738517488829543}{29183852619060598793006400}, \\
x_4 &= \frac{6756734701093279907841433}{9727950873020199597668800}, \\
x_5 &= -\frac{48692585275121857798870727}{9727950873020199597668800}, \\
x_6 &= -\frac{54910091522543267916301993}{9727950873020199597668800}, \\
x_7 &= -\frac{154989116083906895631306937}{29183852619060598793006400}.
\end{aligned}
\tag{4.8}
$$

We note that the abscissae of the four points P_1, P_2, P_3, P_4 are positive. Since the four points P_1, P_2, P_3, P_4 lie on the parabola $y = x^2$, they cannot be collinear, and since the sum of their abscissae is positive, it follows from Lemma 2.2 that these four points are also not concyclic.

We also note that when α_2, β_2, α_3, β_3 are functions of α_1, β_1 defined by (4.5), $\phi(\alpha_i)$, $\phi(\beta_i)$, $i = 1, 2, 3$ are functions of α_1, β_1 that are continuous at $(\alpha_1, \beta_1) = (2, 10)$. Therefore, if in the parametric solution (4.5) of the diophantine chain (4.1), we choose rational values for α_1 and β_1 sufficiently close to 2 and 10 respectively and proceed as above, we will obtain a new set of seven points whose abscissae will be sufficiently close to the values of x_i given by (4.8), and hence the first four points of the new set will have positive abscissae. Since we can choose rational values for the parameters α_1 and β_1 sufficiently close to 2 and 10 in infinitely many ways, we can obtain infinitely many examples of seven points $P_1, \ldots, P_7$ such that the first four of them have positive abscissae and are hence non-concyclic.

Further, using the parametric solution (4.6), we may derive a condition such that the set of five points $P_1,\ P_2,\ P_3,\ P_4,\ P_j$ where $j = 5$ or 6 or 7, has all of its 10 mutual distances rational. As the diophantine chain (4.1) yields 8 different sets of the points $P_1, \ldots, P_7$, we may obtain 24 conditions such that if any one of the conditions is satisfied, we get the desired set of five points. However, each condition results in an equation of very high degree in the parameters α_1 and β_1. While limited trials on one such equation, of degree 80, did not yield any rational solution, the possibility of five points being at rational distances from each other cannot be ruled out. It is, however, unlikely that there are six points on the half parabola $y = x^2$, $x > 0$, all of whose mutual distances are rational.

We also note that, in general, no two of the seven points $P_1, \ldots, P_7$, are reflections of each other across the y-axis. Taking the reflections of the seven points across the y-axis, we get 8 points with 16 distances rational, 10 points with 23 distances rational, 12 points with 30 distances rational and 14 points with 37 distances rational.

4.2 A set of $n+1$ pairs of points on the given parabola. We now give, for an arbitrary positive integer n, a method of obtaining $n+1$ pairs of points $P_0,\ Q_0,\ P_i,\ Q_i$, $i = 1, 2, \ldots, n$ such that, for each i, the six mutual distances of the points $P_0,\ Q_0,\ P_i,\ Q_i$ are rational. Thus these $2n+2$ points are such that $5n+1$ of their $(n+1)(2n+1)$ mutual distances are rational. The abscissae of these points are expressed in parametric terms and, with a choice of parameters, we get $5n+2$ distances rational. As before, taking the reflections of these points across the y-axis, we get $4n+4$ points with $12n+6$ mutual distances rational.

To obtain the $n+1$ pairs of points mentioned above, we will use the solution of the diophantine chain (3.1). Let k denote the common sum $\phi(\alpha_i) + \phi(\beta_i)$, and let ξ be an arbitrary rational number. We now choose $x_{i1},\ x_{i2}$, $i = 0, 1, 2, \ldots, n$ as follows:

$$\begin{aligned} x_{01} &= \{\phi(\xi) + k\}/2, \\ x_{02} &= \{\phi(\xi) - k\}/2, \\ x_{i1} &= \{\phi(\alpha_i) - \phi(\beta_i) - \phi(\xi)\}/2, \\ x_{i2} &= \{-\phi(\alpha_i) + \phi(\beta_i) - \phi(\xi)\}/2, \end{aligned} \tag{4.9}$$

so that

$$
\begin{aligned}
&x_{01}+x_{02}=\phi(\xi),\\
&x_{i1}+x_{i2}=-\phi(\xi)=\phi(-\xi),\\
&x_{01}+x_{i1}=\phi(\alpha_i),\quad i=1,\,2,\dots,\,n,\\
&x_{01}+x_{i2}=\phi(\beta_i),\quad i=1,\,2,\dots,\,n,\\
&x_{02}+x_{i1}=-\phi(\beta_i)=\phi(-\beta_i),\quad i=1,\,2,\dots,\,n,\\
&x_{02}+x_{i2}=-\phi(\alpha_i)=\phi(-\alpha_i),\quad i=1,\,2,\dots,\,n.
\end{aligned}
\tag{4.10}
$$

Taking $x_{i1},\ x_{i2},\, i=0,\,1,\,2,\dots,\,n$ as the abscissae of the points $P_i,\ Q_i$, $i=0,\,1,\,2,\dots,\,n$, respectively, we get $n+1$ pairs of points such that for each i, the six mutual distances of the set of points $P_0,\ Q_0,\ P_i,\ Q_i$ are rational. In this solution, $\xi,\ \alpha_1$ and β_1 are arbitrary parameters. We may choose these parameters such that the distance between the points P_1 and P_2 is also rational.

We note specifically that when $n=1$, we have $k=\phi(\alpha_1)+\phi(\beta_1)$, and the abscissae of the four points $P_0,\ Q_0,\ P_1,\ Q_1$ may accordingly be written in terms of the arbitrary parameters $\xi,\ \alpha_1$ and β_1 as follows:

$$
\begin{aligned}
x_{01}&=\{\phi(\alpha_1)+\phi(\beta_1)+\phi(\xi)\}/2,\\
x_{02}&=\{-\phi(\alpha_1)-\phi(\beta_1)+\phi(\xi)\}/2,\\
x_{11}&=\{\phi(\alpha_1)-\phi(\beta_1)-\phi(\xi)\}/2,\\
x_{12}&=\{-\phi(\alpha_1)+\phi(\beta_1)-\phi(\xi)\}/2.
\end{aligned}
\tag{4.11}
$$

All the six mutual distances of the four points $P_0,\ Q_0,\ P_1,\ Q_1$ are rational. Since $x_{01}+x_{02}+x_{11}+x_{12}=0$, it follows from Lemma 2.2 that these four points are concyclic.

Finally, we give a numerical example with $n=3$. Taking $\alpha_1=2,\ \beta_1=3$, we get the following diophantine chain of type (3.1):

$$
\frac{3}{4}+\frac{4}{3}=\frac{775}{168}+\left(-\frac{425}{168}\right)=\left(-\frac{50032585199}{73060041600}\right)+\frac{202241005199}{73060041600},
\tag{4.12}
$$

and choosing $\xi=30/7$, we get 8 points whose abscissae are as follows:

$x_{01}=863/420,\qquad x_{02}=-1/35,$

$x_{11}=-137/105,\qquad x_{12}=-101/140,$

$x_{21}=307/120,\qquad x_{22}=-3851/840,$

$x_{31}=-200153575439/73060041600,\qquad x_{32}=52120014959/73060041600.$

The points $P_i,\ Q_i$ with abscissae x_{i1} and x_{i2}, $i = 0, 1, 2, 3$, have 17 of their 28 mutual distances rational, 16 distances being rational by virtue of the relations (4.10) while the distance between P_1 and P_2 has been made rational by choice of the parameters $\xi,\ \alpha_1$ and β_1.

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