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ON THE JULIA SETS OF TWO PERMUTABLE ENTIRE FUNCTIONS

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ABSTRACT. In 1958, Baker posed a question that if f and g are two permutable transcendental entire functions, must their Julia sets be the same? Since then several classes of entire functions have been exhibited to support an affirmative answer to the question. In this paper, we shall complement or improve some of these results by exhibiting some new classes of entire functions.

1. Introduction and main results. Let f be a nonconstant entire function, and denote by f^n the nth iterate of f. The Fatou $F(f)$ set of f is the set of $z \in \mathbf{C}$ (the whole complex plane) where the family $\{f^n\}$ is normal in a neighborhood of z. Denote by $J(f)$ the complement of $F(f)$, which is called the Julia set of f. An obvious property of a Julia set for an entire or rational function f is that $J(f) = J(f^n)$. More of the basic results of the iteration of rational functions can be found in [**3, 4, 11, 19**] for transcendental entire or meromorphic functions. Factorization theory of entire or meromorphic functions is a subject which studies when an entire or meromorphic function F can be expressed as the composition of two or more simpler entire or meromorphic functions. Here a function h is said to be simpler than another function k means that h has a growth much slower than that of k's. If F can be expressed as $F = f \circ g$, then f and g are called left and right factors of F, respectively. For more of the details, developments and related results of the factorization theory, we refer the reader to [**5, 8**]. As iteration is a special case of the composition, from this it is easily understood that factorization theory and complex dynamics are closely related to each other. An entire or meromorphic function is called prime (pseudo-prime) f if, whenever $f = g \circ h$ for some meromorphic functions g and h, then either g or h is linear (g rational

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or h polynomial). An entire or meromorphic function F is called left-prime if and only if, whenever $F = f \circ g$ with g transcendental, then f must be linear. Moreover, we will say that a factorization is in entire sense if only entire factors are to be considered in the compositions.

Julia [**13**] and Fatou [**7**] independently proved that for any two rational functions f and g of degree at least two such that f and g are permutable, i.e., $f \circ g = g \circ f$, then $J(f) = J(g)$. Baker [**2**] raised the following natural question:

Question 1. Let f and g be two transcendental entire functions. If f and g are permutable, is $J(f) = J(g)$?

Baker [**1**] and Iyer [**12**] proved independently that, if a nonconstant polynomial f is permutable with a transcendental entire function g, then $f(z) = e^{(2m\pi i)/n}z + b$, for some integers $m, n \in \mathbf{N}$ and $b \in \mathbf{C}$. Moreover, in [**1**], Baker also characterized all nonlinear entire functions that are permutable with an exponential function and proved the following two theorems.

Theorem A. *Let g be a nonlinear entire function permutable with $f(z) = ae^{bz} + c$, $(ab \neq 0, a, b, c \in \mathbf{C})$. Then $g = f^n$ for some $n \in \mathbf{N}$. Hence $J(f) = J(g)$.*

Theorem B. *If f and g are transcendental entire functions and if ∞ is neither a limit function of any subsequence of $\{f^n\}$ in a component of $F(f)$, nor of any subsequence of $\{g^n\}$ in a component of $F(g)$, then $J(f) = J(g)$.*

Note that, except Theorem A, all the results mentioned above require some sort of conditions on both f and g. Recently as a further investigation of Question 1, Ng [**16**] came up with the following:

Question 2. Is there a complete classification of all pairs of nonlinear permutable entire functions?

Ng [**16**] proved the following two theorems which impose conditions on only one of the two permutable functions, say f.

Theorem C. *Let f be a transcendental entire function which satisfies the following conditions*:

A1) *f is not the form $H \circ Q$, where H is periodic entire and Q is a polynomial.*

A2) *f is left-prime in entire sense.*

A3) *f' has at least two distinct zeros.*

A4) *There exists a natural number N such that for any complex number c, the simultaneous equations $f(z) = c$, $f'(z) = 0$ has at most N solutions.*

A5) *The orders of zeros of f' are bounded by M for some $M \in \mathbf{N}$.*

Let g be a nonlinear entire function permutable with f. Then $g(z) = af^n(z)+b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.

Theorem D. *Let q be a nonconstant entire function and p a polynomial with at least two distinct zeros. Suppose that $f(z) = p(z)e^{q(z)}$ is prime in entire sense. Then any nonlinear entire function g which permutes with f is of the form $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.*

Remark. For two permutable transcendental entire functions f and g, one will not be able to conclude, in general, that either g and f are linearly related or $g = af^n + b$ for some $n \geq 2$. For instance, see [**10**], let

$$\begin{aligned} f(z) &= ic\bigg[\exp\bigg(\frac{(4k+3)\pi}{8c^2}\, iz^2\bigg) + \exp\bigg(-\frac{(4k+3)\pi}{8c^2}\, iz^2\bigg)\bigg], \\ g(z) &= c\bigg[\exp\bigg(\frac{(4k+3)\pi}{8c^2}\, iz^2\bigg) - \exp\bigg(-\frac{(4k+3)\pi}{8c^2}\, iz^2\bigg)\bigg], \end{aligned}$$

where k is an integer. Then $f(g) = g(f)$. Moreover, it is easily verified that $g \not\equiv af + b$ for any constants a, b and as $r \to \infty$,

$$T(r, f) \sim T(r, g) \sim 2T\bigg(r, \exp\frac{(4k+3)\pi}{8c^2}\, iz^2\bigg),$$

where $T(r,h)$ denotes the Nevanlinna's characteristic function of h, see e.g., [**9**]. The above growth relationship between f and g also excludes the possibility that $g = af^n + b$ for any constants a, b and $n \geq 2$.

In this paper, we shall generalize Theorems A and D as well as complement Theorem C.

Theorem 1. *Let f be a transcendental entire function satisfying the following conditions.*

(B1) *f' has only one zero.*

(B2) *f is pseudo-prime.*

(B3) *f is not of the form $H \circ Q$, where H is periodic entire and Q a polynomial with degree ≥ 2.*

(B4) *f is not of the form $(z-a)^n e^{h(z)} + A$, where a is the zero of f', $n \geq 2$ and $h(z)$ is a transcendental entire function with infinitely many zeros such that $n + (z-a)h'(z)$ has no zeros.*

Let g be a nonlinear entire function which permutes with f. Then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.

Theorem 2. *Let f be a transcendental entire function satisfying the following conditions.*

(B′1) *f' has only finitely many zeros.*

(B′2) *f is pseudo-prime.*

(B′3) *f is not of the form $H \circ Q$, where H is periodic entire and Q a polynomial with degree ≥ 2.*

(B′4) *f is not of the form $p(z)^n e^{h(z)} + A$, where $p(z)$ is a nonconstant polynomial, $n \geq 2$ and $h(z)$ is a transcendental entire function with infinitely many zeros such that $np'(z) + p(z)h'(z)$ has only finitely many zeros.*

Let g be a nonlinear entire function which permutes with f. Then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.

As a variation of Theorem C, we have

Theorem 3. *Let f be a transcendental entire function which satisfies the following conditions*:

(C1) *f' has at least two zeros.*

(C2) *For any complex number c, the simultaneous equations $f(z) = c$, $f'(z) = 0$ has only finitely many solutions.*

(C3) *f is not of the form $H \circ Q$, where H is periodic entire and Q a polynomial of degree ≥ 2.*

(C4) *f is left-prime in an entire sense.*

Assume further that g is a nonlinear entire function satisfying the following condition

(C5) *For any complex number c, the simultaneous equations $g(z) = c$, $g'(z) = 0$ has only finitely many solutions.*

If g is permutable with f, then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.

Finally we shall prove the following result, which is a generalization of Theorem D.

Theorem 4. *Let $f(z) = p(z)e^{\alpha(z)} + a$, where $p(z)$ is a nonconstant polynomial and not of the form $p_1(z)^n$, where $p_1(z)$ a polynomial and $n \geq 2$, $\alpha(z)$ is a nonconstant entire function such that if $p(z) = A(z-a)$ then $1+(z-a)\alpha'(z)$ has at least one and only finitely many zeros and $a \in \mathbf{C}$. Suppose that $f(z)$ is pseudo-prime in an entire sense. If a nonlinear entire function g is permutable with f, then $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$. Hence $J(f) = J(g)$.*

2. Some lemmas.

Lemma 1 (Poon-Yang [**17**]). *Let f and g be transcendental entire functions such that $g(z) = af(z) + b$, $a, b \in \mathbf{C}$. If g permutes with f, then $J(f) = J(g)$.*

Remark. The proof of Lemma 1 is essentially that of Baker's [**1**] which dealt with the case that $g = f + b$.

Definition 1. Let $F(z)$ be a nonconstant entire function. An entire function $g(z)$ is a generalized right factor of F, denoted by $g \leq F$, if there exists a function f, which is analytic on the range of g, such that $F = f \circ g$. If $h \leq f$ and $h \leq g$, we say that h is a generalized common right factor of f and g.

Ng [**16**] obtained the following two lemmas by essentially adopting the arguments used by Eremenko-Rubel [**6**, Theorem 1.1] in their investigations of the existence of possible common right factors of two transcendental entire functions.

Lemma 2. *Let f and g be two entire functions and $z_1, \dots, z_k$ be $k \geq 2$ distinct complex numbers such that*

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_k) = A, \\ g(z_1) = g(z_2) = \cdots = g(z_k) = B. \end{cases}$$

Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f = g_1 \circ g$ on $\cup_{i=1}^{k} \mathbf{U}_i$, where $\mathbf{U}_i$ is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A and the order of f_1 at A is $K < k$, then there exists an entire function h, which depends on f and g only and is independent of k and z_i, with $h \leq f$, $h \leq g$. Moreover, among the z_is, there exist at least $m = [(k-1)/K] + 1$ distinct points $z_{n1}, \dots, z_{nm}$ such that $h(z_{n1}) = \cdots = h(z_{nm})$.

Lemma 3. *Let f and g be two entire functions and $\{z_k\}_{k \in \mathbf{N}}$ an infinite sequence of distinct complex numbers such that $f(z_k) = A$ and $g(z_k) = B$ for all $k \in \mathbf{N}$. Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\cup_{i=1}^{\infty} \mathbf{U}_i$, where $\mathbf{U}_i$ is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A, then there exists a transcendental entire function h with $h \leq f, h \leq g$.*

Lemma 4 (Ng [**16**]). *Let h, k be transcendental entire functions. Suppose that h has infinitely many zeros. Then for each $n \in \mathbf{N}$, there exists a zero a_n of h such that $k(z) = a_n$ has at least n distinct roots which are not zeros of h.*

The following is an easy consequence of the well-known Borel's theorem, see e.g., [**14**, p. 116].

Lemma 5. *Let $p_0(z), p_1(z), \ldots, p_n(z)$ be $n+1$ polynomials and $g_1(z), g_2(z), \ldots, g_n(z)$ be $n+1$ nonconstant entire functions. If the following identity holds*:

$$\sum_{i=1}^{n} p_i(z)e^{g_i(z)} \equiv p_0(z),$$

then $p_0(z) \equiv 0$.

Lemma 6. *Let $f(z) = p(z)e^{e^{q(z)}}$, where $p(z), q(z)$ are nonconstant polynomials. Then $f(z)$ is pseudo-prime.*

Proof. Assume that $f = g \circ h$, where g is transcendental meromorphic and h is transcendental entire. It is easy to derive that g has at most one pole which h omits. If g has one pole, say a, then $h = e^{k(z)} + a$ and $g(w) = (g_1(w))/(w-a)^n$. If g_1 has a zero, then f has infinitely many zeros. If g_1 has no zero, then f has no zero. All these situations are contradicting with the assumption. If g is entire, it follows that $g \circ h = p(z)e^{eq(z)}$ with $g(w) = (w-b)^n e^{q_1(w)}$ and $h(z) = p_2(z)e^{q_2(z)} + b$, where p_2 is a nonconstant polynomial and q_1, q_2 are two nonconstant entire functions. Thus we have

$$g \circ h = p_2(z)^n e^{nq_2(z)} e^{q_1(p_2(z))} = p(z)e^{e^{q(z)}}.$$

It follows from this that

$$e^{q(z)} = q_1(p_2(z)e^{q_2(z)}) + nq_2(z) + A, \tag{1}$$

where A is a constant. Noting the growth of $e^{q(z)}$, we conclude that both q_1 and q_2 are nonconstant polynomials, which is impossible by Lemma 5. $\square$

3. Proofs of Theorems 1, 2, 3 and 4.

Proof of Theorem 1. First of all, we note the simple fact that if a nonlinear entire function g is permutable with the transcendental entire function f, then g must be transcendental itself. From $f \circ g = g \circ f$ we have

$$f'(g(z))g'(z) = g'(f(z))f'(z).$$

Now we prove either $g = f^n$ which leads to $J(f) = J(g)$ or there exists a transcendental entire function H such that $H \leq f$ and $H \leq g$. Let a be the zero of f'. Now we will discuss the following four cases.

Case 1. g' has no zero. Thus, both of f' and $f'(g)$ have only one zero. Hence

$$f'(z) = (z-a)^n e^{h(z)},$$
$$g(z) - a = (z-a)e^{k(z)},$$

where $h(z)$ and $k(z)$ are nonconstant entire functions, n a positive integer. If $f - a$ has no zero, then $f(z) - a = e^{m(z)}$. Since $f(z)$ is pseudo-prime, $m(z)$ must be a polynomial. By condition (B3), we obtain $m(z) = cz + d$. Thus we have, from Theorem A, $g = f^n$ and $J(f) = J(g)$. If $f - a$ has only one zero, then $f - a = (z-a)^k e^{h_1(z)}$ for some integer k. If $k = 1$, then

$$f'(z) = (1 + (z-a)h_1'(z))e^{h_1(z)}.$$

This contradicts $f'(a) = 0$. If $k \geq 2$, noting that $f'(z) = (z-a)^{k-1}(k + (z-a)h_1'(z))e^{h_1(z)}$ and $f'(z)$ has only one zero, it is derived that $k + (z-a)h_1'(z)$ has no zero, i.e., there an entire function $l(z)$ such that $k + (z-a)h_1'(z) = e^{l(z)}$. If $h_1(z)$ is a polynomial, it is easy to see that $k + (z-a)h_1'(z)$ has at least one zero ($\neq a$). If $h_1'(z)$ is a transcendental entire function with only finitely many zeros, then there exist a polynomial $p(z)$ and an entire function $m(z)$ such that $k + p(z)e^{m(z)} = e^{l(z)}$, which is impossible by Lemma 5. Thus $h_1(z)$ is transcendental entire such that $k + (z-a)h_1'(z) = e^{l(z)}$ with $h_1'(z)$ having infinitely many zeros, which, however, contradicts the condition (B4).

If $f - a$ has at least two zeros, by comparing the zeros on both sides of $(f-a) \circ g = (g-a) \circ f$, we obtain that $f - a$ has infinitely many zeros and all zeros of $(f-a) \circ g$ are the zeros of $f - a$, which is impossible according to Lemma 4.

Case 2. g' has one and only one zero, say b. Thus, we have

$$g'(z) = (z-b)^m e^{k(z)},$$

where $k(z)$ is nonconstant entire function and m a positive integer. Hence

$$(g(z)-a)^n e^{h(g(z))}(z-b)^m e^{k(z)} = (f(z)-b)^m e^{k(f(z))}(z-a)^n e^{h(z)}.$$

It follows that either both of $f-b$ and $g-a$ have only finitely many zeros or they have infinitely many common zeros. If $f-b$ and $g-a$ have infinitely many common zeros, then by Lemma 3, there exists a transcendental entire function $H(z)$, which depends on f and g only, with $H \leq f$, $H \leq g$. If $f-b$ and $g-a$ have only finitely many zeros, then there exist polynomials p, q and entire functions k, h such that

$$g(z) - a = p(z)e^{k(z)},$$
$$f(z) - b = q(z)e^{h(z)}.$$

If $q(z)$ is constant, we have $g = f^n$ and $J(f) = J(g)$ by the same arguments as are used in Case 1. Thus, we only need to consider the case that $q(z)$ is a nonconstant polynomial. If $a \neq b$, then $g(z)-b$ has infinitely many zeros. It follows from the facts $(f-b)\circ g = (g-b)\circ f$ and $f-b$ has finitely many zeros that there exist a zero B of $f-b$ and a zero A of $g-b$ such that $f(z)-A$ and $g(z)-B$ have infinitely many common zeros. Hence, by Lemma 3, there exists a transcendental entire function $H(z)$ (which depends on f and g only) with $H \leq f$, $H \leq g$.

Now we discuss the case that $q(z)$ is a nonconstant polynomial and $a = b$. In this case, we have $p(f)e^{k(f)} = q(g)e^{h(g)}$. If $q(z)$ has a zero different from a, then we see easily that $p(z)$ has a zero $A \neq a$ which is different from a, and a zero B of $q(z)$ such that $f(z)-A$ and $g(z)-B$ have infinitely many common zeros. Hence, by Lemma 3, there exists a transcendental entire function $H(z)$, which depends on f and g only, with $H \leq f$, $H \leq g$. Thus combining the above analysis, we have

$$f(z) - a = (z-a)^{n+1}e^{h(z)},$$
$$g(z) - a = (z-a)^{m+1}e^{k(z)},$$

where $n \geq 1$ and $h(z)$ is a nonconstant entire function such that $n+1+(z-a)h'(z)$ has no zero. By the same arguments used in Case 1, we will arrive at a contradiction by the condition (B4).

Case 3. g' has $k \geq 2$ distinct zeros. Noting $f'(g)g' = g'(f)f'$, there exists a zero b of g' such that $f(z) = b$ and $g(z) = a$ have infinitely many common roots. Hence, by Lemma 3, there exists a transcendental entire function $H(z)$, which depends on f and g only, with $H \leq f$, $H \leq g$.

Case 4. g' has infinitely many zeros. By Lemma 4, for any $N \geq n+2$, there exists a zero a_N of g' such that $f(z) = a_N$ and $g(z) = a$ has at least N common roots $z_1, z_2, \ldots, z_N$. Thus, by Lemma 2, there exists an entire function H, which depends on f and g only, with $H \leq g$, $H \leq g$. Moreover, among $z_1, z_2, \ldots, z_N$, there exist at least $m = [N/(n+1)]$ distinct points at which H takes the same value. Since N as well as m can be arbitrarily large, H must be transcendental.

Combining the above four cases, we have that either (A) $g = f^n$, $J(f) = J(g)$ or (B) there exists a transcendental entire function H such that $H \leq f$, $H \leq g$. In case (A), we have proved the theorem. In case (B), $f = f_1 \circ H$ and $g = g_1 \circ H$ for some f_1 and g_1 which are analytic on the range of H. By Little Picard's theorem, H can omit at most one complex number. If the range of H is $\mathbf{C} - \{c\}$ for some $c \in \mathbf{C}$, then $H = c + e^{h(z)}$ for some entire function h and $f(z) = f_1(c + e^w) \circ h(z)$. Since $f(z)$ is pseudo-prime, $h(z)$ must be a polynomial. By assumption (B3), $h(z)$ must be linear. Thus, f and hence f' both are periodic functions. It follows that f' has either infinitely many or no zeros, which contradicts the assumption that f' has only one zero. So the range of H is the whole $\mathbf{C}$. This implies that both f_1 and g_1 are entire. Since f is pseudo-prime and H is transcendental, f_1 must be a polynomial. If the degree of f_1 is greater than 1, noting f' has only one zero and the range of H is the whole complex plane, it follows that $f_1' = (w-b)^m$ and $H(z) = (z-a)^n e^{k(z)}$. Hence $f(z) = A(z-a)^{n(m+1)} e^{(m+1)k(z)} + B$. By applying the same arguments as above, we shall arrive at a contradiction by condition (B4). Thus, f_1 is linear. Then by adopting the same arguments used in the proof of Theorem C, we have that $g = af^n + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$ by Lemma 1. This also completes the proof. $\square$

Proof of Theorem 2. We first prove that either $g = f^n$ and $J(f) = J(g)$ or there exists a transcendental entire function H such that $H \leq f$

and $H \leq g$. If f' has only one zero, then we have done that in the proof of Theorem 1. If f' has at least two distinct zeros, then $f'(g)g' = g'(f)f'$ implies that $g'(f)$ has infinitely many zeros and g' has at least one zero. If g' has only finitely many zeros, it is derived that there exist a zero A of g' and a zero B of f' such that $f - A$ and $g - B$ has infinitely many common zeros. Hence there exists a transcendental entire function H such that $H \leq f$ and $H \leq g$. If g' has infinitely many zeros, it is derived that there exists a transcendental entire function H such that $H \leq f$ and $H \leq g$, by applying the same arguments as in Case 4 above. The conclusion follows by applying similar arguments as used in the proof of Theorem 1. $\square$

Proof of Theorem 3. We first prove that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. In this case, again note that g is transcendental and from $f \circ g = g \circ f$ that

$$f'(g(z))g'(z) = g'(f(z))f'(z). \tag{2}$$

Now we will discuss the following three cases.

Case 1. $g'(f(z))$ has only finitely many zeros. In this case, g' has at most one zero. Since f' has at least two zeros and g is transcendental, $f'(g(z))$ has infinitely many zeros. It follows that f' has infinitely many zeros. Thus, by Lemma 4, for any $N \in \mathbf{N}$, there exists a zero a_N of f' such that $g(z) - a_N = 0$ has at least N distinct roots which are not zeros of f' thus are zeros of $g'(f(z))$. This is a contradiction.

Case 2. $g'(z)$ has only finitely many zeros, but $g'(f(z))$ has infinitely many zeros. In this case, $g'(z)$ has at least one zero. Let b be a zero of g'. By comparing the zeros of the both sides of (2), we see that $f(z) - b$ and $f'(g(z))$ has infinitely many distinct common zeros, say $\{z_n\}_{n \in \mathbf{N}}$. Note that $f(z_n) = b$ implies $f(g(z_n)) = g(f(z_n)) = g(b)$. Thus, for all $n \in \mathbf{N}$

$$\begin{cases} f(g(z_n)) = g(b), \\ f'(g(z_n)) = 0. \end{cases}$$

By condition (C2), there exist a subsequence $\{z_{nk}\}_{k=1}^{\infty}$ and a constant B such that

$$\begin{cases} g(z_{nk}) = B, \\ f(z_{nk}) = b. \end{cases}$$

Thus, by Lemma 3, there is a transcendental entire function h such that $h \leq f$, $h \leq g$.

Case 3. g' has infinitely many zeros. If there exists a zero a_k of g' such that $f(z) - a_k = 0$ and $f'(g(z)) = 0$ have infinitely many common roots, i.e., there is a sequence of $\{z_n\}_{n=1}$ such that

$$\begin{cases} f(z_n) = a_k, \\ f'(g(z_n)) = 0. \end{cases}$$

Noting $f(g(z_n)) = g(f(z_n)) = g(a_k)$ and, by the same discussion as in Case 2, one can conclude that there is a transcendental entire function h such that $h \leq f$, $h \leq g$.

If, for any zero a_k of g', $f(z) - a_k = 0$ and $f'(g(z)) = 0$ have only finitely many common roots, then for a_k, all (except finitely many) zeros of $f - a_k$ are the zeros of g'. Thus, by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} &2T(r,f) \\ (3) \quad &\leq \sum_{k=1}^{3} \overline{N}\left(r, \frac{1}{f - a_k}\right) + S(r,f) \leq \overline{N}\left(r, \frac{1}{g'}\right) + S(r,f) + O(1) \\ &\leq T(r,g') + S(r,f) + O(1) \leq T(r,g) + S(r,f) + S(r,g) + O(1). \end{aligned}$$

It this case, for all (at most except 2) zeros b_k of f', $g(z) - b_k$ and $g'(f(z))$ have infinitely many common zeros. Otherwise, there exist 3 zeros of f', say b_1, b_2, b_3, such that all (except finitely many) zeros of $g - b_k$, $k = 1, 2, 3$, are the zeros of f'. Thus, by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} &2T(r,g) \\ (4) \quad &\leq \sum_{k=1}^{3} \overline{N}\left(r, \frac{1}{g - b_k}\right) + S(r,f) \leq \overline{N}\left(r, \frac{1}{f'}\right) + S(r,g) + O(1) \\ &\leq T(r,f') + S(r,g) + O(1) \leq T(r,f) + S(r,f) + S(r,g) + O(1). \end{aligned}$$

Combining (3) and (4), we obtain

$$T(r,f) + T(r,g) \leq S(r,f) + S(r,g) + O(1),$$

which is impossible. Now if, for some zero b_k of f' and some zero a_m of g', $g - b_k$ and $f - a_m$ have infinitely common zeros, then $f - a_m$ and $f'(g(z))$ have infinitely many common zeros, which contradicts with the assumption. Thus, for some fixed b_k, there exist a sequence of zeros $\{a_m\}_{m=1}^{\infty}$ of g' and a sequence $\{z_m\}_{m=1}^{\infty}$ such that

$$\begin{cases} f(z_m) - a_m = 0, \\ g(z_m) - b_k = 0. \end{cases}$$

Thus we have

$$\begin{cases} g(a_m) = g(f(z_m)) = f(g(z_m)) = f(b_k), \\ g'(a_m) = 0, \end{cases}$$

which contradicts the condition (C5). Combining the above three cases, it can be concluded that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. Noting that f is left-prime and not of the form $f = H \circ Q$, where H is a periodic function, Q a polynomial and by arguing similarly as in the proof of Theorem 1, there exist entire functions f_1 and g_1 such that $f = f_1 \circ h$, $g = g_1 \circ h$. Moreover, f_1 must be linear as f is left-prime and h is transcendental. Hence, $h = f_1^{-1} \circ f$ and $g = g_1 \circ f_1^{-1} \circ f = g_2 \circ f$ where $g_2 = g_1 \circ f_1^{-1}$. Since $g = g_1 \circ h$, it follows that, for any constant c, $g_1(z) - c$ and $g_1'(z)$ have only finitely many common zeros. Otherwise, $g(z) - c$ and $g'(z)$ have infinitely many common roots, which is impossible according to condition (C5). It is derived that $g_2(z) - c$ and $g_2'(z)$ have only finitely many common zeros, for any complex number c. From $f \circ g = g \circ f$, we have $f \circ g_2 \circ f = g_2 \circ f \circ f$. Noting that the ranges of f and h are the same, which is the whole complex plane, we have $f \circ g_2 \equiv g_2 \circ f$ on $\mathbf{C}$. If g_2 is nonlinear, then f, g_2 satisfy the conditions in the theorem. By repeating the same arguments, we can find an entire function g_3 such that $g_2 = g_3 \circ f$ and $g = g_3 \circ f^2$. By using arguments similar to the proof of Theorem C, we can derive that $g = af^n + b$, where n is an integer, a a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.

Proof of Theorem 4. If $p(z)$ has at least two distinct zeros, then $p(g(z))$ has infinitely many zeros. It follows from $p(g(z))e^{\alpha(g(z))} = (g - a) \circ f(z)$ that $(g - a) \circ f(z)$ has infinitely many zeros. Thus there exist a zero A of $p(z)$ and a zero B of $g - a$ such that $f(z) - B$ and

$g(z) - A$ have infinitely many common zeros. Again by Lemma 3, there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If $p(z)$ has only one zero, then $f(z) = (z-b)^n e^{\alpha(z)} + a$ and $(g-b)^n e^{\alpha(g)} = (g-a) \circ f$. Now if $g-a$ has at least two distinct zeros, by applying the same arguments as above, we conclude that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If $g-a$ has only one zero, say c, then $g(z) = (z-c)^m e^{k(z)} + a$, where $k(z)$ is a nonconstant entire function. Thus, $f \circ g = g \circ f$ implies that $(g-b)^n e^{\alpha(g)} = (f-c)^m e^{k(f)}$. If $b \neq a$, then $g(z) - b$ has infinitely many zeros which are also the zeros of $f(z) - c$. Hence there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$, by Lemma 3. If $c \neq a$, we will arrive at the same conclusion. If $a = b = c$, then

$$f(z) = (z-a)^n e^{\alpha(z)} + a, \tag{5}$$

$$g(z) = (z-a)^m e^{k(z)} + a, \tag{6}$$

where $n \geq 1$ and $m \geq 1$. If $n \geq 2$, this contradicts with the assumption. If $n = 1$, then by the assumption, f' has at least one and only finitely many zeros, which are different from a (a Picard exceptional value of g). Thus, $f'(g)g' = g'(f)f'$ implies that $g'(f)$ has infinitely many zeros. If g' has only finitely many zeros, it is easy to derive that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If g' has infinitely many zeros, by applying arguments similar to that used in Case 4 in the proof of Theorem 1, we can conclude that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. The last case is that $g - a$ has no zero. It is derived that, if f is not of the form (5), then there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If f is of the form (5), then f' has at least one zero different from a, which will lead to the same conclusion. By combining the above cases, we conclude that there exist a transcendental entire function h and functions f_1 and g_1 which are analytic on the range of h. Since h is transcendental entire, h omits at most a finite value. If h omits a finite complex number c, then $h(z) = c + e^{\alpha_1(z)}$, where $\alpha_1(z)$ is a nonconstant entire function. Then $G(w) = f_1(c + e^w)$ is transcendental entire and periodic and $f(z) = G \circ \alpha_1(z)$. Noting f is pseudo-prime, $\alpha_1(z)$ must be nonconstant polynomial. Thus G either has infinitely many or no a-points at all, so does f. This contradict with the hypothesis that f has finitely many a-points. Thus, the range of h is whole complex plane and f_1, g_1 are entire. Noting that f is pseudo-prime and h is transcendental, f_1 must be a polynomial. If $f_1 - a$ has at least two distinct zeros, then f has infinitely many a-points, a contradiction. Thus, $f_1(w) = A(w-c)^k + a$. If $k \geq 2$, it is derived from $p(z)e^{\alpha(z)} = A(h(z)-c)^k$ that $h(z) = p_1(z)e^{\alpha_1(z)} + c$ and $p(z) = Ap_1(z)^k$; this contradicts the assumption. Hence, f_1 is linear. Repeating the arguments in the proof of Theorem C, one can conclude that $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$, and hence $J(f) = J(g)$.

4. Corollaries.

Corollary 1. *Let f be a transcendental entire function such that f' has only one zero and f is prime in an entire sense. Let g be a nonlinear entire function permutes with f, then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.*

Corollary 1 is an easy consequence of Theorem 1, and from it we have the following generalization of Theorem A.

Corollary 2. *If g is a nonlinear entire function permutable with $f(z) = (a_1 z + b_1)e^{a_2 z} + b_2$, $(a_1, a_2, b_1, b_2 \in \mathbf{C}$, $a_2 \neq 0$, a_1, b_1 are not zero at the same time), then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbf{C}$. Hence $J(f) = J(g)$.*

As an application of Theorem 1, we have

Corollary 3. *Let f be a transcendental entire function of finite order such that f' has only one zero and f is not of the form $Ae^{B(z-a)^n} + C$, where $A \neq 0$, $B \neq 0$, $n \geq 2$. Let g be a nonlinear entire function which permutes with f. Then $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$. Hence $J(f) = J(g)$.*

Proof. First of all, it is known that if f is of finite order and f' has only one zero, then f is pseudo-prime. In order to apply Theorem 1, we need to show that f satisfies the condition (B3). Otherwise, we have

$f(z) = f_1(q(z))$, where f_1 is periodic and $q(z)$ is a polynomial with degree ≥ 2. Noting that f' has only one zero and q' has at least one zero, we conclude immediately that f_1' has at most one zero. Since f_1' is periodic, f_1' cannot have any zeros. It follows that $f_1'(w) = e^{a_1 w + b_1}$. From this it is easy to derive that $q(z) = a_2(z-a)^n + b_2$. Thus $f(z) = f_1 \circ q = Ae^{B(z-a)^n} + C$. By the assumption, n can only be equal to 1, and the conclusion follows immediately by Theorem A. □

As an illustration of Corollary 3, we have

Corollary 4. *If g is a nonlinear entire function permutable with the function f:*

$$f(z) = \int^z (z-a)^n e^{p(z)}\, dz,$$

where $a \in \mathbf{C}$ and $p(z)$ is a nonconstant polynomial and $p(z) \not\equiv A(z-a)^{n+1} + B$, for any constants A, B, then $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$. Hence $J(f) = J(g)$.

The following result follows immediately from Theorem 4.

Corollary 5. *Let $f(z) = p(z)e^{q(z)} + a$, where p, q are two nonconstant polynomials, $p(z)$ is not of the form $p_1(z)^n$, where $p_1(z)$ is a polynomial, $n \geq 2$ and $a \in \mathbf{C}$. If a nonlinear entire function g is permutable with f, then $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$. Hence $J(f) = J(g)$.*

By an application of Theorem 4 and Lemma 6, we have

Corollary 6. *Let $f(z) = p(z)e^{e^{q(z)}} + a$, where $p(z), q(z)$ are nonconstant polynomials, $p(z)$ has at least two distinct zeros and is not of the form $p_1(z)^n$, where $p_1(z)$ is a polynomial and $n \geq 2$, $a \in \mathbf{C}$. If nonlinear entire function g is permutable with f, then $g(z) = a_1 f^n(z) + b_1$, where a_1 is a kth root of unity and $b_1 \in \mathbf{C}$. Hence $J(f) = J(g)$.*

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